A Braneless Approach to AdS/QCD

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Brief Reminder: AdS/CFT

The canonical example of gauge/gravity duality is Maldacena's duality (hep-th/9711200): a correspondence between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and string theory on $AdS_5 \times S^5$. The AdS space has a negative cosmological constant associated with some flux through the extra dimensions.

Maldacena's duality gives a concrete realization of 't Hooft's old idea about $1/N$ expansions: $\alpha' \sim 1/\sqrt{\lambda}$, $g_s \sim g_{YM}^2$. The string theory is *perturbative* when the field theory is *nonperturbative*.

Ordinary QCD has $N = 3$, so α' is large. The dual should be highly curved, so naively it seems hopeless.

Warped Space Notation

The metric of the AdS space can be written in an explicitly conformally flat form as:

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2), \tag{1}$$

or (in different coordinates) $e^{-2y/R}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$ (with $e^{y/R} = z/R$).

More generally, we will consider a warped space with metric

$$ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{2}$$

and we will take z to be defined by $e^{y/R} = z/R$.

The corresponding Ricci scalar is

$$\mathcal{R} = -8A''(y) + 20A'(y)^2. \tag{3}$$

The AdS/CFT Correspondence

AdS/CFT instructs us to add a field φ_j to the 5D theory for every *gauge-invariant* operator $\mathcal{O}_j$ of the 4D theory. The 5D action corresponds to the generating functional of the 4D theory, in the sense that $\exp S_{5D}|_\partial = Z[J_j]$ where J_j are sources for the operators $\mathcal{O}_j$ in 4D, and the left-hand side is evaluated with the boundary condition $\varphi_j \to J_j$ at the boundary.

It turns out that the x-independent solutions for an AdS scalar field of mass m are z^Δ and $z^{4-\Delta}$, where $m^2 = \Delta(\Delta - 4)$. The dictionary associates to a scalar operator $\mathcal{O}$ of dimension Δ a field φ with precisely this mass squared.

Why Might AdS/QFT Work?

In Wilsonian RG, demanding that the final physical answers are invariant produces a set of differential equations where the derivatives are with respect to μ.

Recall that, in string theory, one takes *worldsheet β-functions* and reinterprets them as *spacetime equations of motion*, which then must arise from some effective action. Analogously, holography reinterprets the RG equations of a theory as *equations of motion* in a theory where the RG scale is a new dimension.

When the underlying field theory is conformal, the holographic dual will live on pure AdS; QCD is only conformal in the far UV (where it has a Gaussian fixed point), so it should have a dual that is only asymptotically AdS.

SVZ and OPEs

Shifman, Vainshtein, and Zakharov (1979) (hereafter **SVZ**) pointed out that nonperturbative QCD generates a large *gluon condensate*, $\left\langle \alpha_s G^a_{\mu\nu} G^{a\mu\nu} \right\rangle$. It is a manifestation of confinement and has calculable phenomenological implications. Consider a two-point function for some current $J^\mu = \bar{q}\gamma^\mu q$. *Assuming* the validity of a Wilsonian OPE, we can calculate in perturbation theory:

$$i \int d^4x e^{iqx} \langle J^\mu(x) J^\nu(0) \rangle = (q^\mu q^\nu - q^2 g_{\mu\nu})(-\frac{1}{4\pi^2} \log Q^2 (1 + \frac{\alpha_s}{\pi}) + \frac{2m_q}{Q^4} \langle \bar{q}q \rangle + \frac{1}{12\pi Q^4} \left\langle \alpha_s \mathrm{Tr} G^2 \right\rangle + \cdots) \quad (4)$$

This is valid for *deep Euclidean* momenta $Q^2 = -q^2$ and breaks down due to instanton contributions, which in pure Yang-Mills only show up at Q^{-11}.

SVZ pointed out a powerful consequence of the OPE: through dispersion relations, it directly relates to properties of the *lightest* mesons. It is true in general that:

$$-\frac{d}{dQ^2}\Pi(Q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\Pi(s)ds}{(s+Q^2)^2}. \tag{5}$$

The LHS is given by the OPE, while the RHS is an experimentally measurable quantity. With this SVZ did something clever: they take a Borel transform, which produces a result of the form:

$$\frac{1}{\pi M^2}\int \mathrm{Im}\Pi(s)e^{-s/M^2}ds = h_0 + \frac{h_2}{2!(M^2)^2} + \frac{h_3}{3!(M^2)^3} + \cdots \tag{6}$$

The h_i are calculable using the OPE, while the exponential in the integral damps out contributions from high s. The integral is dominated by a contribution from the lightest resonance, and the right-hand-side is determined by the condensates.

AdS/QCD

If we want to try to model QCD holographically, we have to break the conformal symmetry. The easiest way is to just put a hard wall in the space, also known as an IR brane, at some finite $z_c \approx \Lambda_{QCD}^{-1}$. This automatically gives a spectrum of resonances located at zeroes of Bessel functions.

Detailed calculations of mesons were carried out by Erlich et.al. (hep-ph/0501128) and da Rold and Pomarol (hep-ph/0501218). The results are surprisingly good. Let's try to understand why. They considered 5D fields $A_{L\mu}$ and $A_{R\mu}$ corresponding to currents of the $SU(2)_L \times SU(2)_R$ flavor symmetry (considering only u, d quarks), as well as a scalar X with vev proportional to $m_q z + \langle \bar{q} q \rangle z^3$.

The ρ meson mass is given by 2.4 z_c^{-1}, fixing the location of the IR wall. The $\rho - a_1$ mass splitting is determined by the condensate $\langle \bar{q}q \rangle$. The pion mass then determines m_q: it must obey the Gell-Mann – Oakes – Renner relation since this model doesn't violate the basic current algebra assumptions.

With three inputs, the model then gets various other quantities right to roughly 10% to 15% accuracy, including couplings $g_{\rho\pi\pi}$, decay constants for the mesons, and various coefficients of the chiral Lagrangian. These aren't all determined by chiral symmetry and hidden local symmetry; what's going on? Is some large-N approximation getting them right? Or *maybe* it's all a numerical accident?

Another possibility: AdS gets the leading conformal (log) term in the OPE right, the hard wall somehow mimics the gluon condensate, we have the proper $\langle \bar{q}q \rangle$ condensate.... Matching these things can enforce certain properties of light mesons as in SVZ.

But, if we consider more mesons (e.g. glueballs), they couple to the gluon condensate differently. This is an insight of Novikov, Shifman, Vainshtein, Zakharov (1981): not all hadrons are alike. There are tensor $\bar{q}q$ and glueball states of the same dimension, with *very different* couplings to $\left\langle \alpha_s G^2 \right\rangle$. The hard-wall model will give them the same mass. So we need some more systematic approach to matching.

The “Braneless” Approach

Cutting off the space in the IR with a hard wall might be a reasonable approximation in some cases, but it’s *ad hoc*. We already *know* that, in the perturbative regime, deviations from conformality in QCD come from two sources: α_s corrections (log running) and condensates (power corrections).

SVZ told us that these corrections constrain the lightest mesons. If we want a good approximation, it’s best to not add other artificial sources of corrections!

A condensate means a field that in 5D has power law growth, $\varphi(z) \sim z^{\Delta}$. As z grows, this corresponds to a large energy density! It has a *backreaction* on the metric and at some point forces the space to cut off. So all we have to do is solve the coupled Einstein – scalar equations of motion.

Einstein Equations Made Easy

In some cases analytic solutions to the coupled Einstein – scalar equations are easy to find using the superpotential method. (See e.g. DeWolfe/Freedman/Gubser/Karch, hep-th/9909134). Writing the metric as $ds^2 = e^{-2A(y)}dx^2 + dy^2$, and assuming the scalars have a potential $V(\varphi_1, \varphi_2, \ldots)$,we look for a function $W(\varphi_1, \varphi_2, \ldots)$ such that

$$V(\varphi_1, \varphi_2, \ldots) = 18 \sum_i \left(\frac{\partial W}{\partial \varphi_i} \right)^2 - 12 W(\varphi_1, \varphi_2, \ldots)^2. \qquad (7)$$

Then the background is determined by solving:

$$\begin{aligned} A'(y) &= W(\varphi_1(y), \varphi_2(y), \ldots) && (8) \\ \varphi_i'(y) &= 6 \frac{\partial W}{\partial \varphi_i}. && (9) \end{aligned}$$

(Note that we're *not* assuming supersymmetry.)

An Asymptotically Free Background

Motivated by the idea of the *renormalization group* interpretation of holography, we add a field ϕ and demand that $e^{b\phi}$ (for some b) be interpretable as a running gauge coupling:

$$e^{b\phi} = \frac{1}{\log(z_0/z)} = \frac{R}{y_0 - y}. \tag{10}$$

The superpotential is then given by $W(\phi) = \frac{1}{6Rb^2} e^{b\phi} + W_0$. There is a cancellation in the associated potential if $b = \sqrt{2/3}$, in which case

$$V(\phi) = -6e^{\sqrt{\frac{2}{3}}\phi} - 12. \tag{11}$$

Amusingly, if one takes a noncritical $D = 5$ string theory, the central charge term, after going to Einstein frame and converting to our normalization for the scalar field, provides a potential going as $e^{\sqrt{2/3}\phi}$. The extra constant term in our potential is presumably something like the RR flux that provides the cosmological constant in Maldacena's duality, though we don't have an associated S^5. We won't worry more about the string theory interpretation for now....

The metric we get from this superpotential has

$$A(y) = y + \frac{1}{4} \log \frac{R}{y_0 - y}. \tag{12}$$

Where the coupling blows up at y_0, the warp factor goes to $-\infty$ and hence the metric goes to zero. In particular, there is a *curvature singularity*.

We can find a discrete spectrum of normalizable modes by demanding that they not blow up at the singularity; this plays the role formerly played by boundary conditions on the IR wall. They lie at 2.52, 5.45, 8.16, 10.81, ... The ratio of the first two is 2.16, rather larger than the lattice estimate of about 1.7. The higher modes are equally spaced, not having the Regge-like behavior one expects for radial excitations. If we identify $2.52z_0^{-1}$ with the estimate of the first glueball mass as 1600 MeV, we obtain $z_0^{-1} \approx 635$ MeV.

One can also calculate that there is a gluon condensate proportional to z_0^{-4} in this background. To do this, we need to use the known OPE for the two-point function to set the normalization for $\frac{1}{2\kappa^2}$ (analogous to how $\frac{1}{g_5^2}$ was fixed in the hard-wall AdS/QCD model). After this, we simply compute

$$S_{5D} = \frac{1}{2\kappa^2} \int dz \left(\frac{R}{z}\right)^5 \log \frac{z_0}{z} \left(-\mathcal{R} - \frac{1}{2} z^2 \phi'(z)^2 - \frac{12}{R^2} - \frac{6}{R^2} e^{\sqrt{2/3}\phi(z)}\right), \tag{13}$$

which after canceling UV-divergent pieces with counterterms reduces to $S(z_0) = \frac{1}{\kappa^2 z_0^4}$. Now, we identify this with $W[J]$, where we interpret $\frac{1}{4g_{YM}^2}$ as a source for the operator $F^{a\mu\nu}F^a_{\mu\nu}$. However, we also have a definite relationship between g_{YM}^2 and z_0.

We have $e^{-\sqrt{2/3}\phi} = \log\frac{z_0}{z}$, whereas $g_{YM}^{-2} = \frac{\beta_0 \log\frac{z_0}{z}}{8\pi^2}$. Here $\beta_0 = \frac{11}{3}N_c$. This tells us that taking a derivative with respect to g_{YM}^{-2} is the same as taking:

$$\left\langle \frac{1}{4}F^2 \right\rangle = -\frac{24\pi^2}{11N_c} z_0 \frac{d}{dz_0}\left(\frac{1}{\kappa^2 z_0^4}\right). \tag{14}$$

Using the estimate of z_0^{-1} from the first glueball mass, this tells us:

$$\left\langle \frac{1}{2\pi^2}F^2 \right\rangle = 0.05\text{GeV}^4, \tag{15}$$

which is roughly the size expected on the basis of lattice results.

(Work in progress – don't trust the number yet!)

Analyticity

Ordinary perturbation theory has an unphysical pole at Λ_{QCD}, which does not go away at higher orders. Such a pole *cannot be physical*, on general grounds; it corresponds to no physical particle. Shirkov and Solovtsov have proposed an analytic perturbation theory which matches onto the usual perturbation theory but uses dispersion relations and the Källén-Lehmann representation to enforce good analytic properties (hep-ph/9604363, hep-ph/9909305).

Interestingly, we can re-interpret the holographic solution in a similar light. In the metric the coefficient of dx^2 is $e^{-2A(y)}$, so rather than interpreting μ as z^{-1}, we can interpret it as $e^{-A(y)}$ (these agree, with small corrections, in the UV).

Once we interpret $e^{-A(y)}$ as μ, we obtain an expression for $\alpha_s(\mu)$ which doesn't blow up until $\mu = 0$. In particular, we find $\alpha_s^{-1}(\mu) \sim W(\mu^4/\Lambda_{QCD}^4)$, where W is the "Lambert W-function", i.e. $W(y)$ is x such that $y = x\exp(x)$.

This function has also arisen in studies of Shirkov et. al.'s "analytic peturbation theory", though the precise functional form of α_s was slightly different in those studies. Asymptotically,

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + \cdots, \tag{16}$$

but for small x, $W(x) = x - x^2 + \frac{3}{2}x^3 + \cdots$.

This would seem to suggest questions about the universality class of functions that look like asymptotic perturbation series but have good analytic properties....

Corrections

There are two kinds of corrections to consider. α_s corrections should appear as terms in our 5D action multiplied by higher powers of $e^{\sqrt{2/3}\phi}$. Since these are essentially perturbative QCD corrections, they are small at small z, but possibly become very important near $z = z_c$.

The other corrections are basically α' corrections. We don't have a string theory to compute these in, but we *do* know the higher-dimension operators of QCD, which have larger bulk mass (recall $m^2 = \Delta(\Delta - 4)$) and so look like stringy excitations in the dual.

The first new operator to consider in pure Yang-Mills is $\mathrm{Tr}F^3$.

We add to the 5D theory a field χ which should couple to some multiple of TrF^3. We look for a solution with the right bulk mass, and with some coupling to $\exp\sqrt{2/3}\phi$ to provide an anomalous dimension. For now we won't try too hard to match the details of perturbation theory, but instead look for an analytically solvable solution. For this we take:

$$W(\phi,\chi) = \frac{1}{4}e^{\sqrt{2/3}\phi} + \cosh(\chi). \tag{17}$$

The corresponding potential turns out to be:

$$\begin{aligned} V(\phi,\chi) &= -15 - 6e^{\sqrt{2/3}\phi}\cosh\chi + 3\cosh(2\chi) \\ &= -12 - 6\exp\sqrt{2/3}\phi + 6\chi^2 - 3\exp\sqrt{2/3}\phi\chi^2 + \mathcal{O}(\chi^4). \end{aligned} \tag{18}$$

The associated solution is:

$$e^{\sqrt{2/3}\phi(z)} = \frac{1}{\log \frac{z_0}{z}} \tag{19}$$

$$\chi(z) = \log \frac{1 - \left(\frac{z}{z_1}\right)^6}{1 + \left(\frac{z}{z_1}\right)^6} \tag{20}$$

$$A(z) = \frac{1}{4} \log \frac{1}{\log \frac{z_0}{z}} - \frac{1}{6} \log \left(\left(\frac{z_1}{z}\right)^6 - \left(\frac{z}{z_1}\right)^6 \right). \tag{21}$$

Here we allow $\phi(z)$ and $\chi(z)$ to blow up at *different* z values, z_0 and z_1. It turns out that the case $z_1 < z_0$ is unacceptable: it violates the Gubser criterion (hep-th/0002160) that demands that the scalar potential, evaluated on the solution, be bounded above. This means that TrF^3 cannot cause a singularity at higher energy than TrF^2, which seems intuitively reasonable.

We have completed a preliminary calculation of the glueball spectrum for this background (in the case $z_0 = z_1$), neglecting mixing effects, which are expected to be small since the modes associated with TrF^3 are significantly heavier than the lightest modes associated with TrF^2. Previously the first three ratios of scalar glueball masses were 2.16, 3.24, and 4.29; now they shift down to 2.08, 3.04, and 3.96, coming into somewhat better agreement with the lattice data.

It is crucial to check that the overall *scale* of glueball masses, relative to the condensate of TrF^2, are not shifted significantly when the new operator is added. If they are, then our attempt to build up a systematic approximation is failing. The condensate calculation is in progress....

Conclusions

- Backreaction from scalar fields dynamically cuts off the space and avoids the *ad hoc* IR wall.

- Despite the singularity, the spectrum on such backreacted backgrounds is calculable and looks similar to the spectrum of backgrounds with a wall.

- Adding the effects of higher-dimension operators appears to make small differences in mass ratios.

- There is an interesting interpretation of the backgrounds as enforcing good analytic properties.

Future

- Add flavor, see if the good results of Erlich et. al. persist.

- Understand what the relation to analytic perturbation theory might be.

- Try to make precise the RGE / holography correspondence and understand what it might tell us about the size of corrections. Can we use existing numerical studies of RGEs to build 5D actions and backgrounds?