(10)

Instantons

$$\frac{\theta_{ym}}{32\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu} = \frac{\theta_{ym}}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \mathrm{Tr}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right)$$

total derivative

no effect in perturbation theory

semiclassical configurations: $A^a_\mu = \dfrac{h^a_{\mu\nu} (x-x_0)^\nu}{(x-x_0)^2 + \rho^2}$

$$\frac{\theta_{ym}}{32\pi^2} \int d^4x\, F^{a\mu\nu} \tilde{F}^a_{\mu\nu} = n\, \theta_{ym}$$

n ↑ winding #

$$|\theta_{ym}\rangle = \sum_n e^{i\theta_{ym} n} |n\rangle$$

$$PI = \int \mathcal{D}A\, \mathcal{D}\lambda\, \mathcal{D}D\, e^{iS}$$

$\theta_{ym} \to \theta_{ym} + 2\pi$ is a symmetry

$$0 \leq \int d^4x\, \mathrm{Tr}(F \pm \tilde{F})^2 = \int d^4x\, 2\,\mathrm{Tr}\, F^2 \pm 2\,\mathrm{Tr}\, F\tilde{F}$$

$$\int d^4x\, \mathrm{Tr}\, F^2 \geq \left| \int d^4x\, \mathrm{Tr}\, F\tilde{F} \right| = 16\pi^2 |n|$$

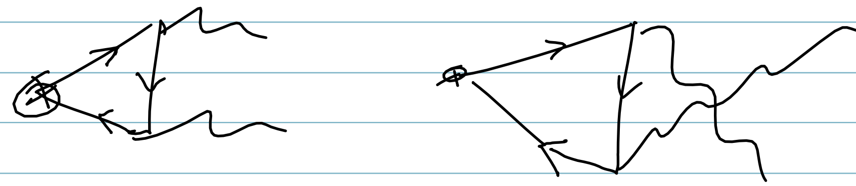
$$e^{-S_{inst}} = e^{-\frac{8\pi^2}{g^2}}$$

$$\int d^4x_0 \int \frac{d\rho}{\rho^5}\, e^{-\frac{8\pi^2}{g(1/\rho)}} = \int d^4x_0 \int_0^\infty \frac{d\rho}{\rho^5} (\rho\Lambda)^{3N}$$

$N \geq 2$ IR divergent

(11)

Adler - Bell - Jackiw



$$\partial_\mu J_A^\mu \propto F^a_{\mu\nu} \tilde{F}^{a\,\mu\nu}$$

$$S = \int d^4x \; i \psi^\dagger \bar{\sigma}^\mu D_\mu \psi$$

$$\psi \to e^{i\alpha(x)} \psi$$

$$S \to S + \int d^4x \; i\psi^\dagger \bar{\sigma}^\mu (\partial_\mu i\alpha) \psi$$

$$= S + \int d^4x \; \alpha(x) \partial_\mu (\psi^\dagger \bar{\sigma}^\mu \psi)$$

classically $\partial_\mu (\psi^\dagger \bar{\sigma}^\mu \psi) = 0$

Path Integra (Fujikawa)

$$\not{D} = i\bar{\sigma}^\mu (\partial_\mu + i A_\mu)$$

$$\bar{\not{D}} = i\sigma^\mu (\partial_\mu - i A_\mu)$$

$$\int d^4x \, \psi^\dagger \not{D} \psi = \int d^4x \, \psi \bar{\not{D}} \psi^\dagger$$

$$D^2 f_n = \bar{\not{D}} \not{D} f_n = -\lambda_n^2 f_n$$

$$\bar{D}^2 g_n = \not{D} \bar{\not{D}} g_n = -\lambda_n^2 g_n$$

$$\sum_n f_n^*(x) f_n(y) = \delta(x-y)$$

$$\mathrm{Tr} \int d^4x \; f_n^*(x) f_m(x) = \delta_{mn}$$

Grassmann variable

$$\psi(x) = \sum_n a_n f_n(x)$$

$$\psi^\dagger(x) = \sum_n b_n g_n(x)$$

$$\not{D} f_n = \lambda_n g_n \qquad \bar{\not{D}} g_n = -\lambda_n f_n$$

in general different # of zero modes

$$\int \mathcal{D}\psi \mathcal{D}\psi^\dagger = \int \prod_{nm} da_n \, db_n$$

chiral rotation $\quad a'_n = \int d^4x \, f_n^* \psi'(x)$

$$b'_n = \int d^4x \, \psi^{\dagger\prime}(x) \, g_n(x)$$

$$a_n \to a'_n = C_{nm} a_m \qquad C_{nm} = \mathrm{Tr} \int d^4x \, e^{i\alpha(x)} f_n^*(x) f_m(x)$$

$$b_n \to b'_n = \bar{C}_{nm} b_m \qquad \bar{C}_{nm} = \mathrm{Tr} \int d^4x \, e^{-i\alpha} g_m^*(x) g_n(x)$$

$$\prod_{nm} da_n \, db_m \to \left(\det C \det \bar{C}\right)^{-1} \prod_{nm} da_n \, db_m$$

$$1 = \int d\theta \, \theta = \int d(c\theta)(c\theta) \qquad d(c\theta) = \frac{1}{c} d\theta$$

expand C_{nm}

$$C_{nm} = \delta_{nm} + \delta C_{nm} + \mathcal{O}(\alpha^2)$$

$$\delta C_{nm} = i \, \mathrm{Tr} \int d^4x \, \alpha(x) f_n^*(x) f_m(x)$$

$$\delta \bar{C}_{nm} = -i \, \mathrm{Tr} \int d^4x \, \alpha(x) g_m^*(x) g_n(x)$$

13

$$\mathrm{Tr}\, M = \sum_i \lambda_i \qquad \mathrm{Det}\, M = \prod_i \lambda_i$$

$$\mathrm{Tr} \log M = \log \mathrm{Det}\, M$$

$$\det C = \exp \mathrm{Tr} \log C \simeq \exp \mathrm{Tr} \log (\delta C)$$

$$(\det C \bar{C})^{-1} = \exp\left(-i \int d^4x\, \alpha(x)\, A(x)\right) + O(\alpha^2)$$

$$A(x) = \sum_n \mathrm{Tr}\left(f_n^*(x) f_n(x) - g_n^* g_n(x)\right)$$
$$= \delta(0) - \delta(0)$$

$$0 = \left.\frac{\delta Z}{\delta \alpha}\right|_{\alpha=0} = \left.\frac{\delta}{\delta \alpha}\right|_{\alpha=0} \int \mathcal{D}\psi \mathcal{D}\psi^\dagger\, e^{iS + \int d^4x\, \alpha(x)(\partial_\mu j_A^\mu - iA)}$$

$$= \int \mathcal{D}\psi \mathcal{D}\psi^\dagger\, e^{iS} \left(\partial_\mu j_A^\mu(x) - i A(x)\right)$$

Regulator function $R(z)$: $R(0) = 1$, $R(\infty) = 0$, $R'(\infty) = 0$, $R'(0) = -1$, $R''(\infty) = 0 \ldots$

eg e^{-z}

$$A(x) = \lim_{M \to \infty} \mathrm{Tr} \sum_n \left(f_n^*(x) R\left(\frac{\lambda_n^2}{M^2}\right) f_n(x) - g_n^*(x) R\left(\frac{\lambda_n^2}{M^2}\right) g_n(x) \right)$$

$$= \lim_{M \to \infty} \mathrm{Tr} \sum_n \left(f_n^*(x) R\left(\frac{-\not{D}^2}{M^2}\right) f_n(x) - g_n^*(x) R\left(\frac{-\bar{\not{D}}^2}{M^2}\right) g_n(x) \right)$$

$$= \lim_{\substack{y \to x \\ M \to \infty}} \mathrm{Tr} \sum_n \left(f_n^*(y) R\left(\frac{-\not{D}^2}{M^2}\right) f_n(x) - g_n^*(y) R\left(\frac{-\bar{\not{D}}^2}{M^2}\right) g_n(x) \right)$$

(14)

$$A(x) = \lim_{\substack{y\to x \\ M\to\infty}} \mathrm{Tr}\left(R\left(\frac{-D^2}{M^2}\right) - R\left(-\frac{\bar{D}^2}{M^2}\right)\right)\delta(y-x)$$

$$D^2 = (\partial^2 + A^2) - \sigma^{\mu\nu}\left(F_{\mu\nu} - 2A_{[\mu}\partial_{\nu]}\right)$$

$$\bar{D}^2 = (\partial^2 + A^2) + \bar{\sigma}^{\mu\nu}\left(F_{\mu\nu} - 2A_{[\mu}\partial_{\nu]}\right)$$

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)$$

$$A(x) = \lim_{\substack{y\to x \\ M\to\infty}} \mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\left(R\left(\frac{p^2 - A^2 + \overbrace{\sigma^{\mu\nu}(F_{\mu\nu} + 2iA_{[\mu}p_{\nu]})}^{F + a^\mu p_\mu}}{M^2}\right) - R\left(\frac{p^2 - A^2 - \underbrace{\bar{\sigma}^{\mu\nu}(F_{\mu\nu} - 2iA_{[\mu}p_{\nu]})}_{F + \bar{a}^\mu p_\mu}}{M^2}\right)\right) e^{ip(y-x)}$$

$$= \lim_{M\to\infty} \mathrm{Tr}\int \frac{d^4p}{(2\pi)^4} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\left(\frac{F + a^\mu p_\mu}{M^2}\right)^n - \left(\frac{-F - \bar{a}^\mu p_\mu}{M^2}\right)^n\right) \times R^{(n)}\left(\frac{p^2 - A^2}{M^2}\right)$$

$$= \lim_{M\to\infty} \mathrm{Tr}\int \frac{d^4\hat{p}\, M^4}{(2\pi)^4}\left(\frac{1}{2}\left(\left(\frac{F}{M^2} + \frac{a^\mu \hat{p}_\mu}{M}\right)^2 - \left(-\frac{F}{M^2} - \frac{\bar{a}^\mu \hat{p}_\mu}{M}\right)^2\right) R^{(2)}\left(\hat{p}^2 - \frac{A^2}{M^2}\right) + \frac{1}{4}\left(\left(\frac{a^\mu \hat{p}_\mu}{M}\right)^4 - \left(\frac{\bar{a}^\mu \hat{p}_\mu}{M}\right)^4\right) R^{(4)}\left(\hat{p}^2 - \frac{A^2}{M^2}\right) + O\left(\frac{1}{M^6}\right)\right)$$

$$= \frac{i}{16\pi^2}\int_0^\infty \hat{p}^2\, d\hat{p}^2\, R^{(2)}(\hat{p}^2)\, \epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\, F_{\mu\nu}F_{\alpha\beta}$$

$$= \frac{i}{16\pi^2} F^{a\mu\nu}\tilde{F}^a_{\mu\nu}$$

(15)

$$\mathrm{Tr} \sum_n \left(f_n^*(x) f_n(x) - g_n^*(x) g_n(x) \right) = \frac{1}{16\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu}$$

$$\partial_\mu j^\mu_A = \frac{i}{16\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu}$$

$$\int d^4x\, A(x) = \lim_{M\to\infty} \int d^4x\, \mathrm{Tr} \sum_n R\left(\frac{\lambda_n^2}{M^2}\right) \left(f_n^*(x) f_n(x) - g_n^*(x) g_n \right)$$

$$= \lim_{M\to\infty} \sum_n R\left(\frac{\lambda_n^2}{M^2}\right) \mathrm{Tr} \int d^4x \left(f_n^*(x) f_n(x) - g_n^*(x) g_n(x) \right)$$

$$= \lim_{M\to\infty} R\left(\frac{0}{M^2}\right) \left(\underset{\substack{\uparrow \\ \text{zeromodes}}}{n_\psi - n_{\psi^+}} \right)$$

$$n_\psi - n_{\psi^+} = \frac{1}{16\pi^2} \int d^4x\, F^{a\mu\nu} \tilde{F}^a_{\mu\nu} = n$$

Instanton Zero Modes

$$\int d^4x\, \partial_\mu j_A^\mu = n\left[\sum_r n_r \cdot 2T(r)\right]$$

$SU(N)$: $T(\square) = 1/2$

$T(Ad) = N$

$$e^{i\alpha}\psi_k \iff \theta_{YM} \to \theta_{YM} - \alpha \sum_r n_r 2T(r)$$

$\bar\sigma^\mu D_\mu \psi$ has $2 \cdot T(r)$ zero eigenvalues in instanton background

consider single fundamental fermion in 1 instaton background

$$\psi = a_0 f_0 + \sum_i a_i f_i$$

$$\psi^+ = \quad + \sum_i b_i g_i$$

$$\int \mathcal{D}\psi \mathcal{D}\psi^+ \exp\left(-\int \psi^+ \bar\sigma^\mu D_\mu \psi\right)$$

$$= \int da_0 \prod_i da_i \prod_j db_j \exp\left(-\sum_i \lambda_i b_i a_i\right)$$

$$= \int da_0 \prod_i da_i \prod_j db_j \prod_i \left(1 - \lambda_i b_i a_i\right)$$

$$= \int da_0 \prod_i \lambda_i = 0$$

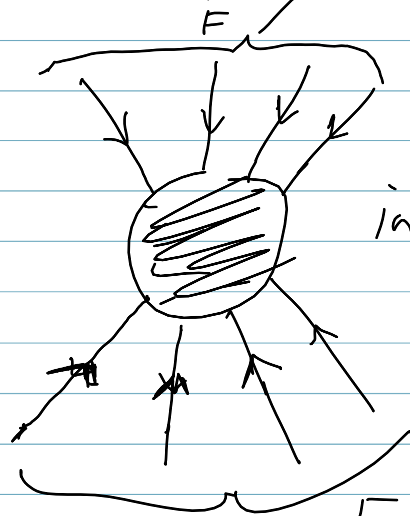
$$\int \mathcal{D}\psi \mathcal{D}\psi^+ \exp\left(-\int \psi^+ \bar\sigma^\mu D_\mu \psi\right) \psi(x)$$

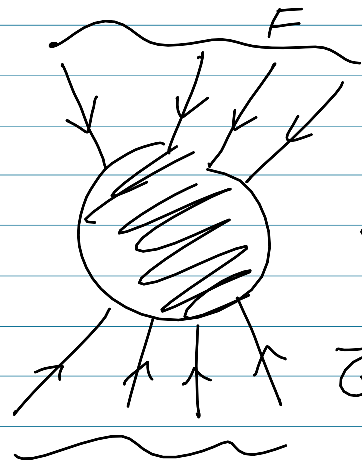
$$= \mathrm{Det}'\left(\bar\sigma^\mu D_\mu\right) \psi_0(x)$$

(17)

for F flavors

at distance $\gg \rho$ ('t Hooft)





$\mu \ll \frac{1}{\rho}$ (fixed size)

$$\mathcal{L} \supset a \left(\det \bar{Q}^{i\alpha} Q_{\alpha j} + \det Q^{+ i\alpha} \bar{Q}^{+}_{\alpha j} \right)$$

$SU(F) \times SU(F) \rightarrow (\bar{Q}' Q')^{*} = V \bar{Q} U Q$

$\det(\bar{Q}' Q') = \det \bar{Q} Q$

$U(1)_A \rightarrow \bar{Q}^{i\prime} Q_i^{\prime} = e^{2i\alpha} \bar{Q}^i Q_i$

$\det(\bar{Q}' Q') = e^{2iF\alpha} \det \bar{Q} Q$

axial symmetry is broken