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$\mathcal{N}=2$ Seiberg - Witten

$\mathcal{N}=1$ SUSY $SO(3)$ one flavor
vector = adjoint actually $\mathcal{N}=2$

$$V \sim \mathrm{Tr}\,[\phi, \phi^\dagger]^2$$

adjoint

classical moduli space $[\phi, \phi^\dagger] = 0$

parameterize with $u = \mathrm{Tr}\,\phi^2$
up to gauge transforms $\phi = \frac{1}{2} a \sigma^3$
classically $u = \frac{1}{2} a^2$

generic point in moduli space $SO(3) \to U(1)$

$SU(2)_R \times U(1)_R$ fermion has $R[\psi] = R(\lambda) - 1$
$R[\phi] = 2$

$U(1)_R$ is anomalous: instantons break $U(1)_R \to Z_4$

VEV $\langle u \rangle$ $Z_4 \to Z_2$ $u \to -u$

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$\mathcal{N}=2$ SUSY W and leading (2 derivatives) Kahler both related to a prepotential

$$\mathcal{L} = \frac{1}{8\pi i}\int d^4\theta \, \frac{\partial P}{\partial A} A^\dagger + \frac{1}{16\pi i}\int d^2\theta \, \frac{\partial^2 P}{\partial A \partial A} W^\alpha W_\alpha + h.c.$$

$\mathcal{N}=2$ supermult. contains $\mathcal{N}=1$ supermult. Scalar component of A is a

$$\tau = \frac{\partial^2 P}{\partial A \partial A}$$

$$P(A) = \underbrace{\frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}}_{\text{pert}} + A^2 \sum_{k=1}^{\infty} p_k \left(\frac{\Lambda}{A}\right)^{4k}$$

take W_α as independent field
impose superspace Bianchi ident.
$\mathrm{Im}\, D^\alpha W_\alpha = 0$ (analog $\partial^\mu \tilde{F}_{\mu\nu} = 0$)
with vector supermult. Lagrange multiplier

$$\frac{1}{4\pi} \mathrm{Im} \int d^4x \, d^4\theta \, V_D D^\alpha W_\alpha = \frac{1}{4\pi}\int d^4x \, d^4\theta \, i D^\alpha V_D W_\alpha$$
$$= \frac{-1}{4\pi} \mathrm{Im} \int d^4x \, d^2\theta \, W_D^\alpha W_\alpha$$

path integral over W_α dual $d^2\theta$ term

$$\frac{1}{16\pi i}\int d^2\theta \left(\frac{-1}{\tau(A)}\right) W_D^\alpha W_{D\alpha} + h.c.$$

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Dual Description

$$A_D = h(A) = \frac{\partial P}{\partial A}$$

scalar component a_D

Kahler $= \frac{1}{8\pi i}\int d^4\theta\, h_D(A_D) A_D^\dagger + h.c.$

$$h_D(-A)^{-1} = h(A) \qquad \tau = h'(A)$$

$$\frac{-1}{\tau(A)} = \frac{-1}{h'(A)} = h_D'(A_D) \equiv \tau_D(A_D)$$

duality implements S
shift symmetry T

full $SL(2,\mathbb{Z})$ on τ

$$\tau = \frac{\partial^2 P}{\partial A\,\partial A} = \frac{\partial a_D}{\partial a}$$

T^n shift τ by n

$a_D \to a_D + na \qquad a \to a$

on (a_D, a)

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

BPS

monopoles and dyons

$$M = 2|Z|^2$$

↑ central charge

classically

$$Z_{cl} = a n_e + a \tau_{cl} n_m$$

adding $\mathcal{N}=2$ hypermult. ($Q, \bar{Q}$ in $\mathcal{N}=1$)
with charge n_e

$$W = \sqrt{2}\, n_e A Q \bar{Q}$$

for charged particle $\quad Z = a n_e$

S-duality monopole $\quad Z = a_D n_m$

in general $\quad Z = a n_e + a_D n_m$

invariant under $SL(2, \mathbb{Z})$

$$(n_m, n_e) \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} \to (n_m, n_e) M^{-1} M \begin{pmatrix} a_D \\ a \end{pmatrix}$$

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Stability

dyons with charges (n_m, n_e)

not relatively prime, only marginally stable

lighter dyons whose charges and masses add up to

$$(n_m, n_e), \ \sqrt{2}\,|a n_e + a_D n_m|$$

stable if n_e, n_m relatively prime

weak coupling

$$\mathcal{F}(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}$$

large $|u|$ $\quad a = \sqrt{2u}$ $\quad a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \ln \frac{a}{\Lambda} + \frac{2ia}{\pi}$

loop in u around ∞

$$\ln u \to \ln u + 2\pi i$$

$$\ln a \to \ln a + \pi i$$

$$a \to -a$$

$$a_D \to -a_D + 2a$$

monodromy on $(a_D, a)^T$ at ∞

$$M_\infty = -T^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

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Singular Points

$\mathrm{Im}\,\tau = \frac{4\pi}{g^2} > 0$ need two more singular points with monodromies that don't commute

Suppose at point u_j: BPS state becomes massless with $(n_m, n_e) = (0,1)$

$$a(u) \approx c_j (u - u_j) \quad \text{near } u_j$$

$U(1)$ coupling $\to 0$

$$\tau(a(u)) \simeq \frac{-i}{\pi} \ln a(u)$$

monodromy $(u - u_j) \to e^{2\pi i}(u - u_j)$

$$a_D(u) \to a_D(u) + 2a(u), \quad a(u) \to a(u)$$

$$M_{u_j} = T^2$$

dyon with (n_m, n_e) massless at u_k

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} = D_{u_k} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \delta n_m - \gamma n_e \\ -\beta n_m + \alpha n_e \end{pmatrix}$$

$$M_{u_k} = D_{u_k}^{-1} T^2 D_{u_k} = \begin{pmatrix} 1 + 2\gamma\delta & 2\delta^2 \\ -2\gamma^2 & 1 - 2\delta\gamma \end{pmatrix}$$

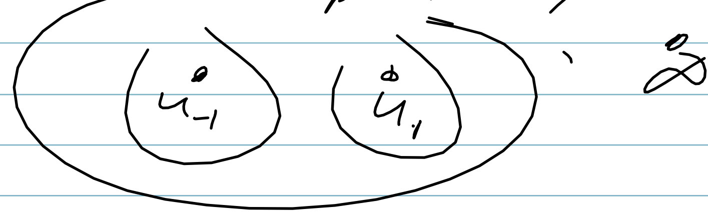
$$= \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ -n_m^2 & 1 - 2n_e n_m \end{pmatrix}$$

Two Singular Points

simplest possibility 2 singular point finite u

$\mathbb{Z}_2$: $u \to -u$

u_1, u_{-1} BPS states with charges (m,n) and (p,q) respectively become massless



$$M_{u_1} M_{u_{-1}} = M_\infty$$

assuming a massless monopole with charge $(1,0)$

$$M_{u_1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{u_{-1}} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

BPS state at u_{-1} has charge $(-1,1)$ or $(1,-1)$ related by $SL(2,\mathbb{Z})$ transformation $-I$

since M_∞ changes electric charge of monopole by 2
obtain dyons $(\pm 1, 2n+1)$ from phase redefinitions of u

$$M_{u_1} = S^{-1} T^2 S \qquad M_{u_{-1}} = (ST^{-1})^{-1} T^2 (ST^{-1})$$

same as $SO(N)$ with $N-2$ flavors

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Consistency Checks

at u, a_D vanishes

low-energy effective theory: monopoles, dual photons

add mass term $m \, Tr \phi^2$

$W_{eff} = \sqrt{2} A_D M \tilde{M} + m f(A_D)$

eqm $\sqrt{2} M \tilde{M} + m f'(A_d) = 0$ $\quad a_D M = 0, \; a_D \tilde{M} = 0$

for $m=0$ $\mathcal{N}=2$ moduli space
$M=0, \; \tilde{M}=0, \; a_D$ arbitrary

$m \neq 0$ $\quad a_D = 0 \quad M^2 = \tilde{M}^2 = -\dfrac{m f'(0)}{\sqrt{2}}$

since M is magnetically charged
=> dual photon gets a mass

electric charge confinement
dual Meisser effect

agrees with gaugino condensation
with a mass gap that we
found in $\mathcal{N}=1$

Seiberg-Witten curve

gives solution for τ and BPS masses

$$y^2 = (x-\Lambda^2)(x+\Lambda^2)(x-u)$$

singularities at $u = \pm\Lambda^2$ $\quad$ Z_2 symmetry
near these points Δ is quadratic in $u \pm \Lambda^2$
so $M_{\pm\Lambda^2} \sim T^2$

singularity at ∞ is subtle, two sets of points approach roots: $(0, \Lambda^4/4u, u)$

rescale $x \to x'(\Lambda^2 - u)$
$y \to y'(\Lambda^2 - u)^{3/2}$

roots at large u: $\left(\pm\frac{\Lambda}{u}, 1\right)$

one pair of branch points converge

for large u $\quad \Delta \sim u^{-2}$ $\quad$ monodromy $\sim T^{-2}$
back in x-y plane, change of variables gives a factor $\sqrt{u}$ to $\frac{dx}{y}$

odd under $u \to e^{2\pi i}u$, so

$$M_\infty = -T^{-2}$$

curve has appropriate singularities and monodromies

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Holomorphic Coupling

$$\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a/\partial u} = \frac{\omega_2}{\omega_1}$$

identify derivs of a, a_D with periods

$$\frac{\partial a_D}{\partial u} = f(u)\,\omega_2 = f(u)\int_b \frac{dx}{y}$$

$$\frac{\partial a}{\partial u} = f(u)\,\omega_1 = f(u)\int_a \frac{dx}{y}$$

define $\frac{d\lambda}{du} = f(u)\frac{dx}{y}$

$$a_D = \int_b \lambda \quad , \quad a = \int_a \lambda$$

adding constants would destroy $SL(2,\mathbb{Z})$ transformation

use $\int_0^1 dx\,(1-zx)^{-\alpha}\,x^{\beta-1}(1-x)^{\gamma-\beta-1} = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} F(\alpha,\beta,\gamma;z)$

$$\omega_1 = 2\int_{-\Lambda^2}^{\Lambda^2} \frac{dx}{y} = \frac{2\pi}{\Lambda\sqrt{1+\frac{u}{\Lambda^2}}}\, F\left(\tfrac{1}{2},\tfrac{1}{2},1;\frac{2}{1+u/\Lambda^2}\right)$$

$$\omega_2 = 2\int_{u}^{\Lambda^2} \frac{dx}{y} = \frac{-\pi i}{\sqrt{2}\,\Lambda} F\left(\tfrac{1}{2},\tfrac{1}{2},1;\tfrac{1}{2}\left(1-u/\Lambda^2\right)\right)$$

large $|u|$, weak coupling

$$\omega_1 = \frac{2\pi}{\sqrt{u}} \quad , \quad \omega_2 = \frac{i}{\sqrt{u}}\ln\left(\frac{u}{\Lambda^2}\right) \Rightarrow f(u) = \frac{\sqrt{2}}{2\pi}$$

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$$a(u) = -\frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2} dx \, \frac{\sqrt{x-u}}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)}}$$

$$= -\sqrt{2(\Lambda^2+u)}\, F\left(-\tfrac{1}{2}, \tfrac{1}{2}, 1; \frac{2}{1+u/\Lambda^2}\right)$$

$$a_D(u) = -\frac{\sqrt{2}}{\pi}\int_{u}^{\Lambda^2} dx \, \frac{\sqrt{x-u}}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)}}$$

$$= \frac{-i}{2}\left(\frac{u}{\Lambda} - \Lambda\right) F\left(\tfrac{1}{2}, \tfrac{1}{2}, 2; \tfrac{1}{2}\left(1 - u/\Lambda^2\right)\right)$$

a_D vanishes at $u = \Lambda^2$

$\Rightarrow$ vanishing monopole mass

at $u = -\Lambda^2$ $\quad a = a_D$

different choice of cycles
yields $SL(2, \mathbb{Z})$ transformed a, a_D

see graphs

Adding Flavors

hypermultiplets in spinor rep, $SU(2)$

$\mathcal{N}=1$ language $\quad W = \sqrt{2}\,\tilde{Q}^i A Q_i$

$\Box$ of $SU(2)$ pseudoreal, "parity" $\tilde{Q} \leftrightarrow Q$

$$R[W]=2 \quad R[Q]=R[\tilde{Q}]=0$$

$U(1)_R$ is anomalous

	$U(1)_R$
Λ_1	2
u	4
x	4
y	6

cf $\Lambda_1 \to 0 \qquad y^2 = x^2(x-u)$

with a mass $\quad R[m]=2$

n-instanton $\propto \Lambda_1^{6n} = \Lambda_1^{3n}$

Λ_1 and m odd under "parity"

$$y^2 = x^3 - ux^2 + t\Lambda_1^6 + m\Lambda_1^3(ax+bu) + cm^3\Lambda_1^3$$

a, b, c, t to be determined

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theory with $\square$ has $\frac{1}{2}$ integer charge
rescale $n_e \to 2n_e$
$a \to \frac{1}{2}a$
$\tau \to 2\tau$

no flavors: $y = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x$

take m large; $\Lambda^4 = m\Lambda_1^3$
$m \to \infty$ Λ fixed
$a = \frac{1}{4}$ $b = c = 0$

one singularity $\to \infty$
singularity at $u \approx \frac{-m^2}{64t}$

field theory $\Rightarrow$ singularity at $u = m^2$ (massless flavor)
$\Rightarrow t = -1/64$

$$y^2 = x^3 - ux^2 + \frac{m}{4}\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6$$

$m \to 0$, two roots coincide when
$$u = e^{i\pi n/3}\,\frac{3}{4}\,\frac{\Lambda^{4/3}}{2^{2/3}}$$

Z_3 symmetry
monodromies conjugate to T