Commutators of SUSY transformations

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi = \delta_{\epsilon_2}\epsilon^\alpha\psi_\alpha - \delta_{\epsilon_1}\epsilon_2^\alpha\psi_\alpha$$

$$= -i\epsilon_1^\alpha(\sigma^\nu\epsilon_2^\dagger)_\alpha\partial_\nu\phi + i\epsilon_2^\alpha(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu\phi$$

$$= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\phi$$

↑spacetime translation

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha$$

$$= -i\,\delta_{\epsilon_2}(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu\phi + i\,\delta_{\epsilon_1}(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu\phi$$

$$= i(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu(\epsilon_2\psi) + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu(\epsilon_1\psi)$$

[Fierz: $\chi_\alpha(\xi\eta) = -\xi_\alpha(\chi\eta) - (\xi\chi)\eta_\alpha$
follows from $\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = -\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma$]

$$= i\epsilon_{2\alpha}(\sigma^\mu\epsilon_1^\dagger\partial_\mu\psi) + i(\epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\psi_\alpha$$

$$-i\epsilon_{1\alpha}(\sigma^\mu\epsilon_2^\dagger\partial_\mu\psi) - i(\epsilon_1\sigma^\mu\epsilon_2^\dagger)\partial_\mu\psi_\alpha$$

$$[\chi\sigma^\mu\xi^\dagger = -\xi^\dagger\bar\sigma^\mu\chi]$$

$$= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\psi_\alpha$$

$$+ i\underbrace{(\epsilon_{1\alpha}\epsilon_2^\dagger\bar\sigma^\mu\partial_\mu\psi - \epsilon_{2\alpha}\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu\psi)}_{\text{vanishes on-shell}}$$

what about off shell?

$$p^\mu = (p,0,0,p) \qquad \bar\sigma^\mu p_\mu = p_0\sigma^0 + p_3\sigma^3$$

$$= \begin{pmatrix} 2p & 0 \\ 0 & 0 \end{pmatrix}$$

projector

	Off shell	On shell
ϕ, ϕ^*	2 d.o.f.	2 d.o.f.
$\psi_\alpha, \psi^\dagger_{\dot\alpha}$	4 d.o.f.	$\longrightarrow$ 2 d.o.f.
		Lagrangian in ∂_t
F, F^*	2 d.o.f.	0 d.o.f.

$$\mathcal{L}_{aux} = F^* F$$

dim 2 on shell $F = 0$

$$\delta F = -i\epsilon^\dagger \bar\sigma^\mu \partial_\mu \psi \quad , \quad \delta F^* = i\partial_\mu \psi^\dagger \bar\sigma^\mu \epsilon$$

$$\delta \mathcal{L}_{aux} = i\partial_\mu \psi^\dagger \bar\sigma^\mu \epsilon F - i\epsilon^\dagger \bar\sigma^\mu \partial_\mu \psi F^*$$

$$\delta \psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi + \epsilon_\alpha F$$

$$\delta \psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\nu)_{\dot\alpha} \partial_\nu \phi^* + \epsilon^\dagger_{\dot\alpha} F^*$$

$$\delta \mathcal{L}_f^{new} = \delta \mathcal{L}_f^{old} + i\epsilon^\dagger \bar\sigma^\mu \partial_\mu \psi F^* + i\psi^\dagger \bar\sigma^\mu \partial_\mu \epsilon F$$

$$= \delta \mathcal{L}_f^{old} + i\epsilon^\dagger \bar\sigma^\mu \partial_\mu \psi F^* - i\partial_\mu \psi^\dagger \bar\sigma^\mu \epsilon F + \partial_\mu (i\psi^\dagger \bar\sigma^\mu \epsilon F)$$

$$S^{new} = \int d^4x \; \mathcal{L}_s + \mathcal{L}_f + \mathcal{L}_{aux}$$

$$\delta S^{new} = 0$$

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha$$

$$= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\psi_\alpha$$

$$+ i(\epsilon_{1\alpha}\epsilon_2^\dagger\bar\sigma^\mu\partial_\mu\psi - \epsilon_{2\alpha}\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu\psi)$$

$$+ \underbrace{\delta_{\epsilon_2}\epsilon_{1\alpha}F - \delta_{\epsilon_1}\epsilon_{2\alpha}F}_{\parallel}$$

$$\epsilon_{1\alpha}(-i\epsilon_2^\dagger\bar\sigma^\mu\partial_\mu\psi) - \epsilon_{2\alpha}(-i\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu\psi)$$

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})F$$

$$= \delta_{\epsilon_2}(-i\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu\psi) - \delta_{\epsilon_1}(-i\epsilon_2^\dagger\bar\sigma^\mu\partial_\mu\psi)$$

$$= -i\epsilon_1^\dagger\bar\sigma^\mu\partial_\mu(-i(\sigma^\nu\epsilon_2^\dagger)\partial_\nu\phi + \epsilon_2F)$$

$$+ \epsilon_2^\dagger\bar\sigma^\mu\partial_\mu(-i(\sigma^\nu\epsilon_1^\dagger)\partial_\nu\phi + \epsilon_1F)$$

$$= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu F$$

$$\underbrace{- \epsilon_1^\dagger\bar\sigma^\mu\sigma^\nu\epsilon_2^\dagger\partial_\mu\partial_\nu\phi + \epsilon_2^\dagger\bar\sigma^\mu\sigma^\nu\epsilon_1^\dagger\partial_\mu\partial_\nu\phi}$$

$$- \epsilon_1^\dagger\epsilon_2^\dagger\partial_\mu\partial^\mu\phi \underset{=0}{+} \epsilon_2^\dagger\epsilon_1^\dagger\partial_\mu\partial^\mu\phi$$

$\therefore$ closes off-shell

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu X$$

$$X := \phi, \phi^\dagger, \psi, \psi^\dagger, F, F^*$$

Noether's Theorem

$$X \to X + \delta X$$

$$\delta \mathcal{L} = \mathcal{L}(X + \delta X) - \mathcal{L}(X) = \partial_\mu V^\mu$$

equ. of motion $\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} \right) = \frac{\delta \mathcal{L}}{\delta X}$

$$\partial_\mu V^\mu = \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta X} \delta X + \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} \delta(\partial_\mu X)$$

$$= \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} \right) \delta X + \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} \partial_\mu \delta X$$

$$= \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} \delta X \right)$$

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\delta \mathcal{L}}{\partial(\partial_\mu X)} - V^\mu \right) = 0$$

Supercurrent

$$\epsilon J^\mu + \epsilon^\dagger J^{\mu\dagger} = \sum_X \delta X \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} - V^\mu$$

$$\partial_\mu V^\mu = \delta \mathcal{L}$$

$$\epsilon J^\mu + \epsilon^\dagger J^{\mu\dagger} = \delta\phi\, \partial^\mu \phi^* + \delta\phi^*\, \partial^\mu \phi$$

$$+ i \psi^\dagger \bar\sigma^\mu \delta\psi - V^\mu$$

$$= \epsilon \psi \partial^\mu \phi^* + \epsilon^\dagger \psi^\dagger \partial^\mu \phi + i \psi^\dagger \bar\sigma^\mu (-i \sigma^\nu \epsilon^\dagger \partial_\nu \phi + \epsilon F$$

$$- \big(\epsilon^\dagger \psi^\dagger \partial^\mu \phi + \epsilon \sigma^\mu \bar\sigma^\nu \psi \partial_\nu \phi^* - \epsilon \psi \partial^\mu \phi^* + i \psi^\dagger \bar\sigma^\mu \epsilon F \big)$$

$$= -\epsilon \sigma^\mu \bar\sigma^\nu \psi \partial_\nu \phi^* + \psi \bar\sigma^\mu \sigma^\nu \epsilon^\dagger \partial_\nu \phi$$

$$+ 2 \epsilon \psi \partial^\mu \phi^*$$

$$= \epsilon \sigma^\nu \bar\sigma^\mu \psi \partial_\nu \phi^* - 2 \epsilon \psi \partial^\mu \phi^* + \psi^\dagger \bar\sigma^\mu \sigma^\nu \epsilon^\dagger \partial_\nu$$

$$+ 2 \epsilon \psi \partial^\mu \phi^*$$

$$J^\mu_\alpha = (\sigma^\nu \bar\sigma^\mu \psi)_\alpha \partial_\nu \phi^*$$

$$J^{\dagger\mu}_{\dot\alpha} = (\psi^\dagger \bar\sigma^\mu \sigma^\nu)_{\dot\alpha} \partial_\nu \phi$$

charges

$$Q_\alpha = \sqrt{2}\int d^3x\, J^0_\alpha \,, \quad Q^\dagger_{\dot\alpha} = \sqrt{2}\int d^3x\, J^{\dagger 0}_{\dot\alpha}$$

check $[\epsilon Q + \epsilon^\dagger Q^\dagger, X] = -i\sqrt{2}\,\delta X$

using equal-time (anti) commutators

$$[\phi(x), \partial_0 \phi^*(y)] = [\phi^*(x), \partial_0 \phi(y)] = i\delta^{(3)}(x-y)$$

$$\{\psi^\dagger_{\dot\alpha}(x), \psi_\alpha(y)\} = -\sigma^0_{\alpha\dot\alpha}\,\delta^{(3)}(x-y)$$

and eqns. of motion

$$[[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \epsilon Q + \epsilon_1^\dagger Q^\dagger], X] = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\cdot[P_\mu, X]$$

$$[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger] = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)P_\mu$$

$$[\epsilon_2 Q, \epsilon_1^\dagger Q^\dagger] = 2\epsilon_2\sigma^\mu\epsilon_1^\dagger P_\mu$$

$$[\epsilon_2^\dagger Q, \epsilon_1 Q] = -2\epsilon_1\sigma^\mu\epsilon_2^\dagger P_\mu$$

$$[\epsilon_2 Q, \epsilon_1 Q] = 0 \qquad [\epsilon_2^\dagger Q^\dagger, \epsilon_1^\dagger Q^\dagger] = 0$$

$$\epsilon_2 Q(\epsilon_1^\dagger Q^\dagger) - (\epsilon_1^\dagger Q^\dagger)(\epsilon_2 Q)$$
$$= \epsilon_2^\alpha Q_\alpha Q^\dagger_{\dot\alpha}\epsilon_1^{\dagger\dot\alpha} - Q^\dagger_{\dot\alpha}\epsilon_1^{\dagger\dot\alpha}\epsilon_2^\alpha Q_\alpha = 2\epsilon_2^\alpha\sigma^\mu_{\alpha\dot\alpha}\epsilon_1^{\dagger\dot\alpha}P_\mu$$

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu$$

$$\{Q_\alpha, Q_\alpha\} = 0 \qquad \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\alpha}\} = 0$$

Interacting WZ model

$\mathcal{L}_{free} = \partial^\mu \phi^{*i} \partial_\mu \phi_i + i \psi^{\dagger i} \bar{\sigma}^\mu \partial_\mu \psi_i + F^{*i} F_i$

$\delta \phi_i = \epsilon^\alpha \psi_{\alpha i}$

$\delta \psi_{\alpha i} = -i (\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi_i + \epsilon_\alpha F_i$

$\delta F_i = -i \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i$

δS_{free}

$\mathcal{L}_{int} = -\frac{1}{2} \underbrace{W^{ij}}_{symm} \psi_i \psi_j + W^i \underbrace{F_i}_{dim 2} + h.c.$

for renormalizable $W^{ij} = W^{ij}(\phi)$ - linear

$W^i = W^i(\phi)$ quad

$\underset{\uparrow}{\mathcal{L}_{boson}} = U(\phi) \xrightarrow{susy} \frac{\partial U}{\partial \phi_i} \epsilon \psi_i + \frac{\partial U}{\partial \phi^{*i}} \epsilon^\dagger \psi^{\dagger i}$

not allowed

$\delta \mathcal{L}_{int}\Big|_{4 spinor} = -\frac{1}{2} \frac{\delta W^{ij}}{\delta \phi_k} (\epsilon \psi_k) \psi_i \psi_j - \frac{1}{2} \frac{\delta W^{ij}}{\delta \phi^{*k}} (\epsilon^\dagger \psi^{\dagger k}) \psi_i \psi_j$

Fierz $(\epsilon \psi_i)(\psi_j \psi_k) + (\epsilon \psi_j)(\psi_k \psi_i) + (\epsilon \psi_k)(\psi_i \psi_j) = 0$ th..

first term vanishes if $\frac{\delta W^{ij}}{\delta \phi_k}$ ~~is~~ symm. in i, j, k

2nd term vanishes $\frac{\delta W^{ij}}{\delta \phi^{*k}} = 0$

W^{ij} analytic in ϕ_i (holomorphic)

$$W^{ij} = \frac{\partial^2 W(\phi)}{\partial \phi_i \partial \phi_j}$$

$$\delta \mathcal{L}_{int}\Big|_{\text{one } \partial} = i W^{ij} \partial_\mu \phi_j \psi_i \sigma^\mu \epsilon^\dagger + i W^i \partial_\mu \psi_i \sigma^\mu \epsilon^\dagger + h.c.$$

note $\partial_\mu \left(\frac{\partial W}{\partial \phi_i} \right) = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j = W^{ij} \partial_\mu \phi_j$

$$= i \partial_\mu \left(\frac{\partial W}{\partial \phi_i} \right) \psi_i \sigma^\mu \epsilon^\dagger + \partial_\mu (W^i \psi_i \sigma^\mu \epsilon^\dagger) - i (\partial_\mu W^i) \psi_i \sigma^\mu \epsilon^\dagger$$

$$\Rightarrow W^i = \frac{\partial W}{\partial \phi^i}$$

$$\delta \mathcal{L}_{int}\Big|_{F, F^*} = -W^{ij} F_i (\epsilon \psi_j) + \frac{\partial W^i}{\partial \phi^j} \epsilon \psi_j F_i + \frac{\partial W^i}{\partial \phi^{*k}} \epsilon^\dagger \psi^\dagger_k F_i + h.c.$$

$$= 0$$

All renormalizable non-gauge interactions are determined by a single function W which is holomorphic in ϕ

W more than cubic $\to$ non-renorm.

$$W = \frac{1}{2} m^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k + E^i \phi_i$$