Extended SUSY

$\mathcal{N}=2$

	state	helicity	#	$SU(2)_R$	
	$\vert\Omega_0\rangle$	0	1	1	$\mathcal{N}=1$ vector
	$Q^+\vert\Omega_0\rangle$	$1/2$	2	$\square$	
	$Q^+Q^+\vert\Omega_0\rangle$	1	1	1	
CPT conjugate	$\vert\Omega_{-1}\rangle$	-1	1	1	
	$Q^+\vert\Omega_{-1}\rangle$	$-1/2$	2	$\square$	
	$Q^+Q^+\vert\Omega_{-1}\rangle$	0	1	1	

$\equiv$ $\mathcal{N}=1$ vector + chiral

state	helicity	#	$SU(2)_R$	
$\vert\Omega_{-1/2}\rangle$	$-1/2$	1	1	χ_L
$Q^+\vert\Omega_{-1/2}\rangle$	0	2	$\square$	ϕ
$Q^+Q^+\vert\Omega_{-1/2}\rangle$	$1/2$	1	1	ψ_R

$\overline{\chi_L}\,\psi_R$ gauge invariant mass term

$\Rightarrow$ vector-like $\equiv$ non-chiral

$\mathcal{N}=3$

state	helicity	d_R	$SU(3)_R$
$\vert\Omega_{-1}\rangle$	-1	1	1
$Q^\dagger\vert\Omega_{-1}\rangle$	$-1/2$	3	$\square$
$(Q^\dagger)^2\vert\Omega_{-1}\rangle$	0	3	$\bar{\square}$
$(Q^\dagger)^3\vert\Omega_{-1}\rangle$	$1/2$	1	1
$\vert\Omega_{-1/2}\rangle$	$-1/2$	1	1
$Q^\dagger\vert\Omega_{-1/2}\rangle$	0	3	$\square$
$(Q^\dagger)^2\vert\Omega_{-1/2}\rangle$	$1/2$	3	$\bar{\square}$
$(Q^\dagger)^3\vert\Omega_{-1/2}\rangle$	1	1	1

vector-like

$\mathcal{N}=4$

		d_R	$SU(4)_R$
$\vert\Omega_{-1}\rangle$	-1	1	1
$Q^\dagger\vert\Omega_{-1}\rangle$	$-1/2$	4	$\square$
$(Q^\dagger)^2\vert\Omega_{-1}\rangle$	0	6	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$(Q^\dagger)^3\vert\Omega_{-1}\rangle$	$1/2$	4	$\bar{\square}$
$(Q^\dagger)^4\vert\Omega_{-1}\rangle$	1	1	1

vector-like

same as $\mathcal{N}=3$

$\equiv$ $\mathcal{N}=1$ vector + 3 chiral

$\equiv$ $\mathcal{N}=2$ vector + hypermultiplet massive

rest frame: $p_\mu (m, 0, 0, 0)$

$$\{ Q_\alpha^a, Q^\dagger_{\dot\beta b} \} = 2m \, \delta_{\alpha\dot\beta} \, \delta^a_b$$

raising op. $Q^\dagger_{\dot\alpha b}$

	spin
$\vert\Omega_s\rangle$	s
$Q^\dagger \vert\Omega_s\rangle$	$s+1/2$
$(Q^\dagger)^2 \vert\Omega_s\rangle$	$s+1$
$\vdots$	
$(Q^\dagger)^{2N} \vert\Omega_s\rangle$	s ~~+~~

$\mathcal{N} = 2$

state	$(d_R, 2j+1)$	$SU(2)_R \; SU(2)$
$\vert\Omega_0\rangle$	$(1, 1)$	$(1, 1)$
$Q^\dagger \vert\Omega_0\rangle$	$(2, 2)$	$(\square, 1,$
$Q^\dagger Q^\dagger \vert\Omega_0\rangle$	$(3,1) + (1,3)$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right) + \left(\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array},\right.$
$Q^\dagger Q^\dagger Q^\dagger \vert\Omega_0\rangle$	$(2, 2)$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger \vert\Omega_0\rangle$	$(1, 1)$	$\left(\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\right)$

$16 = $ 5 spin 0 $\Rightarrow 5$ 4 spin $1/2$ $\Rightarrow 8$

1 spin 1 $\Rightarrow 3$

$\overline{8}$ $\overline{8}$

$\mathcal{N}=4$ state	$(\#_R, 2j+1)$	$SU(4)_R \; SU(2)$
$\lvert\Omega_0\rangle$	$(1,1)$	$(1,1)$
$Q^\dagger\lvert\Omega_0\rangle$	$(\yng(1), \yng(1))$	$(4,2)$
$(Q^\dagger)^2\lvert\Omega_0\rangle$	$(\yng(1,1), \yng(2)) + (\yng(2), \yng(1,1))$	$(6,3)+(10,1)$
$(Q^\dagger)^3\lvert\Omega_0\rangle$	$(\yng(2,1), \yng(2,1)) + (\yng(1,1,1), \yng(3))$	$(\overline{20},2)+(\bar{4},4)$
$(Q^\dagger)^4\lvert\Omega_0\rangle$	$(\yng(2,2), \yng(2,2)) + (\yng(2,1,1), \yng(3,1)) + (\yng(1,1,1,1), \yng(4))$	$(20',1)+(15,3)+(1,5)$
$(Q^\dagger)^5\lvert\Omega_0\rangle$	$(\yng(2,2,1), \yng(3,2)) + (\yng(2,1,1,1), \yng(4,1))$	$(20,2)+(4,4)$
$(Q^\dagger)^6\lvert\Omega_0\rangle$	$(\yng(2,2,2), \yng(3,3)) + (\yng(2,2,1,1), \yng(4,2))$	$(\overline{10},1)+(6,3)$
$(Q^\dagger)^7\lvert\Omega_0\rangle$	$(\yng(2,2,2,1), \yng(4,3))$	$(\bar{4},2)$
$(Q^\dagger)^8\lvert\Omega_0\rangle$	$(\yng(2,2,2,2), \yng(4,4))$	$(1,1)$

8 spin 3/2 = 32
4 spin 1/2 = 96
128

1 spin 2 = 5
27 spin 1 = 81
42 spin 0 = 62
128

256

BPS

$$\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\,\epsilon_{\alpha\beta} Z^{ab}$$ $a, b, 1 \ldots \mathcal{N}$

$$\{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} = 2\sqrt{2}\,\epsilon_{\dot\alpha\dot\beta} Z^\dagger_{ab}$$

$$\epsilon = i\sigma^2$$

antisymmetric central charge

Z skew diagonalizable $\mathcal{N}/2$ eigenvalues

for $\mathcal{N} = 2$ $\quad Z^{ab} \to \epsilon^{ab} Z$

define $A_\alpha = \frac{1}{2}\left[Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right]$

$$B_\alpha = \frac{1}{2}\left[Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right]$$

in rest frame

$$\{A_\alpha, A^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2} Z)$$

$$\{B_\alpha, B^\dagger_\beta\} = \delta_{\alpha\beta}(M - \sqrt{2} Z)$$

mass

$$\langle M,Z|B_\alpha B_\alpha^\dagger|M,Z\rangle + \langle M,Z|B^\dagger_\alpha B_\alpha|M,Z\rangle = M - \sqrt{2} Z$$

positive norm $\quad M \geq \sqrt{2} Z$

massless $\Rightarrow Z = 0$

if $M = \sqrt{2} Z$ state annhilated by B_α

$\frac{1}{2}$ SUSY preserved

$M = \sqrt{2}\, Z$ rep. $\equiv \mathcal{N}=1$ massive

$\sim \mathcal{N}=2$ massless

much smaller than massive rep

	$2j+1$
$\lvert \Omega_0 \rangle$	1
$A^+ \lvert \Omega_0 \rangle$	2
$A^+ A^+ \lvert \Omega_0 \rangle$	1
	4

vs 16

cf $\mathcal{N}=1$ massive chiral mult

$\mathcal{N}=2$ massless hyper

	$2j+1$
$\lvert \Omega_{1/2} \rangle$	2
$A^+ \lvert \Omega_{1/2} \rangle$	1 + 3
$A^+ A^+ \lvert \Omega_{1/2} \rangle$	2
	8

vs 32

cf $\mathcal{N}=1$ massive vector

$\mathcal{N}=2$ massless vector

aka BPS multiplets

$M = \sqrt{2}\, Z$ exact

The Simplest SUSY Model
(Free Wess-Zumino Model)

$$S = \int d^4x \left(\mathcal{L}_s + \mathcal{L}_f \right)$$

$$\mathcal{L}_s = \partial^\mu \phi^* \partial_\mu \phi \qquad \mathcal{L}_f = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = (1, -\sigma^i)$$

Infinitesimal global SUSY transformation

$$\phi \to \phi + \delta\phi \qquad \psi \to \psi + \delta\psi$$

(NW-SE)

$$\delta\phi = \epsilon^\alpha \psi_\alpha \equiv \epsilon\psi$$
$$= \epsilon^\alpha \epsilon_{\alpha\beta} \psi^\beta = -\psi^\beta \epsilon_{\alpha\beta} \epsilon^\alpha$$
$$= \psi^\beta \epsilon_{\beta\alpha} \epsilon^\alpha = \psi^\beta \epsilon_\beta = \psi\epsilon$$

$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\delta\phi^* = \epsilon^\dagger_{\dot{\alpha}} \psi^{\dagger\dot{\alpha}}$$
$$= \epsilon^\dagger \psi^\dagger$$

SW-NE

dimension $(\text{mass})^{-1/2}$

$$\delta\mathcal{L}_s = \epsilon \partial^\mu \psi \partial_\mu \phi^* + \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi$$

$\Rightarrow \delta\psi$ linear in ϵ, one derivative

$$\delta\psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi, \quad \delta\psi^\dagger_{\dot{\alpha}} = i(\epsilon\sigma^\nu)_{\dot{\alpha}} \partial_\nu \phi^*$$

$$\delta\mathcal{L}_f = -(\epsilon\sigma^\nu)\partial_\nu \phi^* \bar{\sigma}^\mu \partial_\mu \psi + \psi^\dagger \bar{\sigma}^\mu (\sigma^\nu \epsilon^\dagger) \partial_\mu \partial_\nu \phi$$

Pauli:
$$\left[\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu\right]_\alpha^{\ \beta} = 2\eta^{\mu\nu} \delta_\alpha^\beta$$
$$\left[\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu\right]^{\dot{\alpha}}_{\ \dot{\beta}} = 2\eta^{\mu\nu} \delta^{\dot{\alpha}}_{\dot{\beta}}$$
$$\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

$$\delta \mathcal{L}_f = \partial_\nu \phi^* \left(\epsilon \sigma^\mu \bar{\sigma}^\nu - 2 \eta^{\mu\nu} \epsilon \right) \partial_\mu \psi$$
$$+ \psi^\dagger \left(\eta^{\mu\nu} \epsilon^\dagger \right) \partial_\mu \partial_\nu \phi$$

$$= - \epsilon \partial^\mu \psi \cancel{\psi} \partial_\mu \phi^* + \psi^\dagger \epsilon^\dagger \partial_\mu \partial^\mu \phi$$
$$+ \partial_\nu \phi^* \left(\epsilon \sigma^\mu \bar{\sigma}^\nu - \eta^{\mu\nu} \epsilon \right) \partial_\mu \psi$$

$$= - \epsilon \partial^\mu \psi \partial_\mu \phi^* - \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi + \partial^\mu \left(\epsilon^\dagger \psi^\dagger \partial_\mu \phi \right)$$
$$+ \partial_\mu \left(\epsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^* \right) - \epsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\mu \partial_\nu \phi^*$$
$$- \partial_\mu \left(\epsilon \psi \partial^\mu \phi^* \right) + \epsilon \psi \partial_\mu \partial^\mu \phi^*$$

$$= - \epsilon \partial^\mu \psi \partial_\mu \phi^* - \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi$$
$$\partial_\mu \left(\epsilon^\dagger \psi^\dagger \partial^\mu \phi + \epsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^* - \epsilon \psi \partial^\mu \phi^* \right)$$
$$- \epsilon \psi \partial_\mu \partial^\mu \phi^* + \epsilon \psi \partial_\mu \partial^\mu \phi^*$$

$$\therefore \quad \delta S = 0$$