White Dwarf

stabilized against collapse by degeneracy pressure of electrons

radius R, e mass m_e, nucleon mass M_p, e's per nucleon q

assume constant density:

$$V = \frac{4}{3}\pi R^3, \qquad \rho = \frac{N M_p}{V}$$

$$E = \frac{\hbar^2}{10\pi^2 m_e}\frac{\left(3\pi^2 N q\right)^{5/3}}{V^{2/3}} = \frac{\hbar^2}{10\pi^2 m_e}\frac{\left(3\pi^2 N q\right)^{5/3}}{(\pi/3)^{2/3} R^2} = \frac{2\hbar^2}{15\pi m_e}\frac{\left(\frac{9}{4}\pi N q\right)^{5/3}}{R^2}$$

$$dE_{grav} = -\frac{GM(r)}{r}dM = -\frac{G\left(\rho\frac{4}{3}\pi r^3\right)}{r}\rho\, 4\pi r^2 dr = -\frac{16}{3}\pi^3 G\rho^2\, r^4 dr$$

$$E_{grav} = -\frac{16}{3}\pi^3 G\rho^2 \int_0^R r^4 dr = -\frac{16}{15}\pi^3 G\rho^2\, R^5 = -\frac{3}{5}\pi^3\frac{GN^2M_p^2}{R}$$

$$E_{tot} = E + E_{grav} = \frac{A}{R^2} - \frac{B}{R}$$

$$\frac{dE_{tot}}{dR} = -2\frac{A}{R^3} + \frac{B}{R^2} = 0$$

$$2A = BR$$

White Dwarf

$$
\begin{aligned}
R &= \frac{2A}{B} = \frac{4\hbar^2}{15\pi m_e}\left(\frac{9}{4}\pi N_q\right)^{5/3} \frac{5}{3\pi^3 G N^2 m^2} \\
&= \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2}{G M_p^2 m_e} \frac{q^{5/3}}{N^{1/3}} = \frac{7.6\times 10^{25}\,\mathrm{m}}{N^{1/3}}
\end{aligned}
$$

for a solar mass $N \approx 1.2 \times 10^{57}$, $R \approx 7 \times 10^6$ m

$$E_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{Nq}{V}\right)^{2/3} = \frac{\hbar^2}{2m_e R^2}\left(\frac{9\pi}{4} Nq\right)^{2/3} = 1.9 \times 10^5 eV$$

$$E_{rest} = m_e c^2 = 5.11 \times 10^5 eV$$

$\uparrow N \Rightarrow R \downarrow\ E_F \uparrow$, more relativistic

UltraRelativistic

$$E = \sqrt{p^2c^2 + m_e^2c^4} - m_ec^2 \approx pc$$

$$dE = E_k \frac{V}{\pi^2} k^2 dk = \hbar ck \frac{V}{\pi^2} k^2 dk$$

$$\begin{aligned} E &= \frac{\hbar cV}{\pi^2} \int_0^{k_F} k^3 dk = \frac{\hbar\, c\, V}{4\pi^2} k_F^4 = \frac{\hbar cV}{4\pi^2} \left(3\pi^2 \frac{Nq}{V}\right)^{4/3} \\ &= \frac{\hbar\, c}{4\pi^2} \frac{\left(3\pi^2 Nq\right)^{4/3}}{V^{1/3}} = \frac{\hbar\, c}{3\pi R} \left(\frac{9\pi}{4} Nq\right)^{4/3} \end{aligned}$$

$$E_{tot} = E + E_{grav} = \frac{C}{R} - \frac{B}{R}$$

$C > B$ expand, $C < B$ contract

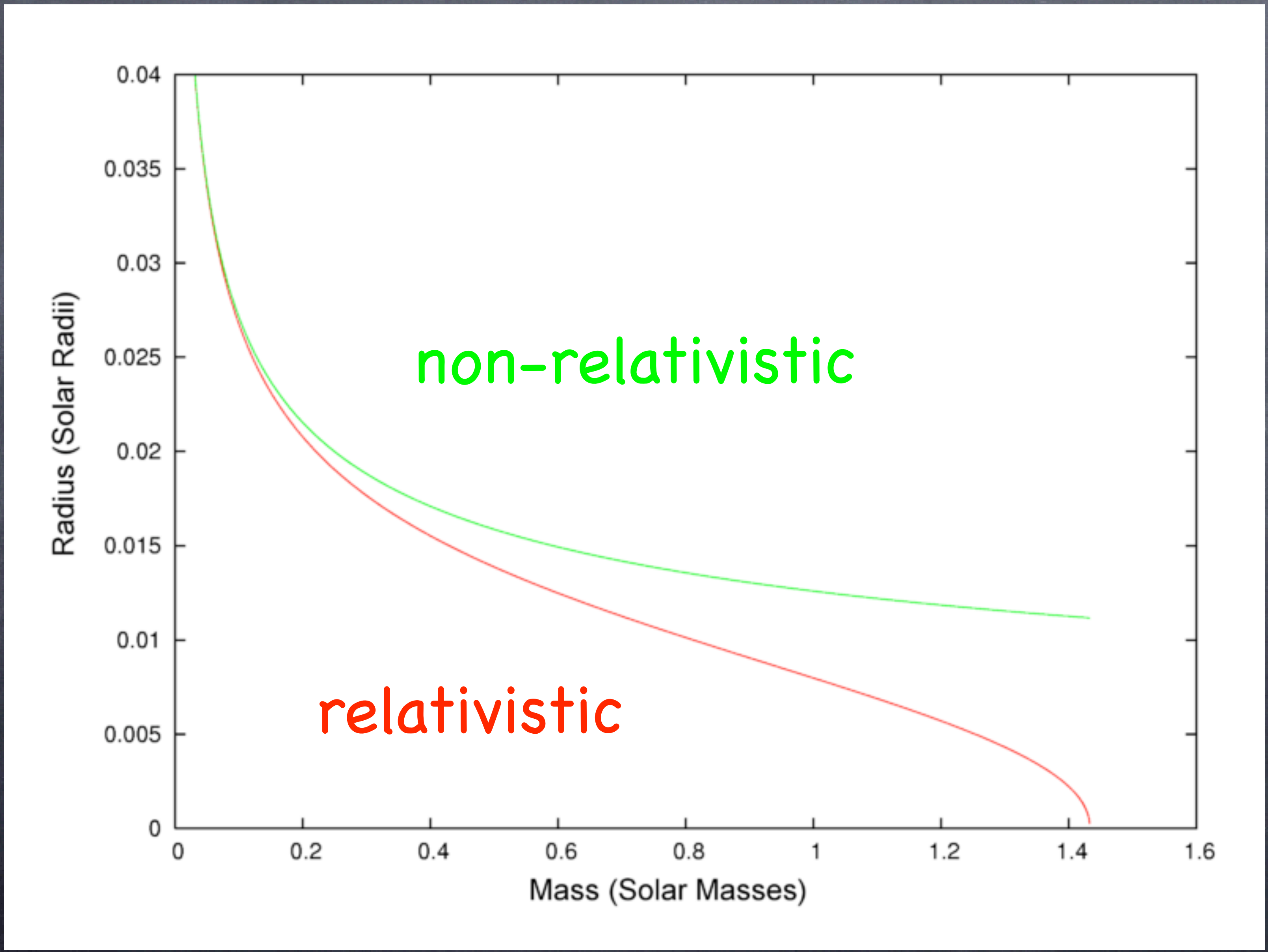
Chandrasekhar Limit

$$\begin{aligned} C &= B \\ \frac{\hbar\, c}{3\pi}\left(\frac{9\pi}{4} N_c q\right)^{4/3} &= \frac{3}{5}\pi^3 G N_c^2 M_p^2 \end{aligned}$$

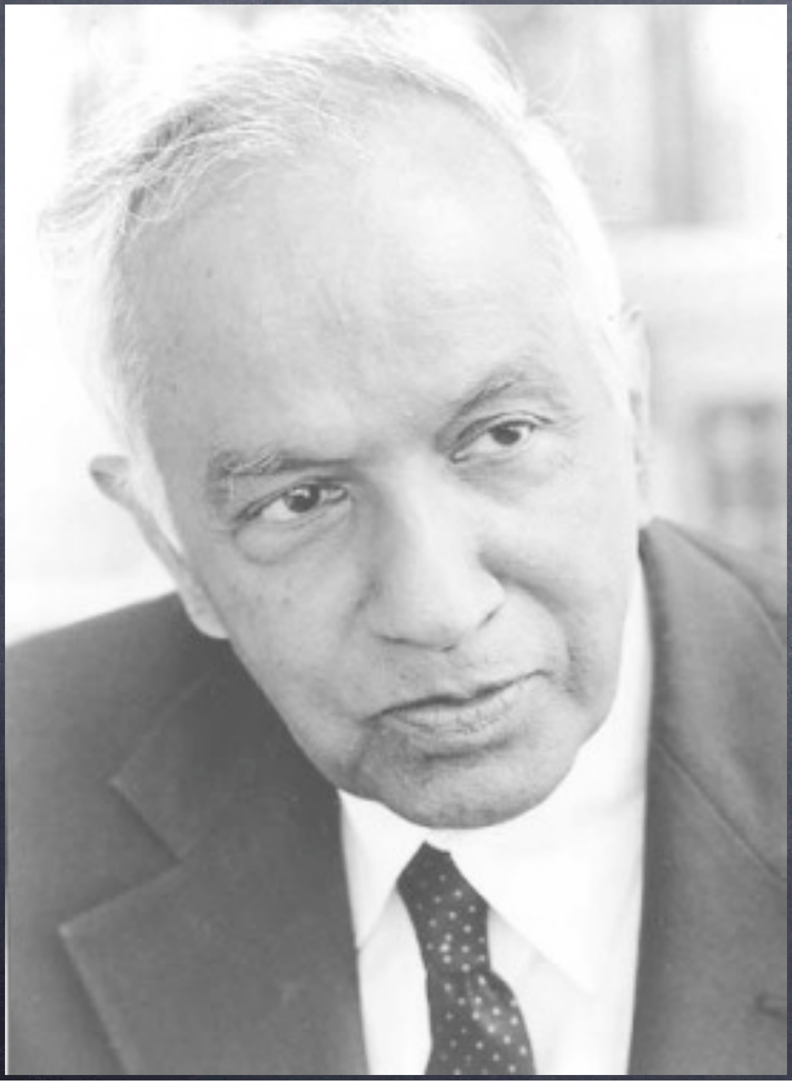
$$N_c = \frac{15}{16}\sqrt{5\pi}\left(\frac{\hbar\, c}{G}\right)^{3/2}\frac{q^2}{M_p^2} \approx 2 \times 10^{57}$$

1.7 solar masses

Chandrasekhar Limit



Subramanyan Chandrasekhar



1983 Nobel Prize

White Dwarf

Neutron Star

from core collapse supernovae

$$p^+ + e^- \rightarrow n + \nu$$

$$m_e \rightarrow m_n \, , q = 1$$

$$N \sim 10^{57}, R \sim 12\,\mathrm{km}$$

$$E_F = \frac{\hbar^2}{2\,m_n\,R^2}\left(\frac{9\pi}{4}\right)^2 = 56\,\mathrm{MeV}$$

$$E_{rest} = m_n\,c^2 = 940\,\mathrm{MeV}$$

non-relativistic

Band Structure

$$V(x+a) = V(x)$$

Bloch's Theorem

$$\psi(x+a) = e^{iKa}\psi(x)$$

$$K = \frac{2\pi j}{Na}$$

Band Structure

$$0 < x < a$$

$$\psi(x) = A\,\sin(kx) + B\,\cos(kx)$$

$$-a < x < 0$$

$$\psi(x) = e^{-iKa}\left[A\,\sin k(x+a) + B\,\cos k(x+a)\right]$$

Band Structure

$$\psi(0_+) = \psi(0_-)$$

$$\psi'(0_+) - \psi'(0_-) = \frac{2m}{\hbar^2} \int_{0_-}^{0_+} V(x)\psi(x)dx$$

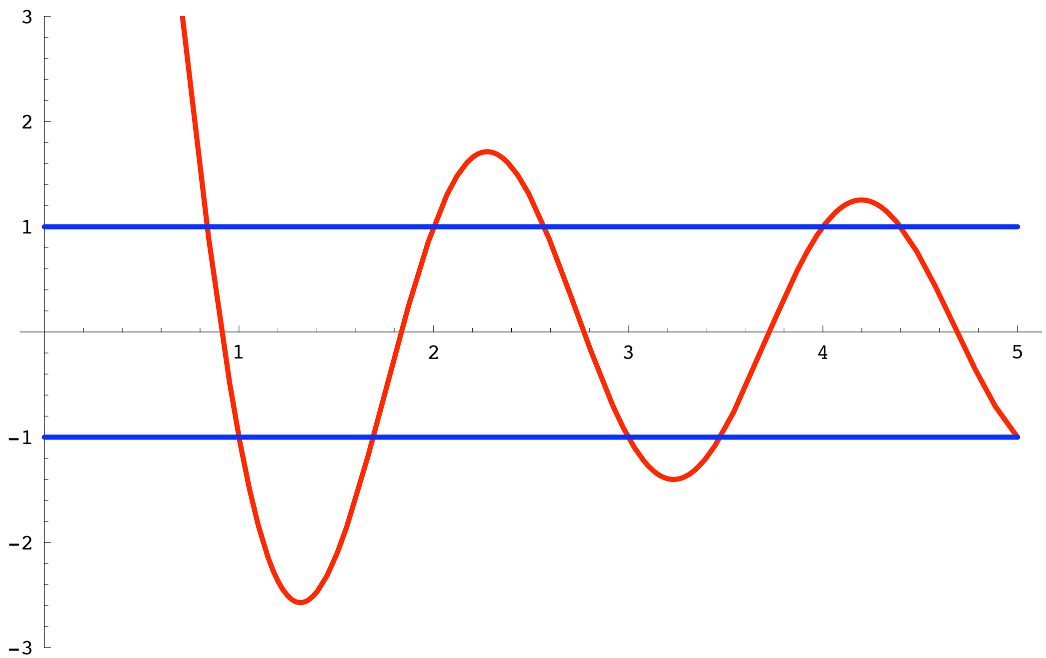
$$\psi'(0_+) - \psi'(0_-) = \frac{2m}{\hbar^2}\,\alpha\, B$$

$$\cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

$$K = n\pi$$

Band Structure

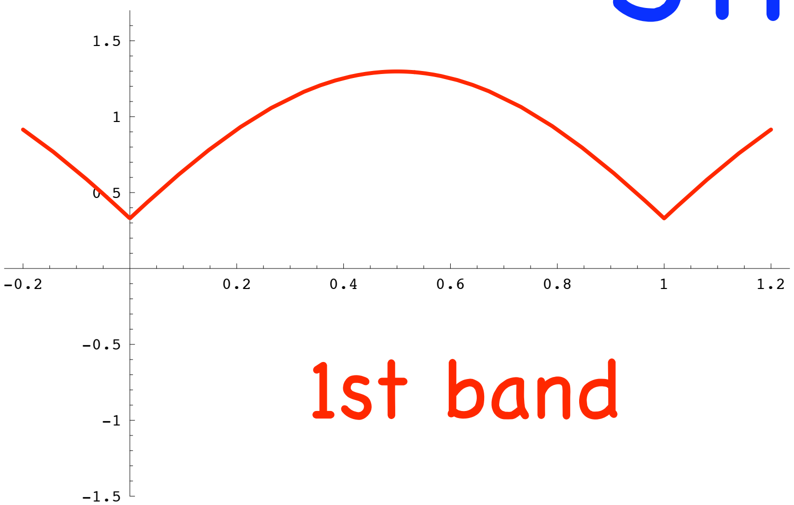
$$\cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

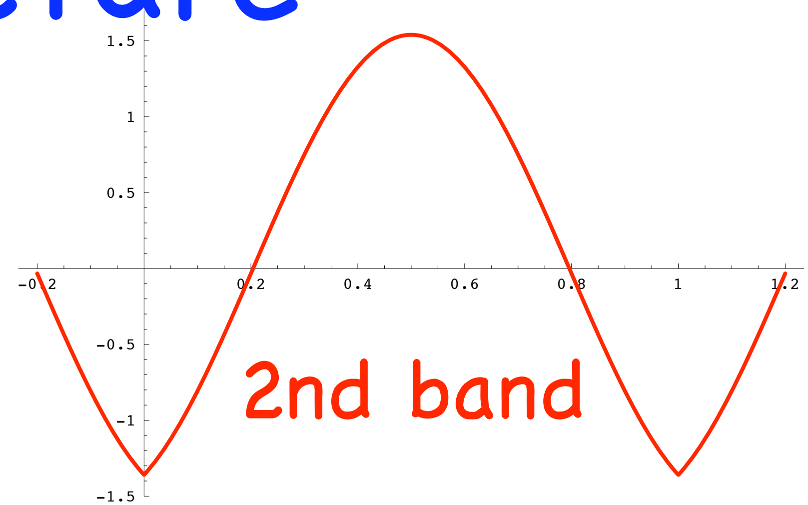


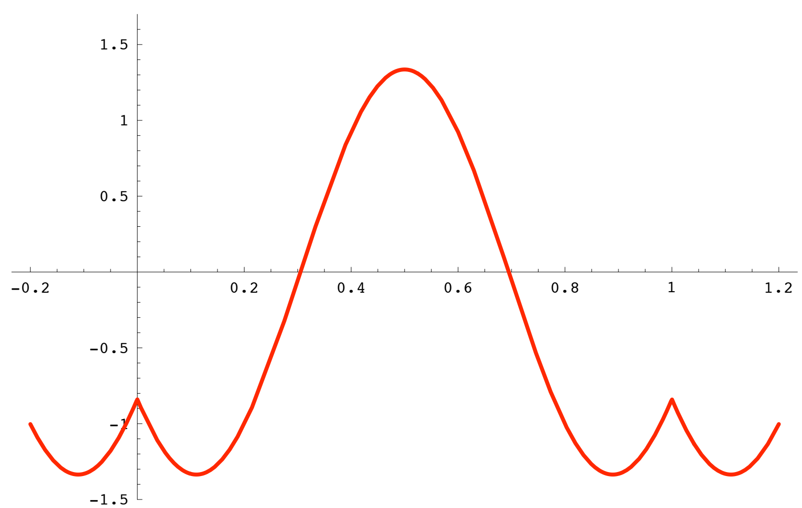
band edge: $K = n\pi$

Bottom of Band Structure

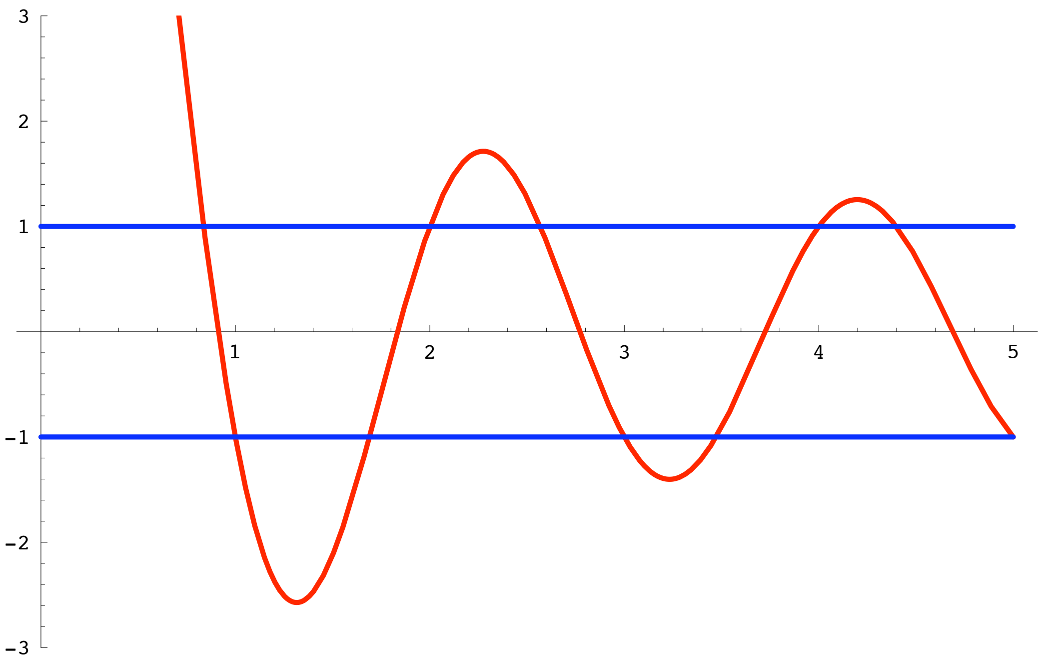
$K = n\pi$



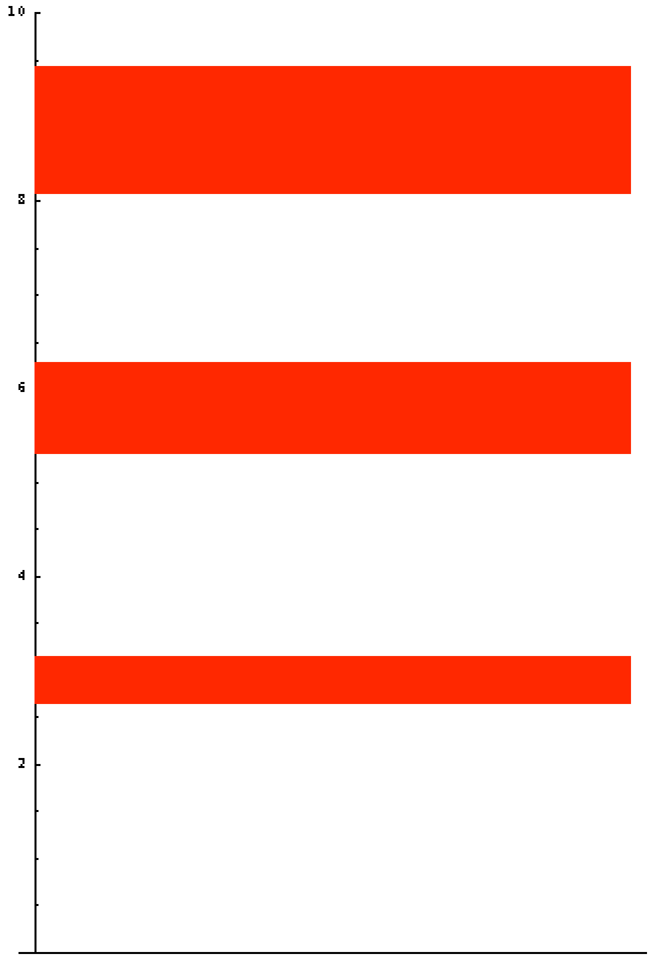




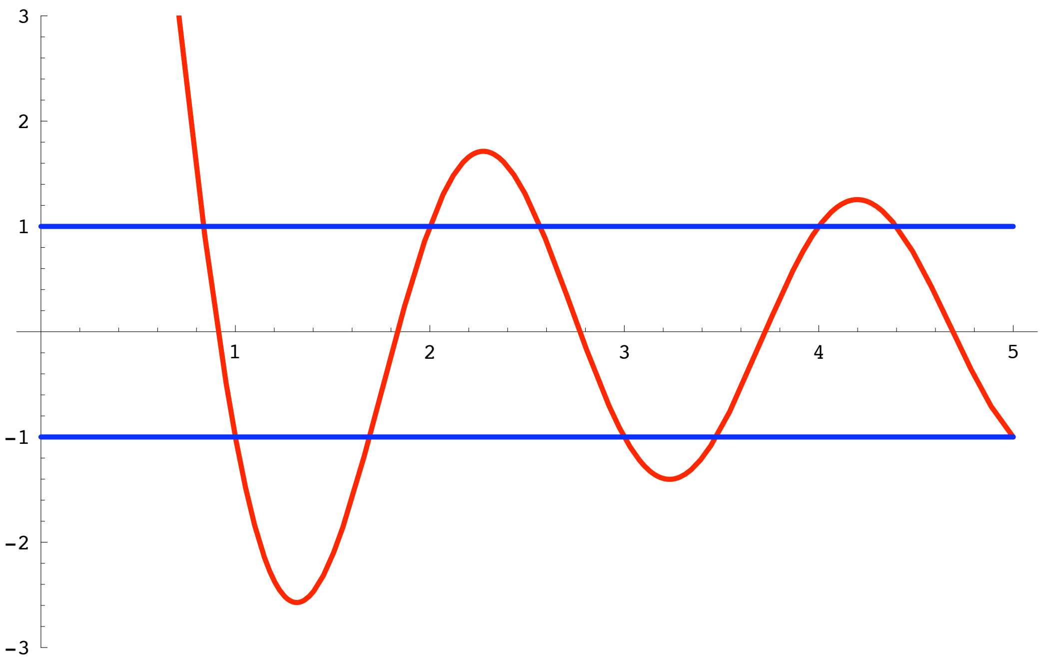
Band Structure



$$\cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$



Band Structure



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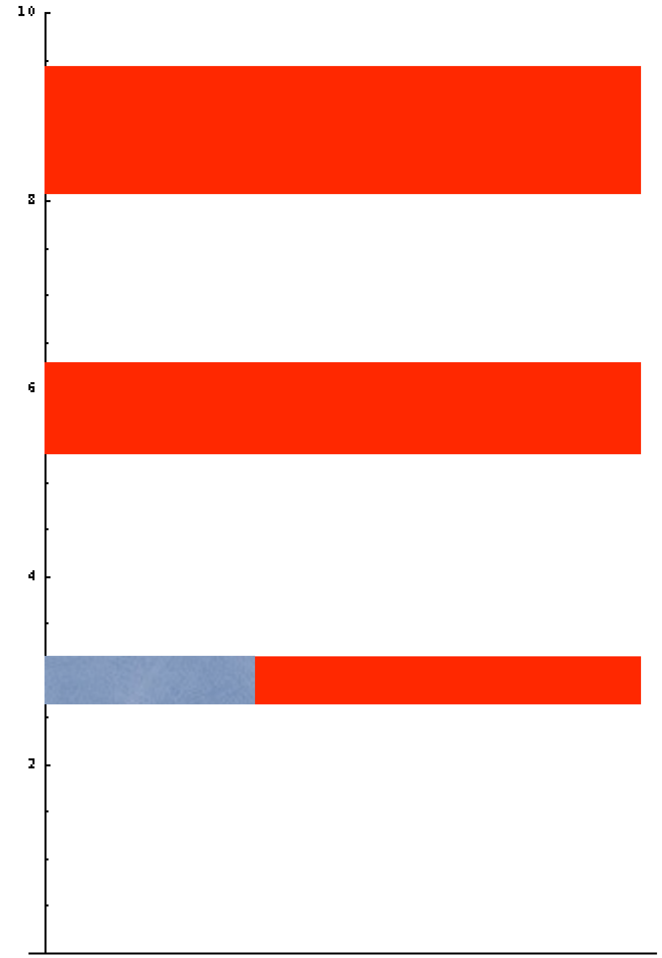
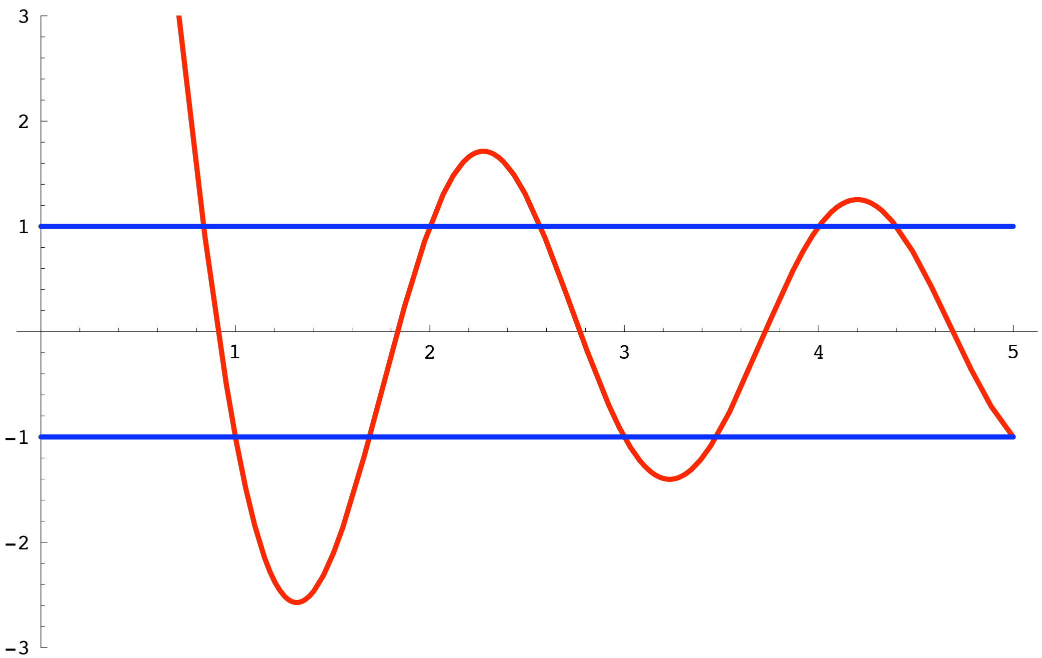
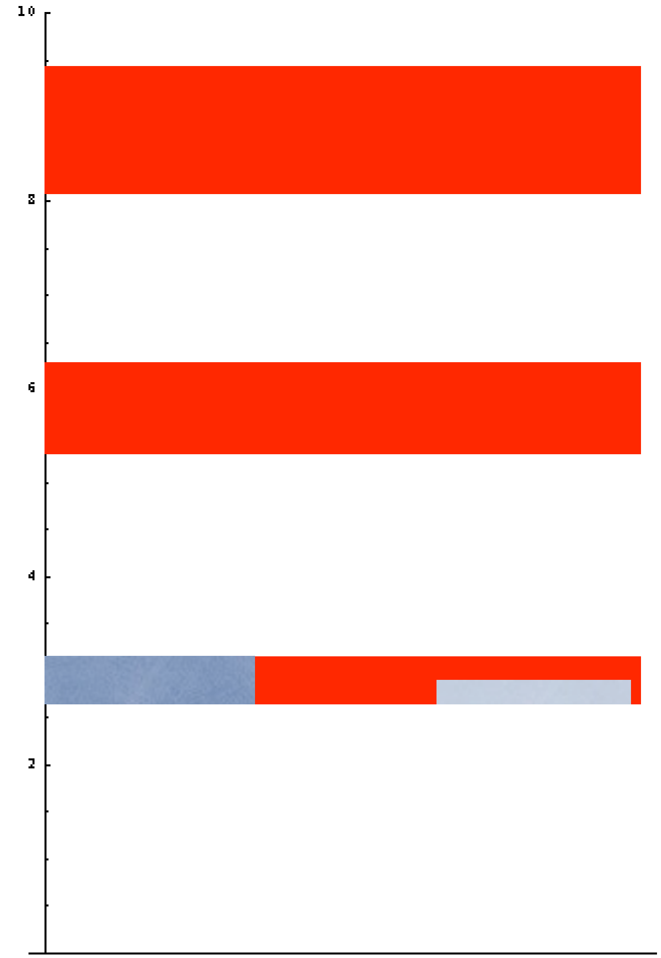


insulator

Band Structure



$$\cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$



insulator conductor