QM 115B Lec 8

Addition of Angular Momentum

two spin ½ particles eg. hydrogen atom

$\uparrow\uparrow$

$\uparrow\downarrow \quad \downarrow\uparrow$

$\downarrow\downarrow$

$$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$$

$$S_z \chi_1 \chi_2 = (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2$$
$$= (S_z^{(1)} \chi_1) \chi_2 + \chi_1 S_z^{(2)} \chi_2$$
$$= (\hbar m_1 \chi_1) \chi_2 \quad \chi_1 \hbar m_2 \chi_2$$
$$= \hbar (m_1 + m_2) \chi_1 \chi_2$$

$\uparrow\uparrow \quad m = \frac{1}{2} + \frac{1}{2} = 1$

$\uparrow\downarrow \quad m = \frac{1}{2} - \frac{1}{2} = 0$?

$\downarrow\uparrow \quad m = -\frac{1}{2} + \frac{1}{2} = 0$

$\downarrow\downarrow \quad m = -\frac{1}{2} - \frac{1}{2} = -1$

$$S_\pm |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} \, |s, m \pm 1\rangle$$

$$S_\pm |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2} \pm 1)} \, |\overset{1,\frac{1}{2}}{s, m \pm 1}\rangle$$

$$= \hbar \sqrt{\tfrac{3}{4} \begin{matrix} -\tfrac{3}{4} \\ +\tfrac{1}{4} \end{matrix}} \, |s, m \pm 1\rangle$$

$$= \begin{cases} 0 \\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{cases}$$

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$$S_- = S_-^{(1)} + S_-^{(2)}$$

$$S_- |\uparrow\uparrow\rangle = (S_-^{(1)}|\uparrow\rangle)|\uparrow\rangle + |\uparrow\rangle S_-^{(2)}|\uparrow\rangle$$

$$= \hbar|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle$$

$s=1$
triplet

$$\begin{cases} |11\rangle = |\uparrow\uparrow\rangle \\ |10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\,{-1}\rangle = |\downarrow\downarrow\rangle \end{cases}$$

symm.

orthoganal state

$s=0$ $\quad |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$
singlet

anti symm.

$$S^2 = (\vec{S}^{(1)} + \vec{S}^{(2)})\cdot(\vec{S}^{(1)} + \vec{S}^{(2)})$$

$$= (\vec{S}^{(1)})^2 + (\vec{S}^{(2)})^2 + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|\downarrow\rangle$$

$$S_y|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} |\uparrow\downarrow\rangle = \Big((S_x^{(1)} |\uparrow\rangle)(S_x^{(2)} |\downarrow\rangle) + (S_y^{(1)} |\uparrow\rangle)(S_y^{(2)} |\downarrow\rangle) + (S_z^{(1)} |\uparrow\rangle)(S_z^{(2)} |\downarrow\rangle) \Big)$$

$$= \left(\frac{\hbar}{2}\right)^2 \Big(|\downarrow\rangle|\uparrow\rangle + i|\downarrow\rangle(-i|\uparrow\rangle) - |\uparrow\rangle|\downarrow\rangle \Big)$$

$$= \frac{\hbar^2}{4}\Big(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle \Big)$$

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} |\downarrow\uparrow\rangle = \left(\frac{\hbar}{2}\right)^2 \Big(2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \Big)$$

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} |1,0\rangle = \frac{\hbar^2}{4}\left(\frac{1}{\sqrt{2}}\right)\Big(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle + 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \Big)$$

$$= \frac{\hbar^2}{4} |1,0\rangle$$

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} |0,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\Big(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle - 2|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \Big)$$

$$= -3\frac{\hbar^2}{4} |0,0\rangle$$

$$S^2 |1,0\rangle = \Big(S^{(1)2} + S^{(2)2} + 2\vec{S}^{(1)} \cdot \vec{S}^{(2)} \Big) |1,0\rangle$$

$$= \hbar^2 \left(\tfrac{1}{2}(\tfrac{1}{2}+1) + \tfrac{1}{2}(\tfrac{1}{2}+1) + \frac{2}{4} \right) |1,0\rangle$$

$$= \underset{1(1+1)}{\underset{\shortparallel}{2}}\hbar^2 |1,0\rangle$$

$$S^2 |0,0\rangle = \hbar^2 \left(\tfrac{1}{2}(\tfrac{1}{2}+1) + \tfrac{1}{2}(\tfrac{1}{2}+1) + 2\left(-\tfrac{3}{4}\right) \right) |0,0\rangle$$

$$= 0$$

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combine spin s_1 and s_2

$\uparrow\uparrow$ $\uparrow\downarrow$

$S = s_1+s_2,\ s_1+s_2-1,\ s_1+s_2-2,\ \dots\ |s_1-s_2|$

example combine $l=1$, $s=1/2$

$\vec{J} = \vec{L} + \vec{S}$ $\quad L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,|l,m-1\rangle$

$\vec{J}_\pm = L_\pm + S_\pm$ $\quad S_-|s,s_m\rangle = \hbar\sqrt{s(s+1)-s_m(s_m-1)}\,|s,s_m-1\rangle$

highest j state $\quad j = 1+1/2 = 3/2$

$$
J_-\left(\begin{array}{l}
|1,1\rangle|1/2,1/2\rangle \\
\frac{1}{\sqrt{2+1}}\left(\sqrt{2}\,|1,0\rangle|1/2,1/2\rangle + |1,1\rangle|1/2,-1/2\rangle\right) \\
\frac{1}{\sqrt{4+8}}\left(\sqrt{2}\sqrt{2}\,|1,-1\rangle|1/2,1/2\rangle + \sqrt{2}\,|1,0\rangle|1/2,-1/2\rangle + \sqrt{2}\,|1,0\rangle|1/2,-1/2\rangle\right) \\
\left(2\,|1,-1\rangle|1/2,-1/2\rangle + 2\sqrt{2}\sqrt{2}\,|1,-1\rangle|1/2,-1/2\rangle\right)
\end{array}\right\}\ \begin{array}{l}2j+1\\=4\end{array}
$$

simplify

	$\lvert j\ j_m\rangle$
$\lvert 1,1\rangle\lvert 1/2,1/2\rangle$	$\lvert 3/2,3/2\rangle$
$\frac{1}{\sqrt{3}}\left(\sqrt{2}\,\lvert 1,0\rangle\lvert 1/2,1/2\rangle + \lvert 1,1\rangle\lvert 1/2,-1/2\rangle\right)$	$\lvert 3/2,1/2\rangle$
$\frac{1}{\sqrt{3}}\left(\lvert 1,-1\rangle\lvert 1/2,1/2\rangle + \sqrt{2}\,\lvert 1,0\rangle\lvert 1/2,-1/2\rangle\right)$	$\lvert 3/2,-1/2\rangle$
$\lvert 1,-1\rangle\lvert 1/2,-1/2\rangle$	$\lvert 3/2,-3/2\rangle$

$\frac{1}{\sqrt{3}}\left(|1,0\rangle|1/2,1/2\rangle - \sqrt{2}\,|1,1\rangle|1/2,-1/2\rangle\right)$ $\quad |1/2,1/2\rangle$

$\frac{1}{\sqrt{3}}\left(\sqrt{2}\,|1,-1\rangle|1/2,1/2\rangle + |1,0\rangle|1/2,-1/2\rangle\right)$ $\quad |1/2,-1/2\rangle$

$$|j,j_m\rangle = \sum_{m+s_m=j_m} C^{l\,s\,j}_{m,s_m,s_j}\,|l,m\rangle|s,s_m\rangle$$