

(23)

QM 115B

Lec. 6

$$\begin{aligned}
 L_{\pm} L_{\mp} &= (L_x \pm iL_y)(L_x \mp iL_y) \\
 &= L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) \\
 &= L^2 - L_z^2 \mp i i \hbar L_z \\
 &= L^2 - L_z^2 \pm \hbar L_z
 \end{aligned}$$

$$L^2 = L_{\pm} L_{\mp} + L_z^2 \mp \hbar L_z$$

$$\begin{aligned}
 L^2 \phi_{\pm} &= (L_{\mp} L_{\pm} + L_z^2 \mp \hbar L_z) \phi_{\pm} \\
 &= (\hbar^2 l(l \mp 1) + \hbar^2 l^2 \mp \hbar^2 l) \phi_{\pm}
 \end{aligned}$$

$$\lambda \phi_{\pm} = \hbar^2 l(l \mp 1) \phi_{\pm}$$

$$L_{-} \phi_b = 0$$

$$L_z \phi_b = \hbar l_b \phi_b \quad L^2 \phi_b = \lambda \phi_b$$

$$\begin{aligned}
 L^2 \phi_b &= (L_{+} L_{-} + L_z^2 - \hbar L_z) \phi_b \\
 &= (\hbar^2 l_b(l_b - 1) - \hbar^2 l_b) \phi_b
 \end{aligned}$$

$$\lambda = \hbar^2 l_b(l_b - 1) = \hbar^2 l(l + 1)$$

$$l_b = l + 1 \text{ or } l_b = -l$$

$$L_z \phi = \hbar m \phi$$

def.

$$m = -l, -l+1, \dots, l-1, l$$

N steps

$$\begin{aligned}
 l &= -l + N \\
 l &= \frac{N}{2}
 \end{aligned}$$

integer
half-integer

Eigen functions of L^2, L_z

$$\vec{L} = \frac{\hbar}{i} \vec{n} \times \vec{\nabla}$$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\vec{n} = r \hat{r}$$

$$\vec{L} = \frac{\hbar}{i} \left(r (\hat{n} \times \hat{r}) \frac{\partial}{\partial r} + (\hat{r} \times \hat{\theta}) \frac{\partial}{\partial \theta} + (\hat{r} \times \hat{\phi}) \frac{\partial}{\partial \phi} \right)$$

$$= \frac{\hbar}{i} \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$$

$$\vec{L} = \frac{\hbar}{i} \left[(-\sin \phi \hat{x} + \cos \phi \hat{y}) \frac{\partial}{\partial \theta} - (\cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}) \frac{\partial}{\partial \phi} \right]$$

$$L_x = \frac{\hbar}{i} \left(-\sin \phi \frac{\partial}{\partial \theta} - \cos \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$L_y = \frac{\hbar}{i} \left(\cos \phi \frac{\partial}{\partial \theta} + \cos \theta \cos \phi \frac{\partial}{\partial \phi} \right)$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

25

$$L_{\pm} = L_x \pm iL_y = \frac{\hbar}{i} \begin{pmatrix} (-\sin\theta \pm i\cos\theta) \frac{\partial}{\partial \theta} \\ -(\cos\theta \pm i\sin\theta) \cot\theta \frac{\partial}{\partial \phi} \end{pmatrix}$$

$$\cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

$$L_{\pm} = \pm \hbar e^{\pm i\theta} \begin{pmatrix} \frac{\partial}{\partial \theta} \pm i \cot\theta \frac{\partial}{\partial \phi} \end{pmatrix}$$

$$L_+ L_- = -\hbar^2 e^{i\theta} \begin{pmatrix} \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \end{pmatrix} e^{-i\theta} \begin{pmatrix} \frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \phi} \end{pmatrix}$$

$$= -\hbar^2 e^{i\theta} e^{-i\theta} \begin{pmatrix} \frac{\partial^2}{\partial \theta^2} - i \left(\frac{-1}{\sin^2\theta} \right) \frac{\partial}{\partial \phi} - i \cot\theta \frac{\partial^2}{\partial \theta \partial \phi} \\ + i \cot\theta (-i) e^{-i\theta} \begin{pmatrix} \frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \phi} \end{pmatrix} \\ + i \cot\theta e^{-i\theta} \begin{pmatrix} \frac{\partial^2}{\partial \phi \partial \theta} - i \cot\theta \frac{\partial^2}{\partial \phi^2} \end{pmatrix} \end{pmatrix}$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \cot^2\theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} \right)$$

$$L^2 = L_+ L_- + L_- L_+ - \hbar L_z$$

$$= -\hbar^2 \begin{pmatrix} \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \cot^2\theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} \\ \frac{\partial^2}{\partial \phi^2} + \frac{1}{i} \frac{\partial}{\partial \phi} \end{pmatrix}$$

$$= -\hbar^2 \left(\frac{1}{\sin^2\theta} \frac{\partial}{\partial \theta} \left(\sin^2\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right)$$

eigen functions $Y_l^m(\theta, \phi)$

no Y_l^m for $l \neq \frac{1}{2}$ integer

$$H\psi = E\psi \quad L^2\psi = \hbar^2 l(l+1)\psi, \quad L_z\psi = \hbar m\psi$$

Schr. Eq

$$\left[\frac{1}{2\mu r^2} \left(-\hbar^2 \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + L^2 \right) + V \right] \psi = E\psi$$

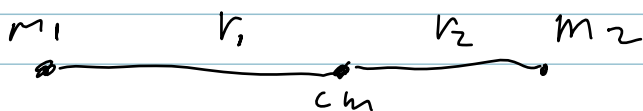
radial kin
 $= \frac{1}{2}\mu \dot{r}^2$

rot. kin

$$\frac{L^2}{2I}$$

$$I = \mu r^2$$

Ex diatomic molecule



$$\begin{aligned} m_1 r_1 &= m_2 r_2 \\ (m_1 + m_2) r_1 &= m_2 (r_1 + r_2) \\ (m_1 + m_2) r_2 &= m_1 (r_1 + r_2) \end{aligned}$$

$$\begin{aligned} I &= m_1 r_1^2 + m_2 r_2^2 \\ &= m_1 \frac{m_2^2 (r_1 + r_2)^2}{(m_1 + m_2)^2} + m_2 \frac{m_1^2 (r_1 + r_2)^2}{(m_1 + m_2)^2} \end{aligned}$$

$$= \frac{m_1 m_2 (m_2 + m_1) r^2}{(m_1 + m_2)^2} = \frac{m_1 m_2}{m_1 + m_2} r^2 = \mu r^2$$

$$H = \frac{L^2}{2I}$$

$$H\psi(\theta, \phi) = \frac{L^2}{2I} \psi(\theta, \phi) = E\psi(\theta, \phi)$$

$$\psi(\theta, \phi) = Y_l^m(\theta, \phi)$$

$$E_l = \frac{\hbar^2 l(l+1)}{2I}$$

$$l = 0, 1, 2, \dots$$

$(2l+1)$ degeneracy