(23)

QM 115B Lec. 6

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y)$$
$$= L_x^2 + L_y^2 \mp i (L_x L_y - L_y L_x)$$
$$= L^2 - L_z^2 \mp i \, i\hbar L_z$$
$$= L^2 - L_z^2 \pm \hbar L_z$$

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

$$L^2 \phi_t = (L_- L_+ + L_z^2 + \hbar L_z)\phi_t$$
$$= (0 + \hbar^2 l^2 + \hbar^2 l)\phi_t$$

$$\lambda \phi_t - \hbar^2 l(l+1)\phi_t$$

$$L_- \phi_b = 0$$

$$L_z \phi_b = \hbar l_b \phi_b \qquad L^2 \phi_b = \lambda \phi_b$$

$$L^2 \phi_b = (L_+ L_- + L_z^2 - \hbar L_z)\phi_b$$
$$= (0 + \hbar^2 l_b^2 - \hbar^2 l_b)\phi_b$$

$$\lambda = \hbar^2 l_b (l_b - 1) = \hbar^2 l(l+1)$$

$l_b = l+1$ no! $\qquad l_b = -l$

$$L_z \phi = \hbar m \phi \qquad m = \underbrace{-l, -l+1, \ldots l-1, l}_{N \text{ steps}}$$

def.

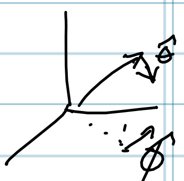
$l = -l + N$

$l = \frac{N}{2}$ integer half-integer

Eigenfunctions of L^2, L_z

$$\vec{L} = \frac{\hbar}{i}\,\vec{r} \times \vec{\nabla}$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$



$$\vec{r} = r\,\hat{r}$$

$$\vec{L} = \frac{\hbar}{i}\left(r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + (\hat{r}\times\hat{\theta})\frac{\partial}{\partial \theta} + \frac{(\hat{r}\times\hat{\phi})}{\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$= \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\vec{L} = \frac{\hbar}{i}\left[(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})\frac{\partial}{\partial \theta} - (\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z})\frac{\partial}{\partial \phi} \right]$$

$$L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial \theta} - \cos\theta\,\overset{\cos}{\cancel{\sin}}\,\phi\frac{\partial}{\partial \phi} \right)$$

$$L_y = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial \theta} \overset{-}{\cancel{+}} \cos\theta\sin\phi\frac{\partial}{\partial \phi} \right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

25

$$L_\pm = L_x \pm i L_y = \frac{\hbar}{i}\left((-\sin\phi \pm i\cos\phi)\frac{\partial}{\partial\theta} - (\cos\theta \pm i\sin\phi)\cot\theta\frac{\partial}{\partial\phi} \right)$$

$$\cos\phi \pm i\sin\phi = e^{\pm i\phi}$$

$$L_\pm = \pm\hbar\, e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_+L_- = -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$= -\hbar^2 e^{i\phi}\Big(e^{-i\phi}\left(\frac{\partial^2}{\partial\theta^2} - i\left(\frac{-1}{\sin^2\theta}\right)\frac{\partial}{\partial\phi} - \cancel{i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi}}\right)$$

$$+ i\cot\theta(-i)e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$+ i\cot\theta\, e^{-i\phi}\left(\cancel{\frac{\partial^2}{\partial\phi\partial\theta}} - i\cot\theta\frac{\partial^2}{\partial\phi^2}\right)\Big)$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$$

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} \right.$$
$$\left. \frac{\partial^2}{\partial\phi^2} + \frac{1}{i}\frac{\partial}{\partial\phi}\right)$$

$$= -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

eigenfunctions $Y_\ell^m(\theta,\phi)$

no Y_ℓ^m for ℓ ½ integer

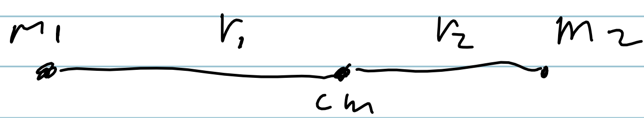
$$H\psi = E\psi \qquad L^2\psi = \hbar^2 l(l+1)\psi, \quad L_z\psi = \hbar m\psi$$

Schr. Eq

$$\left[\frac{1}{2\mu r^2}\left(-\hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + L^2\right) + V\right]\psi = E\psi$$

radial kin $= \frac{1}{2}\mu \dot{r}^2$ rot. kin $\dfrac{L^2}{2I}$ $I = \mu r^2$

Ex diatomic molecule



$$m_1 r_1 = m_2 r_2$$
$$(m_1+m_2) r_1 = m_2 (r_1+r_2)$$
$$(m_1+m_2) r_2 = m_1 (r_1+r_2)$$

$$I = m_1 r_1^2 + m_2 r_2^2$$
$$= m_1 \frac{m_2^2 (r_1+r_2)^2}{(m_1+m_2)^2} + m_2 \frac{m_1^2 (r_1+r_2)^2}{(m_1+m_2)^2}$$
$$= \frac{m_1 m_2 (m_2+m_1) r^2}{(m_1+m_2)^2} = \frac{m_1 m_2}{m_1+m_2} r^2 = \mu r^2$$

$$H = \frac{L^2}{2I} \qquad H\psi(\theta,\phi) = \frac{L^2}{2I}\psi(\theta,\phi) = E\psi(\theta,\phi)$$

$$\psi(\theta,\phi) = Y_l^m(\theta,\phi)$$

$$E_l = \frac{\hbar^2 l(l+1)}{2I} \qquad l = 0, 1, 2, \ldots$$

$(2l+1)$ degeneracy