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QM 115B Lec 3

m is a quantum number of what?

$$\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}$$

$$= -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} + 0$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi} = x\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - y\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

$$= x\hat{p}_y - y\hat{p}_x = \hat{L}_z$$

$$\vec{L} = \vec{r}\times\vec{p} \qquad L_z = xp_y - yp_x$$

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \hat{L}_z^2\Phi = m^2\hbar^2\Phi$$

$$L_z\Phi = \frac{\hbar}{i}\frac{\partial}{\partial\phi}e^{im\phi}$$

$$= m\hbar\Phi$$

Φ is an eigenfunction of $\hat{L}_z$ eigenvalue $m\hbar$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\right]$$

$$= \frac{r^2 2\mu}{\hbar^2}\left(E - V(r)\right)$$

$\uparrow$ const,

$$\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta = C_\theta\,\Theta$$

$$C_\theta = -l(l+1)$$

Solution $\Theta(\theta) = A\,P_l^m(\cos\theta)$

$$P_l^m(x) = (1-x^2)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|} P_l(x)$$

P_l is the Legendre polynomial

$$P_l = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$ (Rodrigues formula)

$l = 0, 1, 2 \ldots$

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l	$\lvert m \rvert$	$P_l^m(\cos\theta)$
0	0	1
1	0	$\cos\theta$
1	1	$\sin\theta$
2	0	$\frac{1}{2}(3\cos^2\theta - 1)$
2	1	$3\cos\theta\sin\theta$
2	2	$3\sin^2\theta$

$$l = 0, 1, 2 \qquad m = 0, \pm 1, \pm 2 \dots \pm l$$

$$Y_l^m(\theta, \phi) = A\, P_l^m(\cos\theta)\, e^{im\phi}$$

Normalization $\quad d^3r = r^2 \sin\theta\, d\theta\, d\phi$

$$1 = \int |\psi|^2 d^3r = \underbrace{\int |R|^2 r^2 dr}_{=1} \underbrace{\int |Y_l^m|^2 \sin\theta\, d\theta\, d\phi}_{=1}$$

for convienence

$$\int_0^{2\pi}\int_0^{\pi} |Y_l^m(\theta,\phi)|^2 \sin\theta\, d\theta\, d\phi = 1$$

Y_l^m is an eigenfunction $\quad L^2 = L_x^2 + L_y^2 + L_z^2$

L_z

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]$$

$$L^2 Y_l^m = -\hbar^2 A \left[\frac{\Phi}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta}{\partial\theta}\right) + \frac{\Theta}{\sin^2\theta}\left(-m^2\Phi\right) \right]$$

$$= -\hbar^2 A \left(\Phi \left(-l(l+1)\right) \Theta \right)$$

$$= \hbar^2 \underbrace{l(l+1)}_{\text{eigenvalue}} Y_l^m$$

Radial Eq

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{r^2\, 2\mu}{\hbar^2}\left(E - V(r)\right)R = \ell(\ell+1)R$$

change variables $\quad u(r) = r\,R(r)$

$$R = \frac{u}{r} \qquad \frac{dR}{dr} = -\frac{u}{r^2} + \frac{1}{r}\frac{du}{dr} = \frac{1}{r^2}\left(r\frac{du}{dr} - u\right)$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(r\frac{du}{dr} - u\right) = \frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr}$$

$$= r\frac{d^2u}{dr^2}$$

$$r\frac{d^2u}{dr} + \frac{2\mu}{\hbar^2}\left(E - V(r)\right)ru = \ell(\ell+1)\frac{u}{r}$$

$$\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} - V(r)u - \frac{\hbar^2}{2\mu}\ell(\ell+1)\frac{u}{r^2} = -Eu$$

equiv 1D Schr. eq with effective pot.

$$V_{eff} = V + \frac{\hbar}{2\mu}\frac{\ell(\ell+1)}{r^2}$$

$$\int_0^\infty |R|^2 r^2\,dr = 1 = \int_0^\infty |u|^2\,dr$$

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Infinite Spherical Well

$$V(|\vec{r}|) = \begin{cases} 0 & \text{if } |\vec{r}| \le a \\ \infty & \text{if } |\vec{r}| > a \end{cases}$$

inside $\quad -\frac{\hbar^2}{2\mu}\left(\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2}u\right) = Eu$

$$\frac{d^2u}{dr} = \left(\frac{\ell(\ell+1)}{r^2} - k^2\right)u$$

$$k = \sqrt{\frac{2\mu E}{\hbar}}$$

$\ell = 0 \qquad \frac{d^2u}{dr} = -k^2 u$

$$u(r) = A\sin kr + B\cos kr$$

$\text{Prob} = |R|^2 = \left|\frac{u(r)}{r}\right|^2 \qquad \frac{\cos kr}{r}$ blows up at $r=0$

$$B = 0$$

$u(r) = A\sin kr \qquad \sin ka = 0$

$ka = n\pi$

$E_{n,0} = \frac{k_{n,0}^2\hbar^2}{2\mu} = \frac{n^2\pi^2\hbar^2}{2\mu a^2}$

$n = 1, 2, 3$

$\int_0^a A^2\sin^2(kr)\,dr = 1$

$A = \sqrt{\frac{2}{a}} \qquad Y_0^0 = \frac{1}{\sqrt{4\pi}}$

$$\psi_{n,0,0} = \frac{1}{\sqrt{2\pi a}}\,\frac{\sin(n\pi r/a)}{r}$$

arbitrary $l \neq 0$

$$u = A r j_l(kr) + B r n_l(kr)$$

spherical Bessel   spherical Neumann

blows up at $r=0$

$\Rightarrow B$

$$u(r) = A r j_l(kr)$$

$$j_l(ka) = 0$$

$$k = \frac{\beta_{nl}}{a}$$ nth zero of the lth sph. Bessel

$$\psi_{nlm}(r,\theta,\phi) = A_{nl}\, j_l(\beta_{nl} r/a)\, Y_l^m(\theta,\phi)$$

for each l $m = 0, \pm 1, \pm \ldots \pm l$

$1 + 2l$

$(2l+1)$ fold degeneracy