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QM 115B Lec 20

Time Dependent Perturbation Theory

Two-level System

$$H^0 \psi_a = E_a \psi_a \quad , \quad H^0 \psi_b = E_b \psi_b$$

$$\langle \psi_a | \psi_b \rangle = \delta_{ab}$$

$$\psi(t) = c_a \psi_a e^{-iE_a t/\hbar} + c_b \psi_b e^{-iE_b t/\hbar}$$

$$|c_a|^2 + |c_b|^2 = 1$$

time dependent perturbation $H'(t)$

$H = H^0 + H'$

$$\psi(t) = c_a(t) \psi_a e^{-iE_a t/\hbar} + c_b(t) \psi_b e^{-iE_b t/\hbar}$$

$$H\psi = i\hbar \frac{\partial}{\partial t}\psi$$

$$c_a E_a \psi_a e^{-iE_a t/\hbar} + c_b E_b \psi_b e^{-iE_b t/\hbar} + c_a H' \psi_a e^{-iE_a t/\hbar} + c_b H' \psi_b e^{-iE_b t/\hbar}$$
$$= i\hbar \left(\dot{c}_a \psi_a e^{-iE_a t/\hbar} + \dot{c}_b \psi_b e^{-iE_b t/\hbar} + c_a \psi_a \left(\frac{-iE_a}{\hbar}\right) e^{-iE_a t/\hbar} + c_b \psi_b \left(\frac{-iE_b}{\hbar}\right) e^{-iE_b t/\hbar} \right)$$

$$c_a H'(t) \psi_a e^{-iE_a t/\hbar} + c_b H'(t) \psi_b e^{-iE_b t/\hbar} = i\hbar \left(\dot{c}_a \psi_a e^{-iE_a t/\hbar} + \dot{c}_b \psi_b e^{-iE_b t/\hbar} \right)$$

inner prod ψ_a

$$c_a \langle \psi_a | H' | \psi_a \rangle e^{-iE_a t/\hbar} + c_b \langle \psi_a | H' | \psi_b \rangle e^{-iE_b t/\hbar} = i\hbar \dot{c}_a e^{-iE_a t/\hbar}$$

$$H'_{ij} = \langle \psi_i | H' | \psi_j \rangle$$

$$\dot{c}_a = \frac{-i}{\hbar}\left(c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right)$$

$$\dot{c}_b = \frac{-i}{\hbar}\left(c_b H'_{bb} + c_a H'_{ba} e^{-i(E_a - E_b)t/\hbar} \right)$$

assume $H'_{aa} = H'_{bb} = 0 \qquad \omega_0 = \frac{E_b - E_a}{\hbar}$

$$\dot{c}_a = \frac{-i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = \frac{-i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a$$

suppose $c_a(0) = 1 \qquad c_b(0) = 0$

for small H' : zeroth order $c_a^{(0)}(t) = 1 \quad c_b^{(0)}(t) = 1$

1st order $\frac{d}{dt} c_a^{(1)} = \frac{-i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b^{(0)} = 0 \Rightarrow c_a^{(1)}(t) = 1$

$$\frac{d}{dt} c_b^{(1)} = \frac{-i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a^{(0)}(t)$$

$$c_b^{(1)} = \frac{-i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$$

2nd order $\frac{d}{dt} c_a^{(2)} = \frac{-i}{\hbar} H'_{ab} e^{-i\omega_0 t} \left(\frac{-i}{\hbar}\right) \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$

$$c_a^{(2)} = 1 - \frac{1}{\hbar^2} \int_0^t H'_{ab}(t') e^{-i\omega_0 t'} \int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''} dt'' dt'$$

etc.

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Sinusoidal Perturbations

$$H'(\vec{r},t) = V(\vec{r})\cos\omega t$$

$$H'_{ab} = V_{ab}\cos\omega t$$

$$V_{ab} = \langle \psi_a | V(\vec{r}) | \psi_b \rangle$$

assume $V_{aa}=0$

first order $$c_b(t) \simeq -\frac{i}{\hbar} V_{ba} \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt'$$

$$= -\frac{i V_{ba}}{2\hbar} \int_0^t e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'} dt'$$

$$= -\frac{i V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0+\omega)t'}}{i(\omega_0+\omega)} + \frac{e^{i(\omega_0-\omega)t'}}{i(\omega_0-\omega)} \right]_0^t$$

$$= -\frac{V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega} \right]$$

near resonance $\omega_0+\omega \gg |\omega_0-\omega|$

$$c_b(t) \simeq -\frac{V_{ba}}{2\hbar} \frac{e^{i(\omega_0-\omega)t/2}}{\omega_0-\omega} \left(e^{i(\omega_0-\omega)t/2} - e^{-i(\omega_0-\omega)t/2} \right)$$

$$\simeq -i\frac{V_{ba}}{\hbar} \frac{\sin[(\omega_0-\omega)t/2]}{\omega_0-\omega} e^{i(\omega_0-\omega)t/2}$$

$$P_{a\to b}(t) = |c_b(t)|^2 \simeq \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$