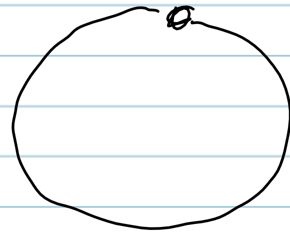
(61)

QM 115B Lec 16

Prob 6.7



bead on a wire

$$-\frac{L}{2} \le x \le \frac{L}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \qquad \psi_n = \frac{1}{\sqrt{L}} e^{2\pi i n x/L}$$

$$E = \frac{2\pi^2\hbar^2 n^2}{mL^2} \qquad n = 0, \pm 1, \pm 2$$

doubly degenerate

$$H' = -V_0\, e^{-x^2/a^2} \qquad a \ll L$$

$$W_{nn} = \langle \psi_n | H' | \psi_n \rangle = -V_0 \int_{-L/2}^{L/2} e^{-x^2/a^2} \underbrace{\psi_n^* \psi}_{= 1/L}\, dx$$

$$\simeq -\frac{V_0}{L}\int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = -\frac{V_0}{L} a\sqrt{\pi}$$

$$W_{-n,-n} = W_{nn}$$

$$W_{n,-n} = -\frac{V_0}{L}\int_{-L/2}^{L/2} e^{-x^2/a^2}\, e^{-4\pi i n x/L}$$

$$= -\frac{V_0}{L} a\sqrt{\pi}\, e^{-(2\pi n a/L)^2}$$

$$E'_{\pm} = \frac{1}{2}\left(W_{nn} + W_{-n-n} \pm \sqrt{(W_{nn} - W_{-n-n}) + 4|W_{n-n}|^2}\right)$$

$$\cdot\ W_{nn} \pm |W_{n-n}| = -\sqrt{\pi}\frac{V_0 a}{L}\left(1 \mp e^{-(2\pi n a/L)^2}\right)$$

"good states"

$$\alpha W_{nn} + \beta W_{n-n} = \alpha E'_{\pm}$$

$$\beta = \alpha \left(\frac{E'_{\pm} - W_{nn}}{W_{n,-n}} \right)$$

$$= \alpha \left(\frac{\pm |W_{n,-n}|}{W_{n,-n}} \right) = \mp \alpha$$

$$\psi_{+} = \alpha \psi_{n} - \alpha \psi_{-n} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{L}} \left(e^{i2\pi n x/L} - e^{-i2\pi n x/L} \right)$$

$$= i \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n x}{L} \right)$$

$$\psi_{-} = \alpha \psi_{n} + \alpha \psi_{-n} = \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L} \right)$$

$$E'_{+} = \langle \psi_{+} | H' | \psi_{+} \rangle = \frac{2}{L}(-V_0) \int_{-L/2}^{L/2} e^{-x^2/a^2} \sin^2\left(\frac{2\pi n x}{L} \right) dx$$

$$E'_{-} = \langle \psi_{-} | H' | \psi_{-} \rangle = \frac{2}{L}(-V_0) \int_{-L/2}^{L/2} e^{-x^2/a^2} \cos^2\left(\frac{2\pi n x}{L} \right) dx$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$E'_{\pm} = -\frac{V_0}{L} \int_{-\infty}^{\infty} e^{-x^2/a^2} \left(1 \mp \cos\left(\frac{4\pi n x}{L} \right) \right) dx$$

$$\simeq -\frac{V_0 a \sqrt{\pi}}{L} \left(1 \mp e^{-(2\pi n a/L)^2} \right)$$

$$P \sin(x) = \sin(-x) = -\sin x$$
$$P \cos(x) = \cos(-x) = \cos x$$

$$P f(x) = f(-x)$$
$$P H'(x) = H'(-x) = H'(x)$$

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Higher Degeneracy

$\alpha W_{aa} + \beta W_{ab} = \alpha E'$
$\alpha W_{ba} + \beta W_{bb} = \beta E'$

$$\begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underset{\uparrow}{E'} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

"good" states eigenvectors of W (eigenvalues)

n-fold

$W_{ij} = \langle \psi_i^0 | H' | \psi_j^0 \rangle$ $\quad n \times n$

construct a basis degenerate subspace that diagonaliz W

$H = H_0\ H'$

$$H = V_0 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad H' = \epsilon V_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\chi_1^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \chi_2^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \chi_3^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$E_1^0 = V_0 \quad E_2^0 = V_0 \quad E_3^0 = V_0$

$\varepsilon \ll 1$

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$$W_{ab} = \langle \chi_a | H' | \chi_b \rangle = \epsilon \cancel{V_0} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$0 = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left((2-\lambda)(1-\lambda) - 1\right) + \lambda - 1$$

$$= (1-\lambda)\left(2 - 3\lambda + \lambda^2 - 1 - 1\right)$$

$$= \lambda(1-\lambda)(\lambda - 3)$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\sqrt{2/3} \\ 1/\sqrt{6} \end{pmatrix}$$

$$E \simeq \begin{cases} V_0 \\ V_0(1+\epsilon) \\ V_0(1+3\epsilon) \end{cases}$$

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3D Harmonic Oscillator

$$H = -\frac{\hbar^2 \vec{\nabla}^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

$$= \hbar\omega\left(a_+^{(x)}a_-^{(x)} + a_+^{(y)}a_-^{(y)} + a_+^{(z)}a_-^{(z)} + 3/2\right)$$

$n = n_x + n_y + n_z$		$\vert n_x n_y n_z\rangle$	d
0	G.S.	$\vert 000\rangle$	1
1	1st Excited	$\vert 1,0,0\rangle$ $\vert 0,1,0\rangle$ $\vert 0,0,1\rangle$	3
2	2nd Excited	$\vert 110\rangle$ $\vert 101\rangle$ $\vert 011\rangle$ $\vert 2,0,0\rangle$ $\vert 020\rangle$ $\vert 002\rangle$	6

$\frac{(n+1)(n+2)}{2}$

$$H' = b\hbar\omega\left(a_+^{(x)}a_-^{(y)} + a_+^{(y)}a_-^{(x)}\right)$$

$\uparrow$ $\hat{n}_z$ conserved

G.S. $\langle 000|H'|000\rangle = 0$

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1st Excited state

$$\langle n_x\, n_y\, n_z | H' | 100\rangle = \begin{cases} b\hbar\omega & \begin{matrix} n_x & n_y & n_z \\ 0 & 1 & 0 \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle n_x\, n_y\, n_z | H' | 010\rangle = \begin{cases} b\hbar\omega & 1\ 0\ 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle n_x\, n_y\, n_z | H' | 001\rangle = 0$$

$$\begin{array}{c|cc:c} & n_z = 0 & & n_z = 1 \\ & 100 & 010 & 001 \\ \hline 100 & 0 & b\hbar\omega & 0 \\ 010 & b\hbar\omega & 0 & 0 \\ \hdashline 001 & 0 & 0 & 0 \end{array}$$

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \qquad \lambda \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = \pm a$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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"good" states

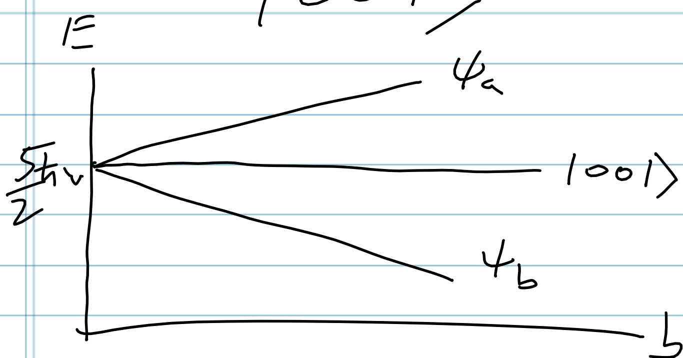
$$\psi_a = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{\sqrt{2}}|010\rangle$$

$$\psi_b = \frac{1}{\sqrt{2}}|100\rangle - \frac{1}{\sqrt{2}}|010\rangle$$

$$|001\rangle$$

$$E_a = E^0 + b\hbar\omega = (1 + 3/2)\hbar\omega + b\hbar\omega = 5/2\,\hbar\omega + b\hbar\omega$$

$$E_b = 5/2\,\hbar\omega - b\hbar\omega$$



2nd Excited State

$$\langle n_x n_y n_z | H' | 002\rangle = 0$$

$$\langle n_x n_y n_z | H' | 020\rangle = \begin{cases} b\hbar\omega\sqrt{1}\sqrt{2} & n_x n_y n_z = 110 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle n_x n_y n_z | H' | 200\rangle = \begin{cases} b\hbar\omega\sqrt{2}\sqrt{1} & 110 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle n_x\, n_y\, n_z | H' | 110\rangle = \begin{cases} b\hbar\omega \sqrt{2}\sqrt{1} & \begin{matrix} n_x & n_y & n_z \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle n_x\, n_y\, n_z | H' | 101\rangle = \begin{cases} b\hbar\omega \sqrt{1}\sqrt{1} & 0\ 1\ 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle n_y\, n_y\, n_z | H' | 011\rangle = \begin{cases} b\hbar\omega \sqrt{1}\sqrt{1} & 1\ 0\ 1 \\ 0 & \text{otherwise} \end{cases}$$

$$W = b\hbar\omega \times \begin{array}{c} \\ 110 \\ 101 \\ 011 \\ 200 \\ 020 \\ 002 \end{array} \begin{array}{c} \begin{matrix} 110 & 101 & 011 & 200 & 020 & 002 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

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$$
\begin{array}{c|cccccc}
 & \multicolumn{3}{c}{n_z=0} & \multicolumn{2}{c}{n_z=1} & n_z=2 \\
 & 110 & 200 & 020 & 101 & 011 & 002 \\
\hline
110 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\
200 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
020 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
101 & 0 & 0 & 0 & 0 & 1 & 0 \\
011 & 0 & 0 & 0 & 1 & 0 & 0 \\
002 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$n_z = 0$

$$0 = \begin{vmatrix} -\lambda & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\lambda & 0 \\ \sqrt{2} & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{vmatrix} + \sqrt{2} \begin{vmatrix} \sqrt{2} & \sqrt{2} \\ -\lambda & 0 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 2) + \sqrt{2}\,\lambda\sqrt{2}$$

$$= -\lambda^3 + 2\lambda + 2\lambda = -\lambda(\lambda^2 - 4)$$

$\lambda = 0 \qquad \lambda = \pm 2$

$$\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ b \end{pmatrix} = \pm 2 \begin{pmatrix} a \\ b \\ b \end{pmatrix}$$

$$\begin{pmatrix} 2\sqrt{2}\, b \\ \sqrt{2}\, a \\ \sqrt{2}\, a \end{pmatrix} = \pm 2 \begin{pmatrix} a \\ b \\ b \end{pmatrix}$$

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$$2\sqrt{2}\, b = \pm 2a$$

$$b = \pm \frac{a}{\sqrt{2}}$$

$$\sqrt{2}\, a = \pm 2b$$

$$b = \pm \frac{a}{\sqrt{2}}$$

$$N = (a\ b\ b)\begin{pmatrix} a \\ b \\ b \end{pmatrix} \qquad \sqrt{N} = \sqrt{a^2+2b^2}$$

$$= \sqrt{a^2 + \frac{2a^2}{2}} = \sqrt{2}\,|a|$$

eigenvectors $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$n_z = 1$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \qquad \lambda = \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = \pm a$$

eigenvectors $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

"good" states

$$\psi_a = \frac{1}{\sqrt{2}}|200\rangle - \frac{1}{\sqrt{2}}|020\rangle$$

$$\psi_b = \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{2}|200\rangle + \frac{1}{2}|020\rangle$$

$$\psi_c = \frac{1}{\sqrt{2}}|110\rangle - \frac{1}{2}|200\rangle - \frac{1}{2}|020\rangle$$

$$\psi_A = \frac{1}{\sqrt{2}}|101\rangle + \frac{1}{\sqrt{2}}|011\rangle$$

$$\psi_B = \frac{1}{\sqrt{2}}|101\rangle - \frac{1}{\sqrt{2}}|011\rangle$$

$|002\rangle$

$$E_a = E_2^0 + b\hbar\omega\cdot 0 = \hbar\omega\left(2+\tfrac{3}{2}\right) = \tfrac{7}{2}\hbar\omega$$

$$E_b = \tfrac{7}{2}\hbar\omega + 2b\hbar\omega$$

$$E_c = \tfrac{7}{2}\hbar\omega - 2b\hbar\omega$$

$$E_A = \tfrac{7}{2}\hbar\omega + \hbar\omega$$

$$E_B = \tfrac{7}{2}\hbar\omega - \hbar\omega$$

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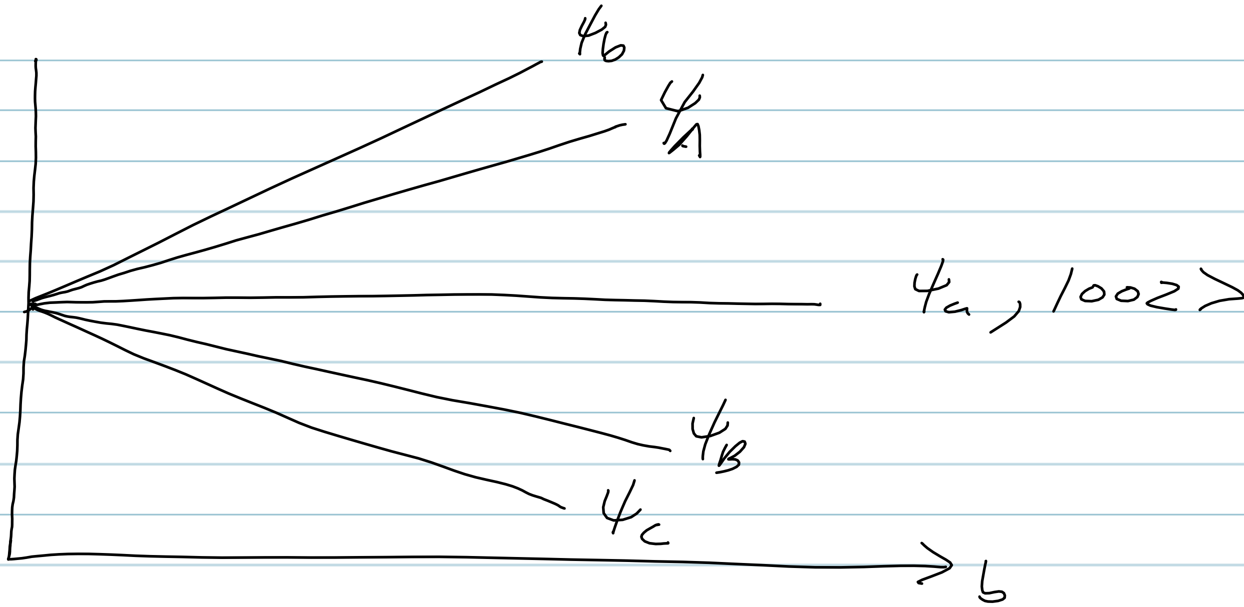

3rd excited state

$n = 3$ $\quad n_z = 0$ subspace

$$
\begin{array}{c|cccc}
 & 120 & 210 & 030 & 300 \\
\hline
120 & & & & \\
210 & & & & \\
030 & & & & \\
300 & & & &
\end{array}
\quad
\begin{array}{c} 120 \\ 210 \\ 030 \\ 300 \end{array}
\begin{pmatrix} 0 & 2 & 3 & 0 \\ 2 & 6 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}
$$

$\lambda = \pm 1 \pm \sqrt{10}$