

E

QM 115B Lec. 14

Linear perturbation of Harmonic Oscillator

$$V = \frac{1}{2} m \omega^2 x^2 - \underbrace{qEx}_{H'}$$

$$x' = x - \frac{qE}{m\omega^2}$$

$$x = x' + \frac{qE}{m\omega^2}$$

$$V = \frac{1}{2} m \omega^2 \left(x'^2 + \frac{2qEx'}{m\omega^2} + \frac{q^2 E^2}{m^2 \omega^4} \right)$$

$$- qE \left(x' + \frac{qE}{m\omega^2} \right)$$

$$= \frac{1}{2} m \omega^2 x'^2 + qEx' - qEx' - \frac{1}{2} \frac{q^2 E^2}{m\omega^2} + \frac{q^2 E^2}{m\omega^2}$$

$$= \frac{1}{2} m \omega^2 x'^2 - \frac{1}{2} \frac{q^2 E^2}{m\omega^2}$$

$$E_n = (n + \frac{1}{2}) \hbar \omega - \frac{1}{2} \frac{q^2 E^2}{m\omega^2}$$

perturbation theory

$$E'_n = \langle \psi_n^0 | H' | \psi_n^0 \rangle$$

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m \omega}} (\mp i p + m \omega x) = -qE \langle n | x | n \rangle$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-)$$

$$a_+ |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$a_- |n\rangle = \sqrt{n} |n-1\rangle$$

$$\begin{aligned}
\langle n|x|n' \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle n|a_+ + a_-|n' \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} (\langle n|a_-|n' \rangle + \langle n|a_+|n' \rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n'} \langle n|n'-1 \rangle + \sqrt{n} \langle n|n'+1 \rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n'} \delta_{n,n'-1} + \sqrt{n} \delta_{n,n'+1})
\end{aligned}$$

$$E_n' = -qE \langle n|x|n \rangle = 0$$

$$\begin{aligned}
E_n'' &= \sum_{m \neq n} \frac{|\langle m|-qEx|n \rangle|^2}{(n+\frac{1}{2})\hbar\omega - (m+\frac{1}{2})\hbar\omega} \\
&= q^2 E^2 \sum_{m \neq n} \frac{|\langle m|x|n \rangle|^2}{(n-m)\hbar\omega} \\
&= \frac{q^2 E^2}{\hbar\omega} \sum_{m \neq n} \frac{\hbar}{2m\omega} \left(\frac{\sqrt{m} \delta_{n,m-1} + \sqrt{n} \delta_{m,n-1}}{n-m} \right)^2 \\
&= \frac{q^2 E^2}{2m\omega^2} \left(\frac{n+1}{n-(n+1)} + \frac{n}{n-(n-1)} \right) \\
&= -\frac{q^2 E^2}{2m\omega^2}
\end{aligned}$$

$$E = (n+\frac{1}{2})\hbar\omega - \frac{q^2 E^2}{2m\omega^2} + \dots$$

(53)

cubic perturbation to S.H.O.

$$H' = 2^{3/2} b \sqrt{\frac{\mu^3 \omega^5}{\hbar}} x^3$$

\uparrow dimensionless units $\frac{\text{kg}^{3/2} (\text{1/s})^{5/2} \text{m}^3}{(\text{kg m}^2/\text{s})^{1/2}} = \text{kg m}^2/\text{s}^2$

$$x^3 = \left(\sqrt{\frac{\hbar}{2\mu\omega}} \right)^3 (a_+ + a_-)^3$$

$$H' = \frac{1}{2} \hbar \omega (a_+ + a_-)^3$$

$$\begin{aligned} (a_+ + a_-)^3 &= (a_+^2 + a_+ a_- + a_- a_+ + a_-^2) (a_+ + a_-) \\ &= a_+^3 + a_+^2 a_- + a_+ a_- a_+ + a_- a_+^2 \\ &\quad + a_-^2 a_+ + a_- a_+ a_- + a_- a_-^2 + a_-^3 \end{aligned}$$

$$\begin{aligned} a_+ a_- |n\rangle & \\ a_+ \sqrt{n} |n-1\rangle & \\ \sqrt{n} \sqrt{n} |n\rangle & \\ \hat{n} = a_+ a_- & \end{aligned}$$

$$\begin{aligned} a_- a_+ |n\rangle & \\ a_- \sqrt{n+1} |n+1\rangle & \\ \sqrt{n+1} \sqrt{n+1} |n\rangle & \\ \hat{n} + 1 = a_- a_+ & \end{aligned}$$

$$[a_-, a_+] = 1$$

$$\begin{aligned} (a_+ + a_-)^3 &= a_+^3 + a_+ \hat{n} + \hat{n} a_+ + \hat{n} a_- \\ &\quad + (\hat{n} + 1) a_+ + (\hat{n} + 1) a_- + a_- (\hat{n} + 1) + a_-^3 \end{aligned}$$

$$[\hat{n}, a_-] = [a_+ a_-, a_-] = a_+ [a_-, a_-] + [a_+, a_-] a_- = -a_-$$

$$[\hat{n}, a_+] = [a_+ a_-, a_+] = a_+ [a_-, a_+] + [a_+, a_+] a_+ = a_+$$

$$\begin{aligned} (a_+ + a_-)^3 &= a_+^3 + (\hat{n} a_+ - a_+) + \hat{n} a_+ + \hat{n} a_- \\ &\quad + (\hat{n} + 1) a_+ + (\hat{n} + 1) a_- + \hat{n} a_- + a_- + a_-^3 \\ &= a_+^3 + 3\hat{n} a_+ + 3(\hat{n} + 1) a_- + a_-^3 \end{aligned}$$

$$E_n' = \langle n | H' | n \rangle = 0$$

$$E_n^2 = \sum_{m \neq n} \frac{|\langle m | H' | n \rangle|^2}{E_n^0 - E_m^0}$$

$$\begin{aligned} \langle m=n-3 | H' | n \rangle &= b \hbar \omega \langle n-3 | \sqrt{n(n-1)(n-2)} | n-3 \rangle \\ &= b \hbar \omega \sqrt{n(n-1)(n-2)} \end{aligned}$$

$$\begin{aligned} \langle m=n-1 | H' | n \rangle &= b \hbar \omega \langle n-1 | 3(n-1)\sqrt{n} | n-1 \rangle \\ &= b \hbar \omega 3n\sqrt{n} \end{aligned}$$

$$\begin{aligned} \langle m=n+1 | H' | n \rangle &= b \hbar \omega \langle n+1 | 3\hat{n} a_+ | n \rangle \\ &= b \hbar \omega 3(n+1)\sqrt{n+1} \end{aligned}$$

$$\langle m=n+3 | H' | n \rangle = b \hbar \omega \sqrt{(n+1)(n+2)(n+3)}$$

$$E_n^2 = \frac{b^2 \hbar^2 \omega^2}{\hbar \omega} \left(\frac{n(n-1)(n-2)}{(n+1/2) - (n-3+1/2)} + \frac{9n^2n}{n - (n-1)} \right. \\ \left. + \frac{9(n+1)^2(n+1)}{n - (n+1)} + \frac{(n+1)(n+2)(n+3)}{n - (n+3)} \right)$$

$$= b^2 \hbar \omega \left(\frac{n(n^2 - 3n + 2)}{3} + 9n^3 \right. \\ \left. + \frac{9(n+1)^3}{-1} + \frac{(n+1)(n^2 + 5n + 6)}{-3} \right)$$

$$= b^2 \hbar \omega \left(\frac{n^3 - 3n^2 + 2n}{3} - (n^3 + 5n^2 + 6n + n^2 + 5n + 6) \right. \\ \left. + 9(n^3 - (n^3 + 3n^2 + 3n + 1)) \right)$$

55

$$\begin{aligned} E_h^2 &= -b^2 h \omega \left(\frac{9h^2 + 9h + 6}{3} + 27h^2 + 27h + 9 \right) \\ &= -b^2 h \omega \left(30h^2 + 30h + 11 \right) \\ &= -b^2 h \omega \left(30 \left(h + \frac{1}{2} \right)^2 + \frac{7}{2} \right) \end{aligned}$$

$$E_h = \left(h + \frac{1}{2} \right) h \omega - b^2 h \omega \left(30 \left(h + \frac{1}{2} \right)^2 + \frac{7}{2} \right)$$