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III, Identical Bosons

$$G = \sum_{n=1}^{\infty}\left(\ln (N_n + d_n - 1)! - \ln N_n! - \ln (d_n-1)!\right)$$

$$+ \alpha\, c_N + \beta\, c_E$$

$N_n \gg 1$

$$G \simeq \sum_{n=1}^{\infty}\Big((N_n + d_n - 1)\ln (N_n + d_n - 1) - (N_n + d_n - 1) - N_n \ln N_n + N_n - \ln (d_n-1)! - \alpha N_n - \beta E_n N_n\Big)$$

$$+ \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_n} \simeq \ln (N_n + d_n - 1) - \ln N_n - \alpha - \beta E_n = 0$$

$$N_n = \frac{d_n - \underbrace{1}_{\text{drop}}}{e^{\alpha + \beta E_n} - 1}$$

in all cases

$$\sum_{n=1}^{\infty} N_n(\alpha,\beta) = N$$

$$\sum_{n=1}^{\infty} N_n(\alpha,\beta) E_n = E$$

to do sum we need to know E_n, d_n

ideal gas: find α, β
non-interacting particles of mass m, in a bo

$$E_k = \frac{\hbar^2 \vec{k}^2}{2m} \qquad \vec{k} = \left(\frac{\pi n_x}{l_x}, \frac{\pi n_y}{l_y}, \frac{\pi n_z}{l_z}\right)$$

approx by continuum

$(2s+1)$ states per volume $\frac{\pi^3}{V}$ of k-space

first octant shell dk

$$d_k = \frac{1}{8} \frac{4\pi k^2 dk}{(\pi^3/V)} (2s+1) = (2s+1) \frac{V}{2\pi^2} k^2 dk$$

1) distinguishable particles

$$N = \sum_n d_n e^{-\alpha - \beta E_n} = (2s+1) \frac{V}{2\pi^2} \int_0^\infty k^2 dk \, e^{-(\alpha + \beta \frac{\hbar^2 k^2}{2m})}$$

$$= (2s+1) V e^{-\alpha} \left(\frac{m}{2\pi \beta \hbar^2}\right)^{3/2}$$

$$e^{-\alpha} = \frac{N}{(2s+1) V} \left(\frac{2\pi \beta \hbar^2}{m}\right)^{3/2}$$

$$E = (2s+1) \frac{V}{2\pi^2} \int_0^\infty k^2 dk \, e^{-(\alpha + \beta \frac{\hbar^2 k^2}{2m})} \frac{\hbar^2 k^2}{2m}$$

$$= (2s+1) \frac{3V}{2\beta} e^{-\alpha} \left(\frac{m}{2\pi \beta \hbar^2}\right)^{3/2} = \frac{3}{2} \frac{N}{\beta}$$

$$\frac{E}{N} = \frac{3}{2} \frac{1}{\beta} = \frac{3}{2} k_B T \qquad \beta = \frac{1}{k_B T}$$

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"chemical potential"

$$\mu(T) = -\alpha k_B T$$

$$e^{-(\alpha+\beta\epsilon)} = e^{-(-\mu+\epsilon)/k_BT} = e^{-(\epsilon-\mu)/k_BT}$$

dividing N_n by d_n

most probable number of particles in a particular state with energy ϵ

$$n(\epsilon) = \begin{cases} e^{-(\epsilon-\mu)/k_BT} & \text{Maxwell-Boltzmann} \quad \text{classi, "distinguishing"} \\ \dfrac{1}{e^{(\epsilon-\mu)/k_BT}+1} & \text{Fermi-Dirac} \quad \text{identical fermions} \\ \dfrac{1}{e^{(\epsilon-\mu)/k_BT}-1} & \text{Bose-Einstein} \quad \text{identical bosons} \end{cases}$$

$$\frac{\epsilon-\mu}{k_BT} \gg 1 \qquad \left\{ \begin{matrix} F\text{-}D \\ B\text{-}E \end{matrix} \right\} \longrightarrow M\text{-}B$$

F-D as $T \to 0$

$$e^{(\epsilon-\mu)/k_BT} \to \begin{cases} 0 & \epsilon < \mu(0) \\ \infty & \epsilon > \mu(0) \end{cases}$$

$$n(\epsilon) \to \begin{cases} 1 & \epsilon < \mu(0) \\ 0 & \epsilon > \mu(0) \end{cases}$$

all states are filled up to energy $\mu(0)$

$$\mu(0) = E_F$$

Identical Bosons or Fermions

$E = \frac{\hbar^2 k^2}{2m}$

$$N = (2s+1)\frac{V}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{e^{\left(\frac{\hbar^2k^2}{2m}-\mu\right)/k_BT} \pm 1}$$

$+$ F-D
$-$ B-E

$$E = (2s+1)\frac{V}{2\pi^2}\frac{\hbar^2}{2m}\int_0^\infty \frac{k^2\,dk}{e^{\left(\frac{\hbar^2k^2}{2m}-\mu\right)/k_BT} \pm 1}$$

Blackbody Spectrum

$E = h\nu = \hbar\omega \qquad k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$

photons two spin states $m \pm 1$
photons is not conserved

$\alpha = 0$

$$N_k = \frac{dk}{e^{\hbar kc/k_BT}}, \quad \text{Also} \quad \frac{2V}{2\pi^2}\frac{k^2\,dk^2}{e^{\hbar\omega/k_BT}-1}$$

$$\rho(\omega) = \frac{1}{\pi^2c^3}\frac{\hbar\omega^3}{e^{\hbar\omega/k_BT}-1}$$

$$\rho(\omega)|d\omega| = \bar{\rho}(\lambda)|d\lambda|$$

$$\bar{\rho}(\lambda) = \rho(\omega)\left|\frac{d\omega}{d\lambda}\right| = \frac{2\pi c}{\lambda^2}\frac{1}{\pi^2c^3}\frac{\hbar\left(\frac{2\pi c}{\lambda}\right)^3}{e^{\frac{\hbar 2\pi c}{\lambda k_BT}}-1}$$

$$= \frac{16\pi^2\hbar c}{\lambda^5}\frac{1}{e^{hc/\lambda k_BT}-1}$$

$$\left.\frac{d\bar{\rho}}{d\lambda}\right|_{\lambda_{max}} = 0 \qquad x_{max} = \frac{hc}{\lambda_{max}k_BT} \simeq 4.966$$

$$\lambda_{max} = \frac{2\pi\hbar c}{4.966\,k_B}\frac{1}{T} = 2.897\times10^{-3}\frac{m\cdot K}{T}$$ Wien displacement law