Zero Modes of Massive Fermions and Axion Strings

[arXiv:2310.01476]

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In collaboration with
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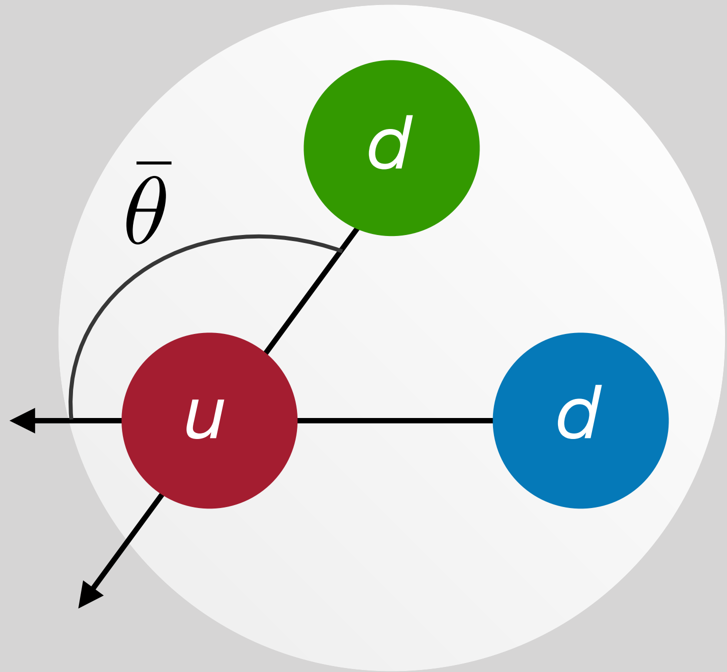
Axions in Particle Theory

Three primary reasons axions draw a lot of attention:

Strong CP Problem



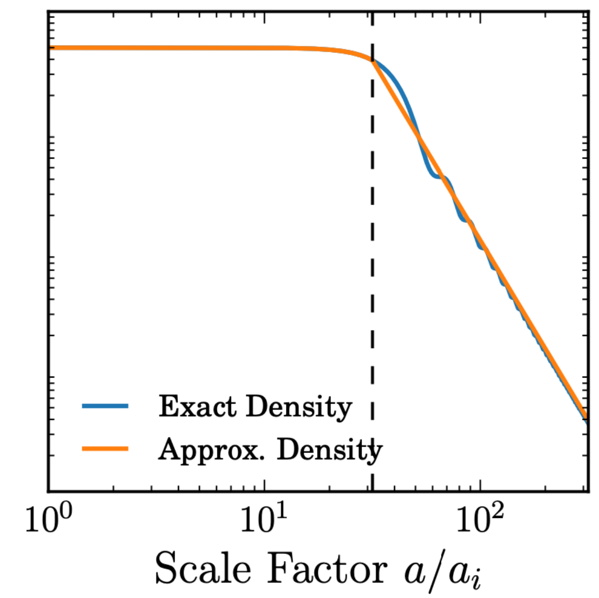
$$\mathcal{L} \supset \frac{\theta}{8\pi^2} G \wedge G$$

$$\bar{\theta} \equiv \theta + \arg\det m_q \lesssim 10^{-10}$$

Axions “promote” $\bar{\theta}$ to a dynamical field, whose potential is minimized at the origin.

(see e.g., Hook [1812.02669])

Dark Matter

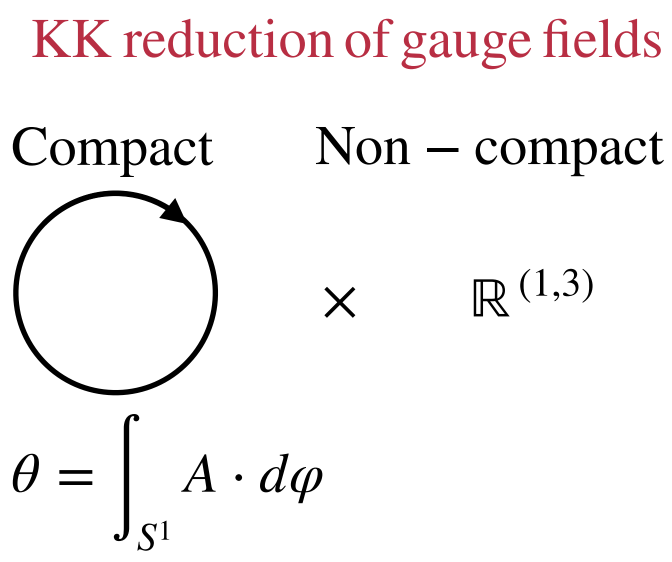


Generic misalignment of $\bar{\theta}$ leads to oscillating axion field that redshifts like matter:

$$\ddot{\theta} + 3H\dot{\theta} + m_a^2 \sin\theta \simeq 0$$

(Marsh, *Phys. Rept.* **643** (2016), 1-79)

String Theory



Axions (periodic scalar fields) arise from compactifications of higher-dimensional fields

$\Longrightarrow$ A low-energy playground for UV physics and quantum gravity

(Arvanitaki, et al., [0905.4720], Svrcek, Witten, [hep-ph/0605206])

Axions are Connected to Topology:

Fundamentally, axions should be thought of as scalar fields with a *periodic field range*: $\theta(x) \sim \theta(x) + 2\pi$

This makes them naturally connected to notions of topology, and makes *topological defects* particularly important for their phenomenology:

- Yang-Mills *instantons* give contributions to the axion potential
- *Monopoles* can also contribute to the potential via the Witten effect in Abelian gauge theory (Fan, Fraser, Reece, Stout, [2105.09950])
- Axion *Strings* and *Domain Walls* lead to important effects in Cosmological history

Axion Strings (or "Global Vortices")

Consider ϕ^4-theory with a complex scalar, that spontaneously breaks the global (PQ) symmetry:

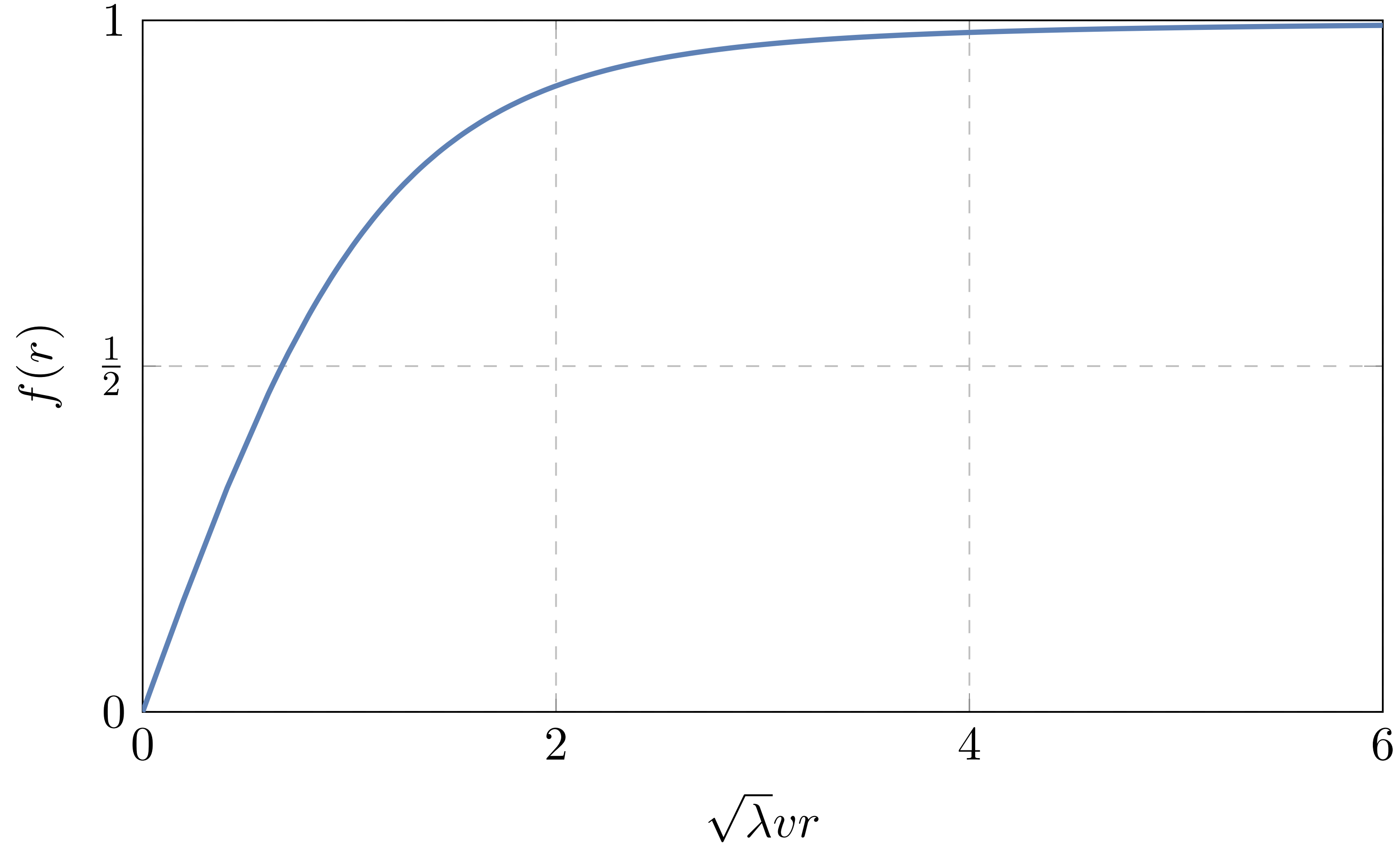
$$\mathcal{L} = \left|\partial_\mu \Phi\right|^2 - \lambda\left(|\Phi|^2 - v^2\right)^2$$

We identify the "axion" $\theta(x) = \arg \Phi(x)$

This theory admits "string" solutions:

$$\Phi_n(x) = f(r) e^{in\varphi}$$

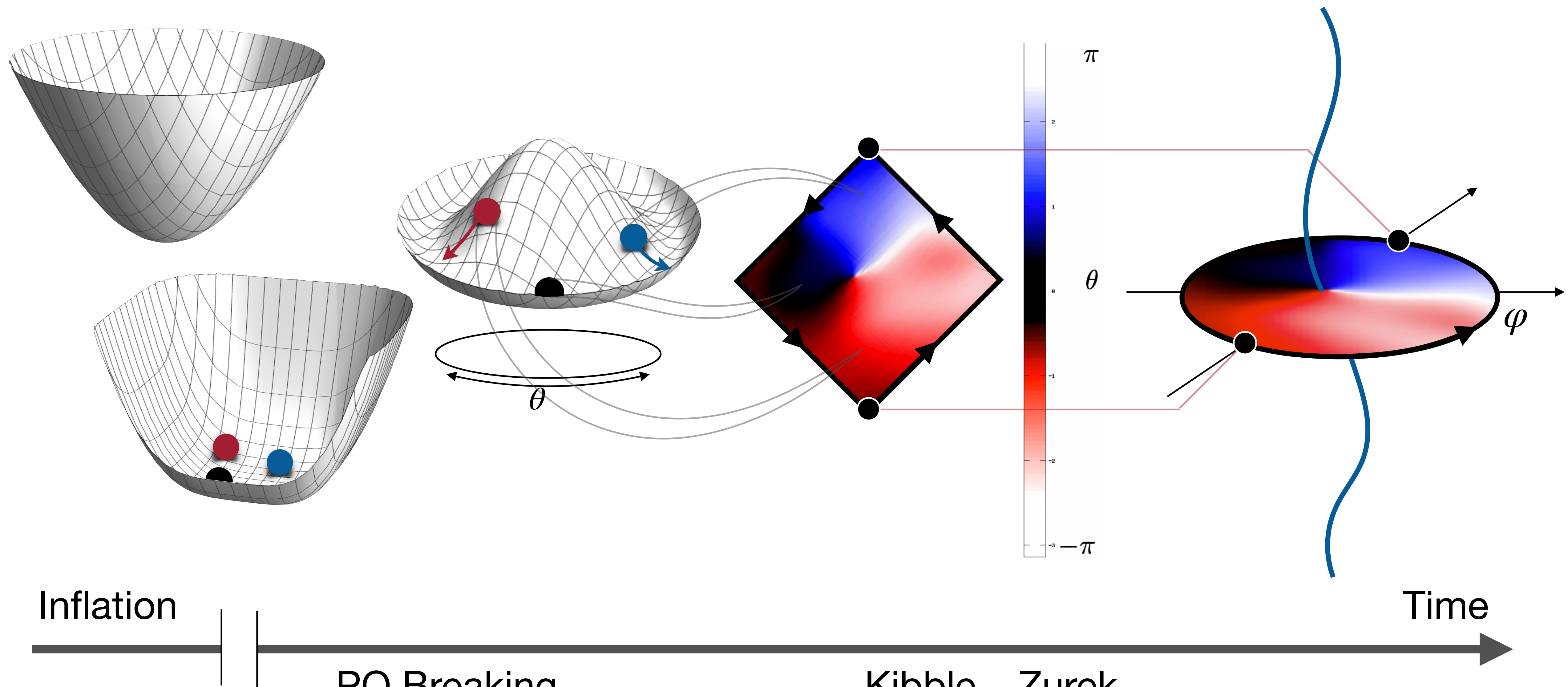
which are *topologically stable.*



Note: unlike gauge vortices, the tension is logarithmically divergent — can regulate by having a string with the opposite orientation at a distance $R \gg (\sqrt{\lambda} v)^{-1}$

Cosmological String Formation

[Kibble, *J. Phys. A.* **9** (1976), 1387-1398,
Zurek, *Nature* **317**, 505 (1985)]



Interactions of strings and formation of loops lead to a *scaling* behavior in Cosmology, which provides a stable energy density of strings

$\Longrightarrow$ influences the primordial axion abundance!

Axion Electrodynamics

The theory and phenomenology becomes more interesting if we add spinor electrodynamics:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi + \left|\partial_\mu\Phi\right|^2$$
$$+ y\bar{\psi}\left(\Phi_1 + i\gamma^5\Phi_2\right)\psi - \lambda\left(|\Phi|^2 - v^2\right)^2$$

Now we have a *chiral* $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry:

$$\Phi \mapsto e^{i\alpha}\Phi, \qquad \psi \mapsto e^{-i\gamma^5\alpha/2}\psi$$

At the quantum level, this symmetry suffers from a *chiral anomaly*:

$$\partial_\mu j^\mu_{\mathrm{PQ}} = \frac{1}{8\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

A chiral rotation can remove the axion coupling to fermions, but due to the anomaly, it reappears as a coupling to $F \wedge F$.

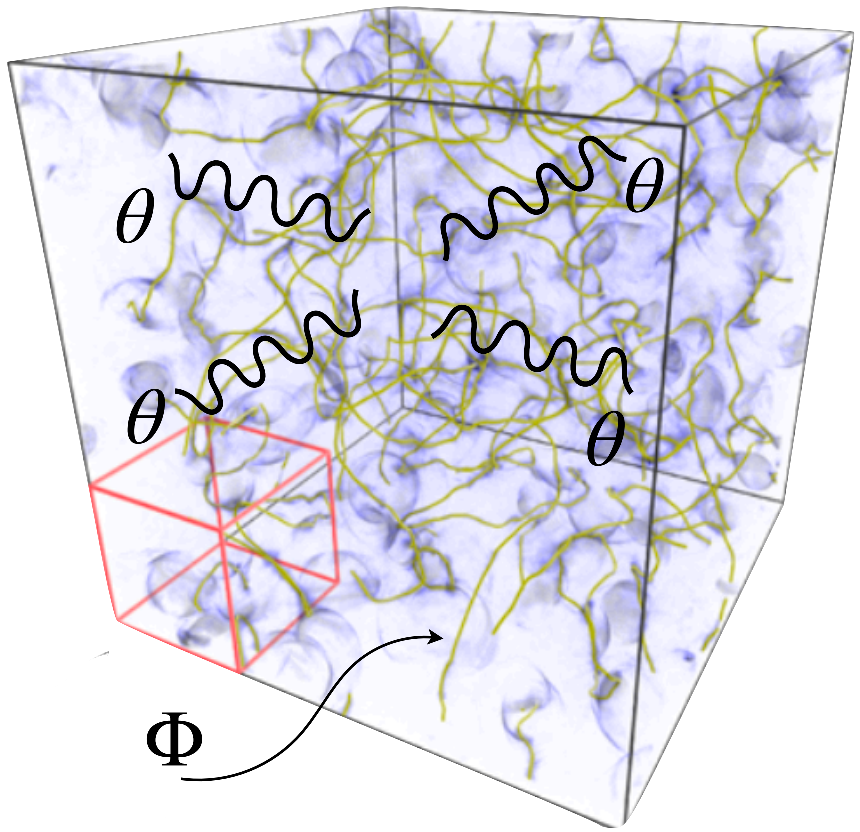
Electrodynamics and Axion Strings

In a *string* background, the fermions admit *chiral zero modes*, which travel along the string with a fixed helicity.

These fermions move at the speed of light, and carry charge with no dispersion — the string becomes a ***superconducting wire***.

There are a number of phenomenological consequences:

- Superconducting axion strings experience a drag force in plasma, enhancing the number of radiated axions.
  (See e.g., Buschmann, *Nat. Commu.* **13** (2022) 1, 1049, Fukuda et al., *JHEP* **06** (2021), 052.)



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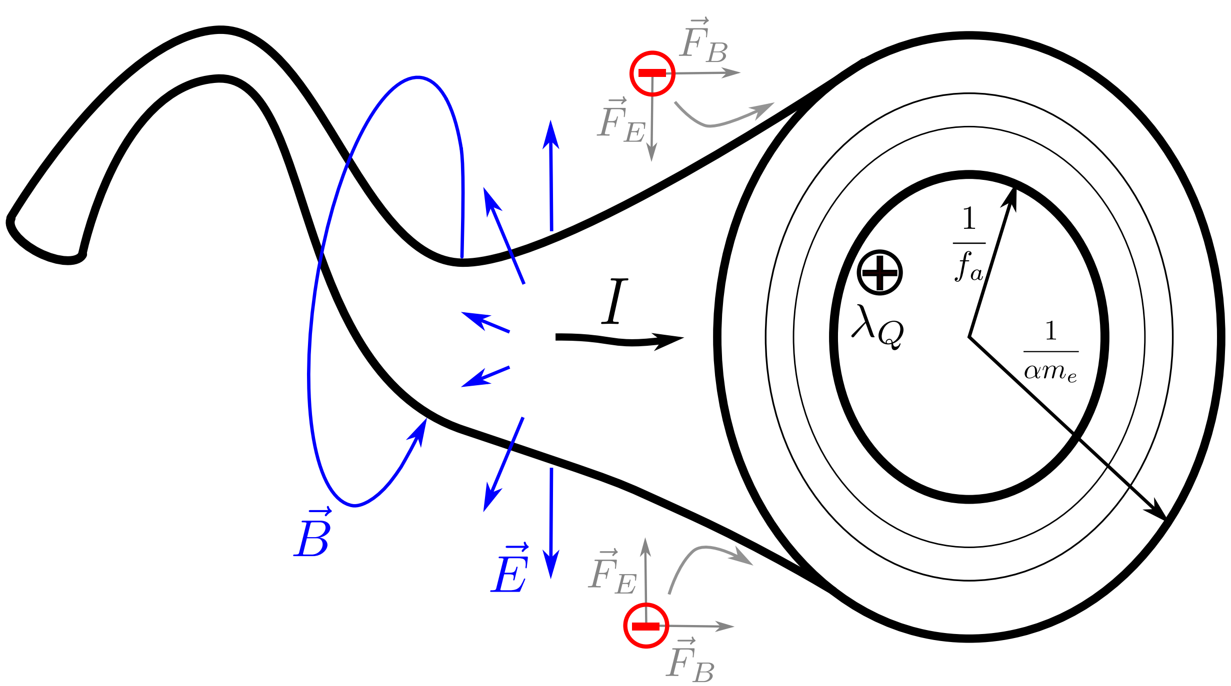
- In the presence of a primordial magnetic field, the strings can charge up and ionize the SM plasma, creating a "cosmological particle collider"
  (Agrawal et al., *JHEP* **01** (2022), 103.)



Existence of Chiral Zero Modes

The Fermion Equations of Motion:

$$\left(i\partial\!\!\!/ + y f(r) e^{i\gamma^5\varphi}\right)\psi = 0$$

Search for solutions with definite helicity, traveling along the $(-z)$-axis at the speed of light:

$$\psi_p(x) = \sqrt{\frac{p}{2}}\begin{pmatrix}1\\0\\0\\\pm i\end{pmatrix} e^{-ip(t+z)} F(r)\,,$$

with $F(r) \propto \exp\left(\pm y \int_0^r \mathrm{d}r' f(r')\right)$

For the mode to be normalizable, we must pick the minus sign…

$\Longrightarrow$ the string enforces the *chirality* of the zero modes!

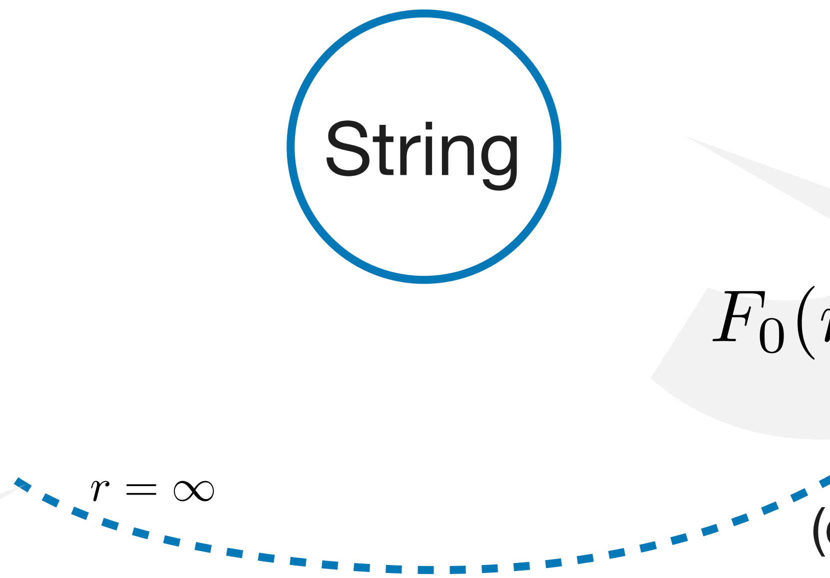
Existence of Chiral Zero Modes

The Fermion Equations of Motion:

Looking at the differential equation for the profile F more closely:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{f'(r)}{f(r)}\frac{\mathrm{d}}{\mathrm{d}r} - y^2 f^2(r)\right] F(r) = 0$$

We can examine the radial dependence in two asymptotic regimes:



$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \dots\right] F(r) = 0$$

$$F_0(r) \sim C_1 + C_2 r^2$$

regular

(due to smoothness of $f(r)$)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \mu^2 + \dots\right] F(r) = 0$$

$$F_\infty(r) \sim C_- e^{-\mu r} + C_+ e^{\mu r}$$

normalizable

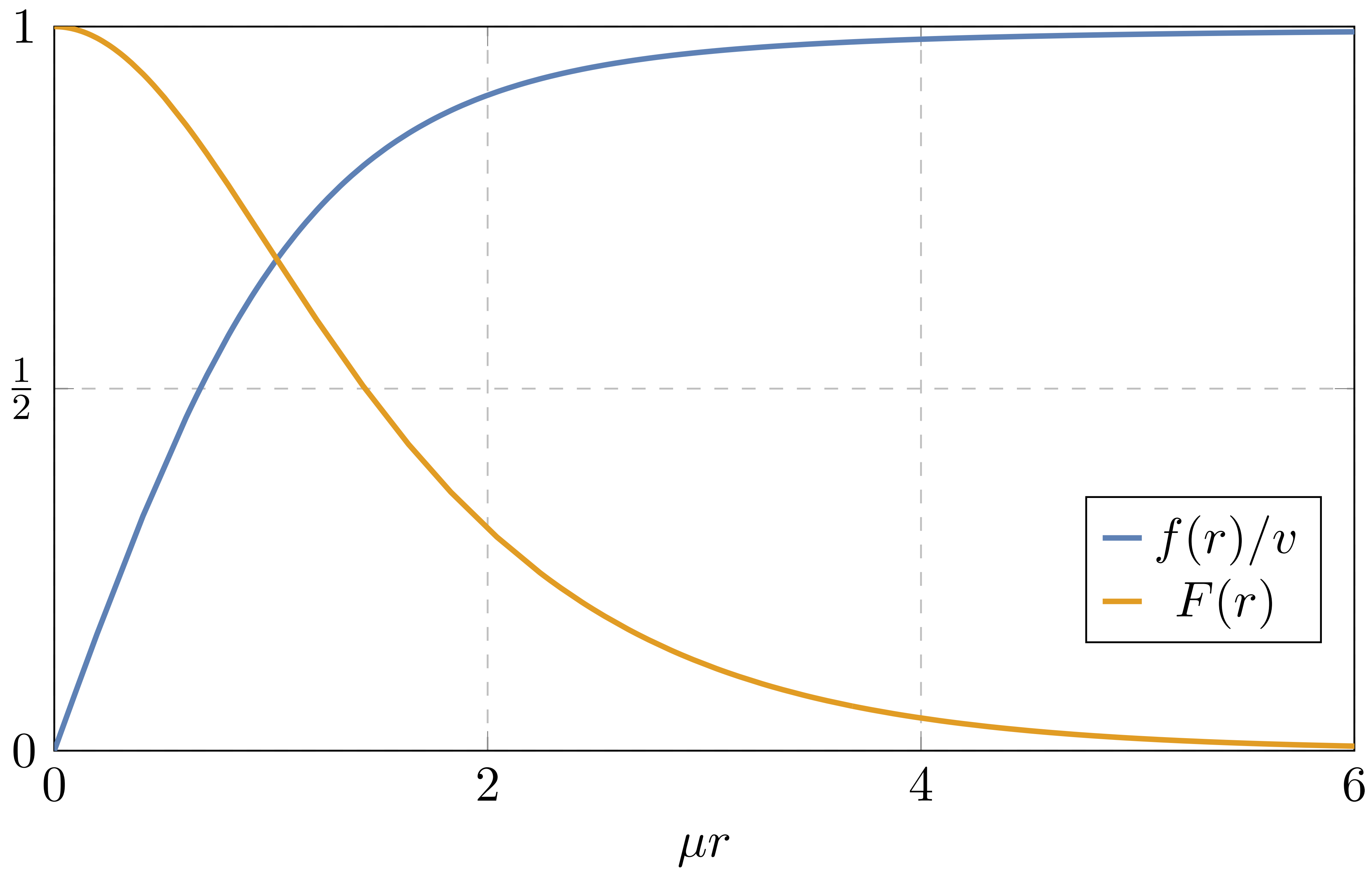
We can write the solution as

$$F(r) = C_- F_- = C_{-,1} F_1(r) + C_{-,2} F_2(r)$$

$\Longrightarrow$ *guaranteed* to exist, because both F_1, F_2 are regular at the origin.

Existence of Chiral Zero Modes

The Fermion Equations of Motion:

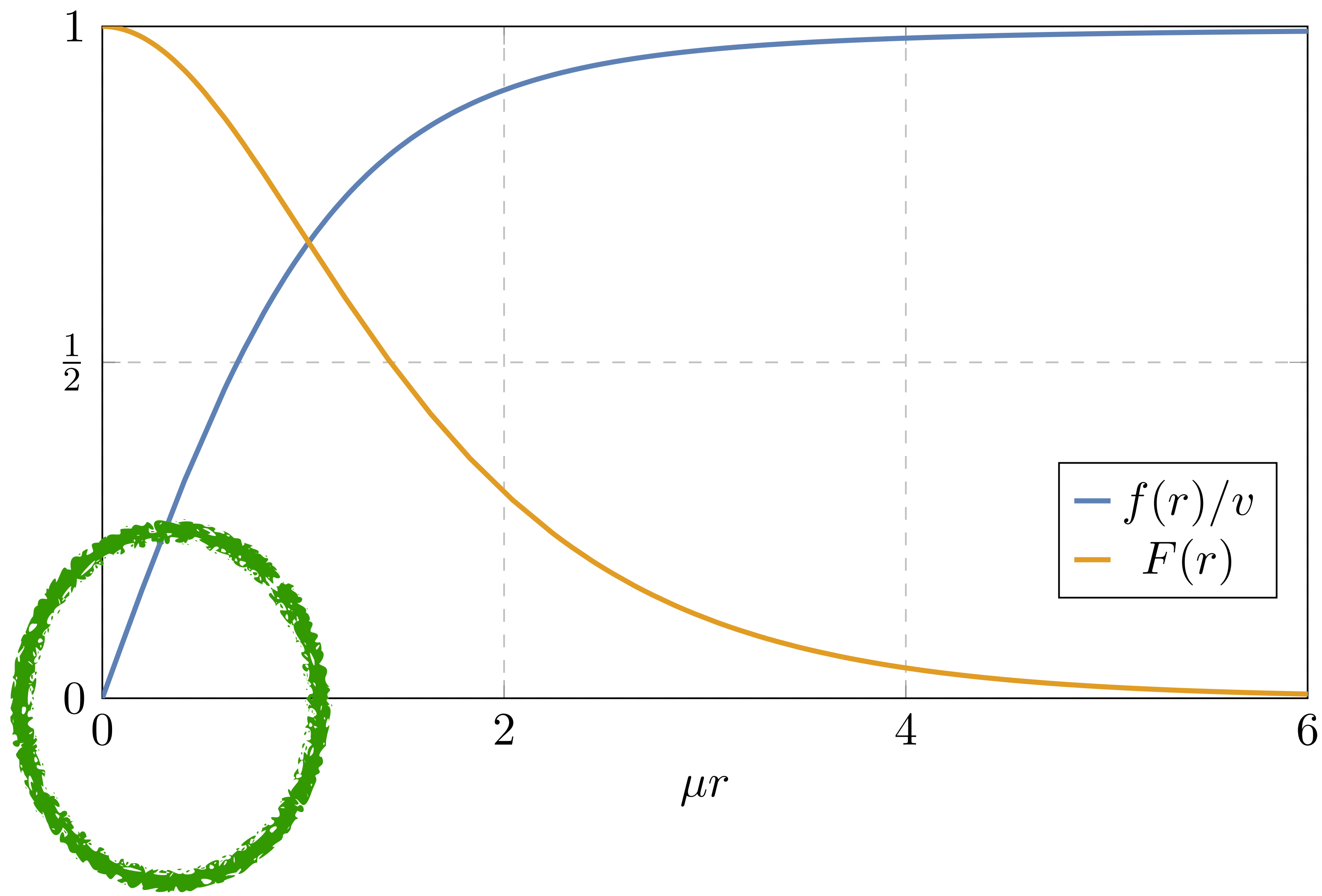


Existence of Chiral Zero Modes

A heuristic picture

Recall that, with an unbroken PQ symmetry, the fermion mass comes entirely from the chiral Yukawa coupling:

$$y\bar{\psi}(\Phi_1 + i\gamma^5\Phi_2)\psi \mapsto \underbrace{yv}_{\equiv\mu}\,\bar{\psi}\psi$$



The massless modes are localized precisely where the string profile, $f \to 0$

This leads to a heuristic understanding that the zero modes "exist" in part because of the vanishing vev at the string core — my goal today is to demonstrate that this is *wrong*!

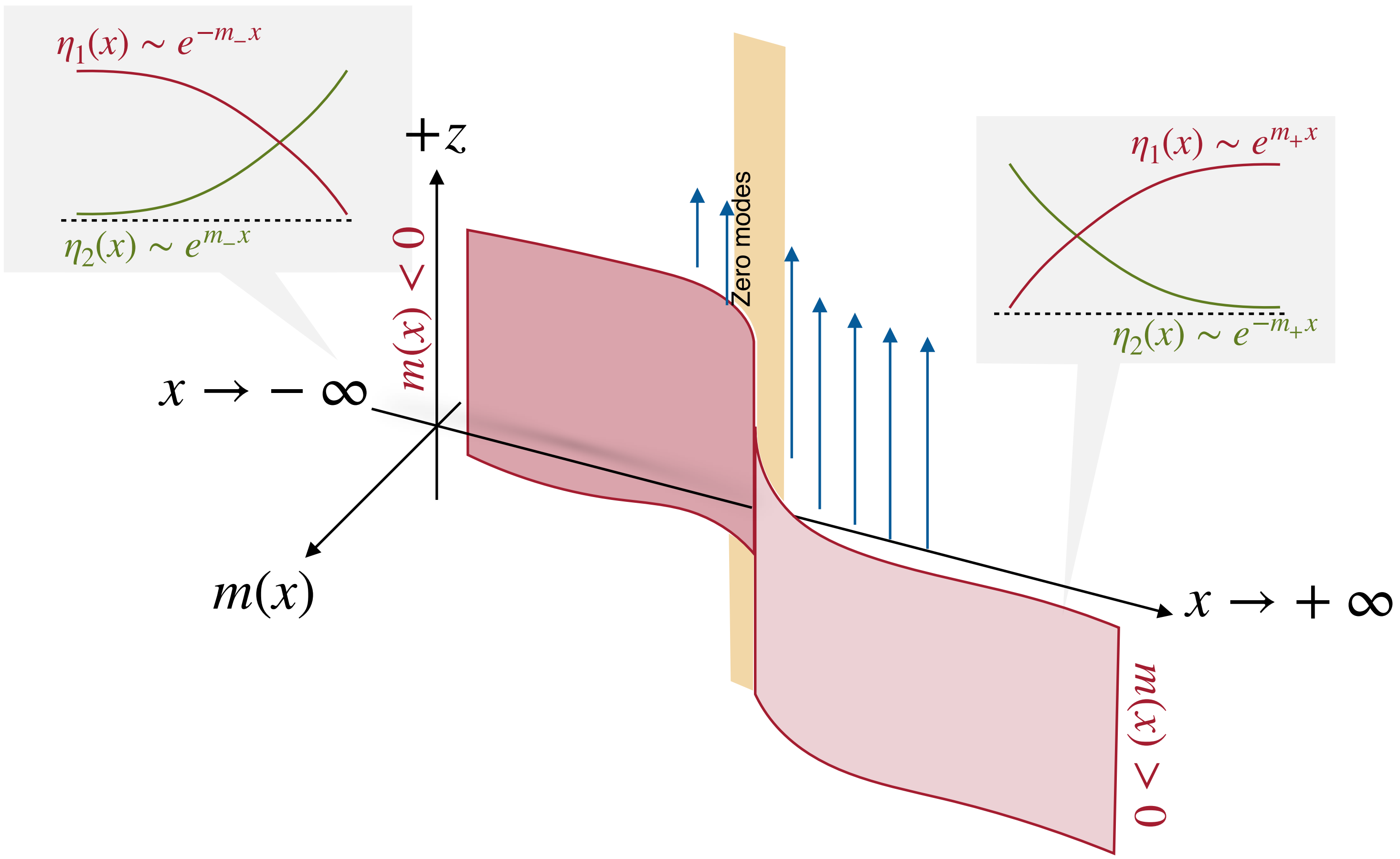
Analogy: Zero Modes on Domain Walls

Consider, by contrast, the Dirac equation on the (x, z)-plane, where the fermion mass is a monotonic function $m(x)$:

$$\mathcal{L} \supset \bar{\psi}\big(i\partial\!\!\!/ - m(x)\big)\psi$$

We can search for zero mode solutions (propagating along $+z$) of the form:

$$\psi(t,x,z) = e^{-ip(t-z)}\begin{pmatrix}\eta(x)\\0\end{pmatrix} \quad \Longrightarrow \quad \eta'' + \big(m'(x) - m^2(x)\big)\eta = 0$$



To match the asymptotic solutions and obtain a ***normalizable*** mode, we must have:

$$\lim_{x\to\pm\infty} m(x) = \pm m_\infty$$

with $m_\infty > 0$. In other words, $m(x)$ must vanish somewhere.

Anomaly Inflow for Axion Strings

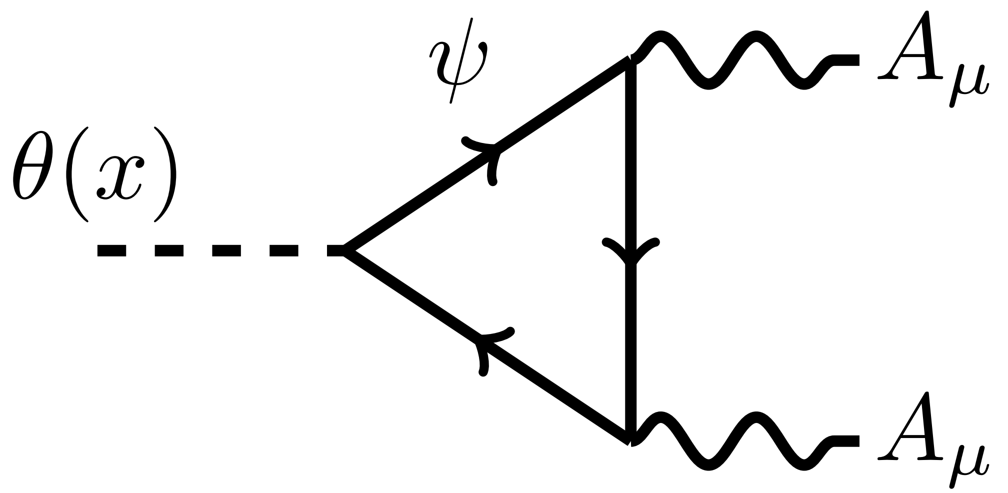
Callan & Harvey, *Nucl. Phys. B* **250** (1985), 427-436

We can also argue for the existence of zero modes on *topological* grounds.

Following Callan, Harvey, we integrate out the massive fermions (ignoring possible zero modes):

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{v^2}{2}\partial_\mu\theta\partial^\mu\theta + \frac{1}{16\pi^2}\theta F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

The θ-term leads to subtleties for gauge-invariance in the string background…



Anomaly Inflow for Axion Strings

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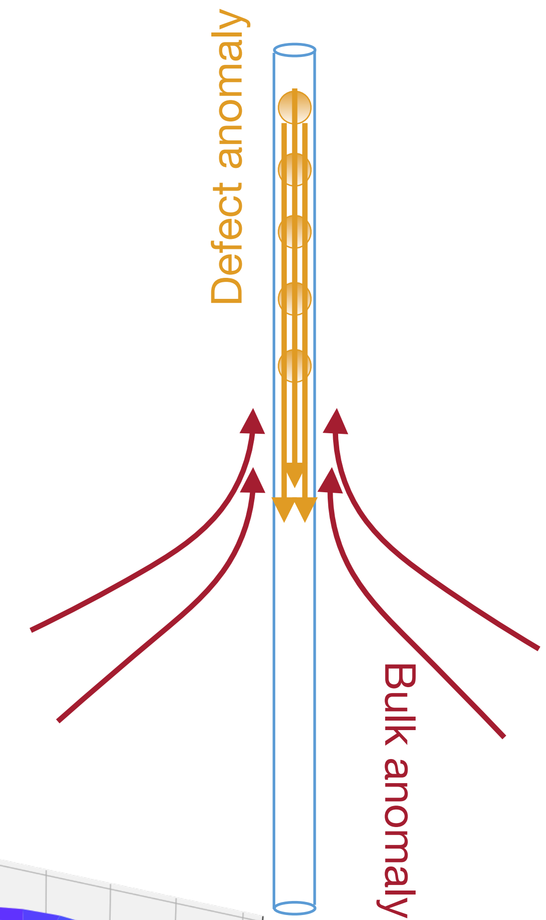


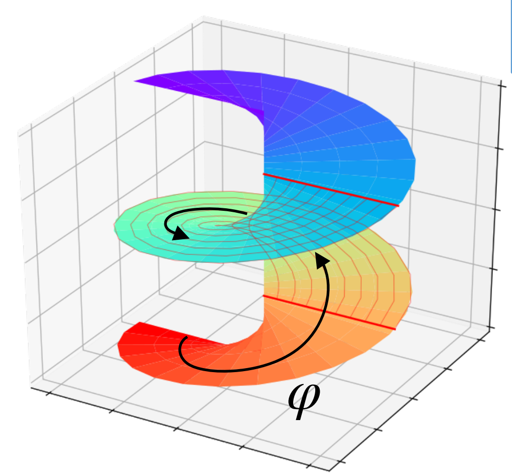
Under a gauge transformation, $\delta_\Lambda A_\mu = \partial_\mu \Lambda$

$$\delta_\Lambda S_{\mathrm{EFT}} = \delta_\Lambda \left[\frac{1}{8\pi^2} \int \theta F \wedge F \right]$$

$$= \delta_\Lambda \left[-\frac{1}{8\pi^2} \int \mathrm{d}\theta \wedge A \wedge F \right]$$

$$= -\frac{1}{8\pi^2} \int \mathrm{d}\theta \wedge \mathrm{d}(\Lambda F) = +\frac{1}{8\pi^2} \int \mathrm{d}^2\theta \wedge \Lambda F$$

In the string background, $\mathrm{d}^2\theta \neq 0$ — the axion has non-trivial winding number around the string — the EFT appears to have a gauge anomaly! But…



Massless, chiral fermions in 2d have precisely the equal and opposite gauge anomaly — they are *required* for consistency of the theory.

Beyond the Minimal Model:

There are a number of reasons to consider zero modes and axion strings beyond the simplest phenomenological model:

- In the DFSZ model for the QCD axion, there are several distinct string solutions, some of which are "electroweak strings" that allow for superconductivity, despite *none* of the scalar profiles vanishing at the core. (Abe et al., [2010.02834]

- In a SM plasma, fermions acquire a *Debye mass*. Can we study the breakdown of superconductivity at high temperatures?

- While the global symmetry is broken, the theory is perturbative for small Dirac masses — how are the underlying degrees of freedom affected?

$\Longrightarrow$ Simplest first step: study the Callan & Harvey Lagrangian with a small, explicit breaking term!

Zero Modes from Massive Fermions:

Perturbative analysis in small m

We now study solutions to the fermion equation of motion,

$$(i\partial\!\!\!/ - m + yf(r)e^{i\gamma^5\varphi})\psi = 0$$

Work perturbatively in m/μ, where $\mu \equiv yv$ is the Yukawa mass far from the string core. We'll look for solutions of the form:

$$\psi(x) = \sqrt{\frac{p}{2}}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix} F(r) + \delta\psi(r,\varphi)\right] e^{-ip(t+z)}$$

Expanding in Fourier modes, this reduces to a set of nonlinear equations for the components $(G, H) = (\delta\psi_{0,\ell=1}, -i\delta\psi_{3,\ell=1})$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r}\right)G(r) = yf(r)H(r)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\right)H(r) = yf(r)G(r) - mF(r)$$

Zero Modes from Massive Fermions:

Perturbative analysis in small m

As before, we apply an asymptotic analysis to $G(r)$ — the presence of the string leads to a regular singularity at $r = 0$, but we can match the two regions and construct a normalizable solution.

Given a solution for $G(r)$, we can solve for the profile function $H(r)$ via

$$H(r) = \frac{1}{yf(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r}\right)G(r)$$

While there are additional potential singularities here, we demonstrate that a normalizable solution is still generically allowed.

$\Longrightarrow$ for small PQ-breaking masses, the zero modes *still exist* — what about arbitrary values?

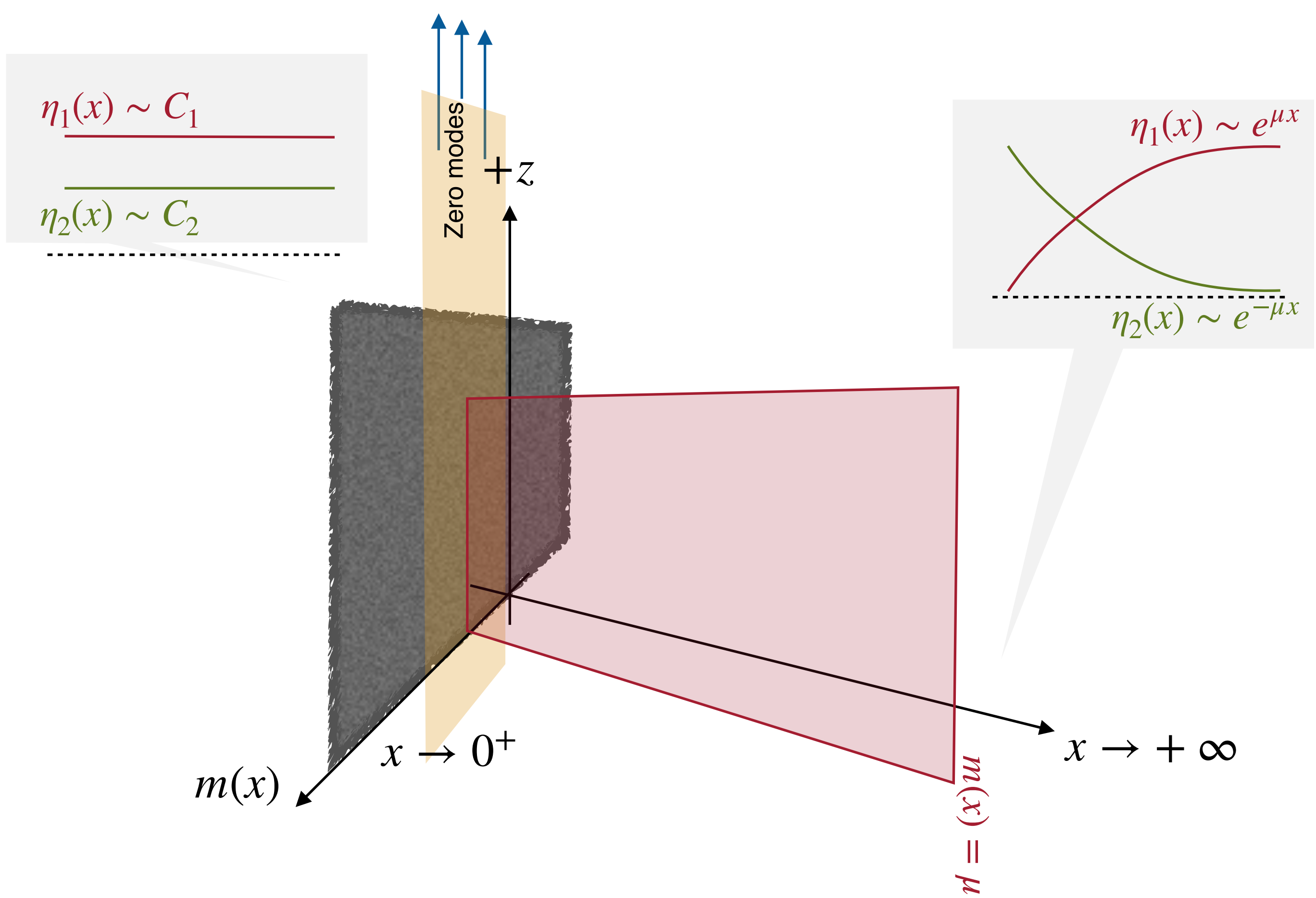
Analogy: Domain Walls on the Half-plane

As a useful analogy, consider a domain wall in the (x, z)-plane again, but now restricted to $x > 0$.

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

Again search for zero mode solutions (propagating along $+z$) of the form:

$$\psi(t, x, z) = e^{-ip(t-z)}\begin{pmatrix}\eta(x)\\ 0\end{pmatrix} \quad \Longrightarrow \quad \eta'' - \mu^2\eta = 0$$



Now, *both* solutions at $x \to 0$ are always regular, so the solution decaying at $x \to \infty$ is *always* normalizable, despite the mass never going to zero!

Anomaly Inflow Revisited

Let's revisit the anomaly inflow arguments used in the massless case, being careful about the chiral rotation in the presence of the mass.

In the string background, we have $\mathcal{L} \supset \bar{\psi}\big(i\not{D} - m + yf(r)e^{i\gamma^5\theta(x)}\big)\psi$

So we are interested in the path integral,

$$Z_\psi(\theta) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[i\int \mathrm{d}^4x\, \bar{\psi}\big(i\not{D} - M(r,\theta)e^{i\gamma^5\alpha(r,\theta)}\big)\psi\right]$$
$$= \frac{\det\left(i\not{D} - M(r,\theta)e^{i\gamma^5\alpha(r,\theta)}\right)}{\det\left(i\not{D} - \widetilde{M}\right)}$$

Where we've separated the modulus and argument of the complex mass:

$$M(r,\varphi) = \sqrt{\big(m - yf(r)\cos\varphi\big)^2 + y^2f^2(r)\sin^2\varphi}$$
$$\alpha(r,\varphi) = \arg\left(m - yf(r)e^{-i\varphi}\right) = i\log\left(\frac{m - yf(r)e^{-i\varphi}}{M(r\varphi)}\right)$$

Anomaly Inflow Revisited

Now we want to eliminate the chiral phase in the functional determinant, so we perform a spatially-dependent, chiral field redefinition:

$$\psi \mapsto e^{-i\gamma^5 \alpha(r,\theta)/2}\psi$$

$$Z_\psi(\theta) \mapsto \frac{\det\left(i\not{D} - M(r,\theta)\right)}{\det\left(i\not{D} - \widetilde{M}\right)} \exp\left[\frac{i}{8\pi^2}\int \alpha(r,\theta)\, F\wedge F\right]$$

result of the *chiral* anomaly

As in the massless case, under a *gauge* transformation:

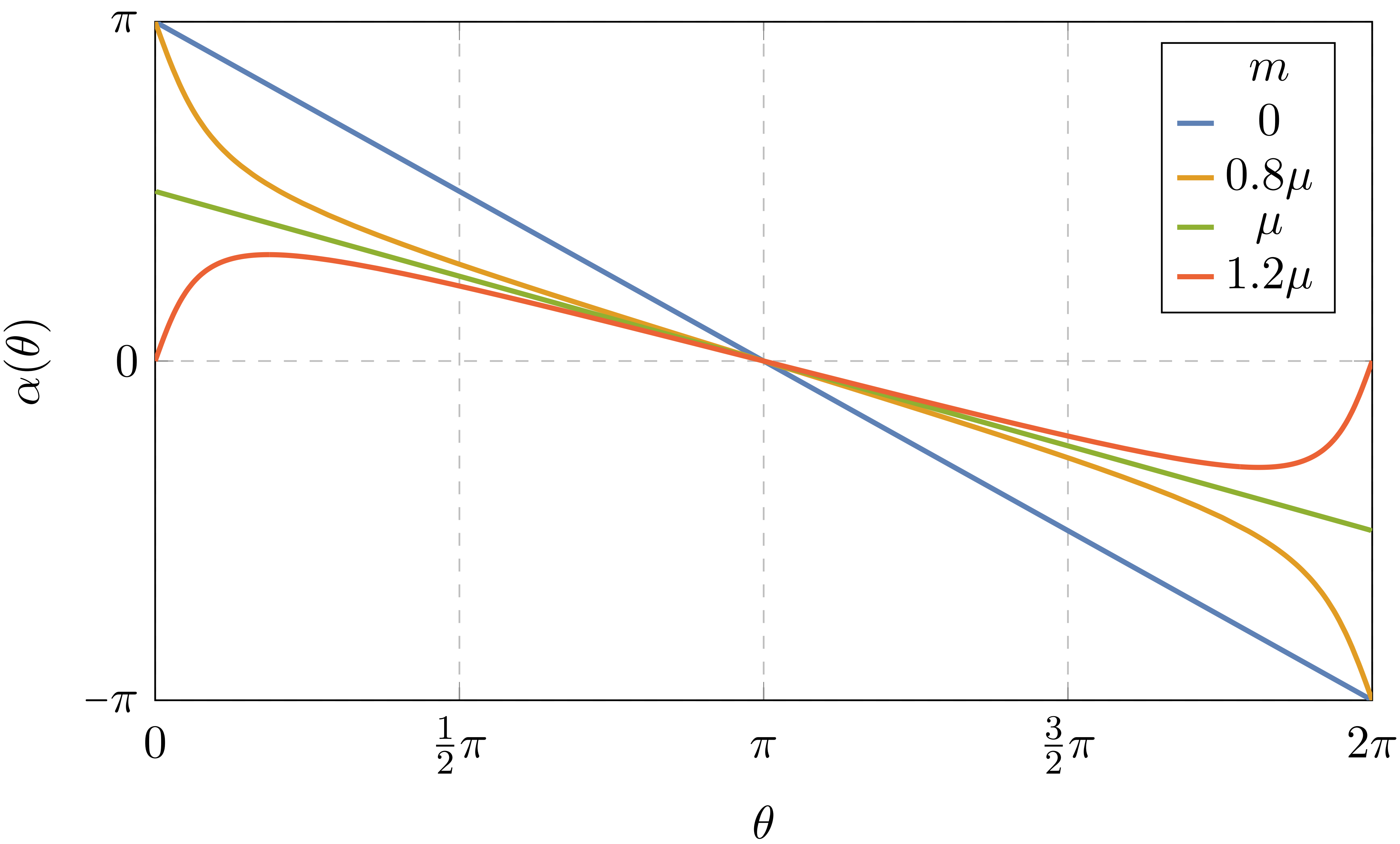
$$\delta_\Lambda Z_\psi(\theta) = +\frac{1}{8\pi^2}\int \mathrm{d}^2\alpha(r,\theta)\wedge(\Lambda F) + \ldots$$

Anomaly inflow depends on whether α is single- or multi-valued

Anomaly Inflow Revisited

Look at the phase at $r \to \infty$:

$$\lim_{r\to\infty} \alpha(r,\theta) = \arg\left(m - \mu e^{-i\theta}\right)$$



Solving for the Profiles Numerically

Clearly we should expect zero modes to exist for all $m < \mu$. As a final check, we find their profiles by solving the fermion equations of motion numerically.

Beginning with $\left(i\partial\!\!\!/ - m + yf(r)e^{i\gamma^5\varphi}\right)\psi = 0$, we seek solutions of the form:

$$\psi_p(x) = \sqrt{p}(1,0,0,\mp i)e^{-ip(t+z)}F(x,y)$$

We discretize this equation by mapping the Functions onto *Chebyshev* polynomials,

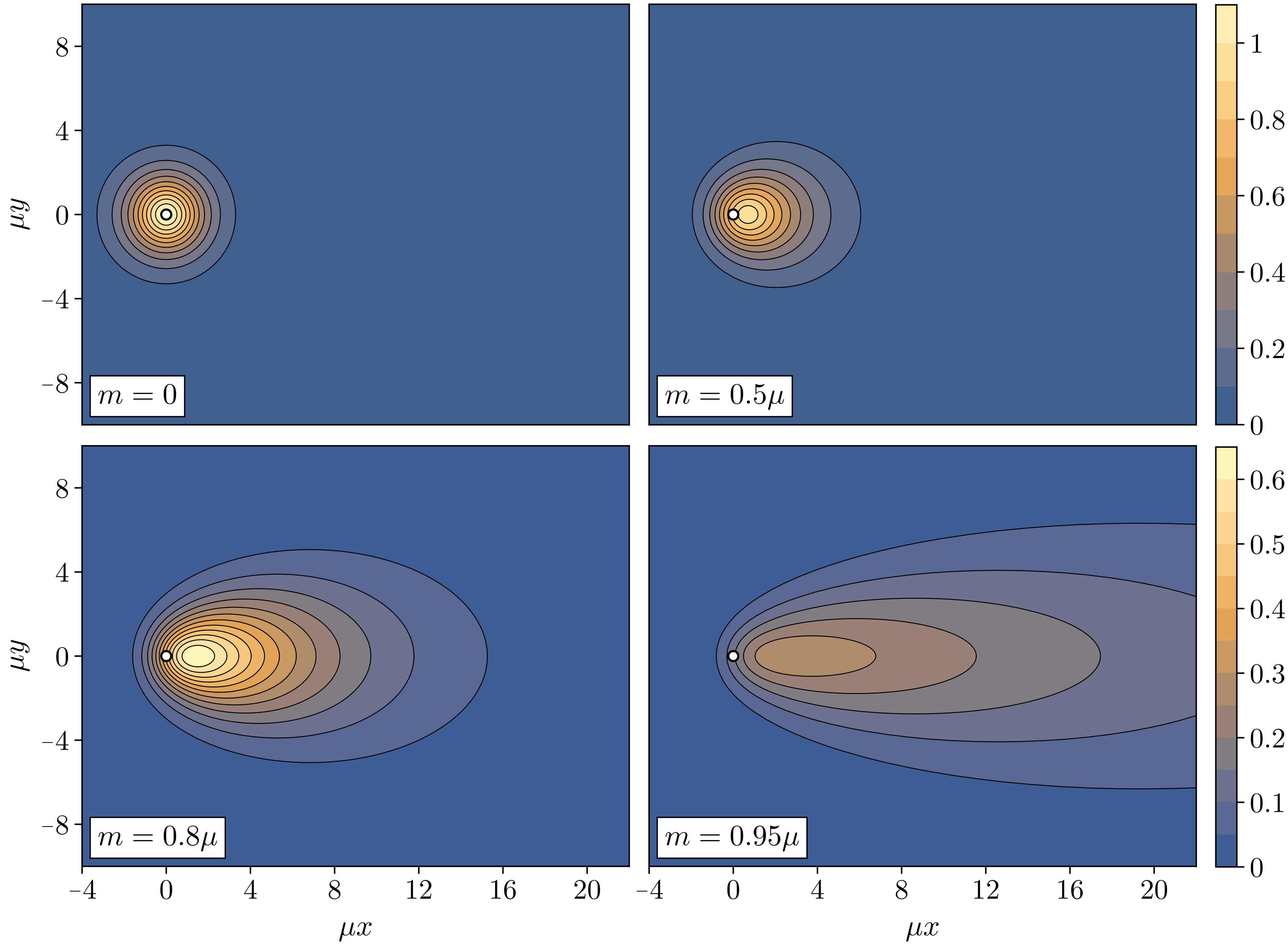
$$F(x,y) = \vec{\mathcal{C}} \cdot \vec{p}(\zeta_x, \zeta_y)$$

The "Chebyshev nodes", $\zeta_{x,y}(x,y)$ are suitable maps from the (x,y)-plane onto an interval, with the convenient property that $\partial_{x,y}F = \vec{\mathcal{M}}(p_i') \cdot \vec{\mathcal{C}}$

This reduces the problem to an algebraic matrix equation:

$$\mathcal{D}[F] = 0 \mapsto \mathcal{D}[\vec{\mathcal{M}}] \cdot \vec{\mathcal{C}} = 0$$

Zero Mode Profiles



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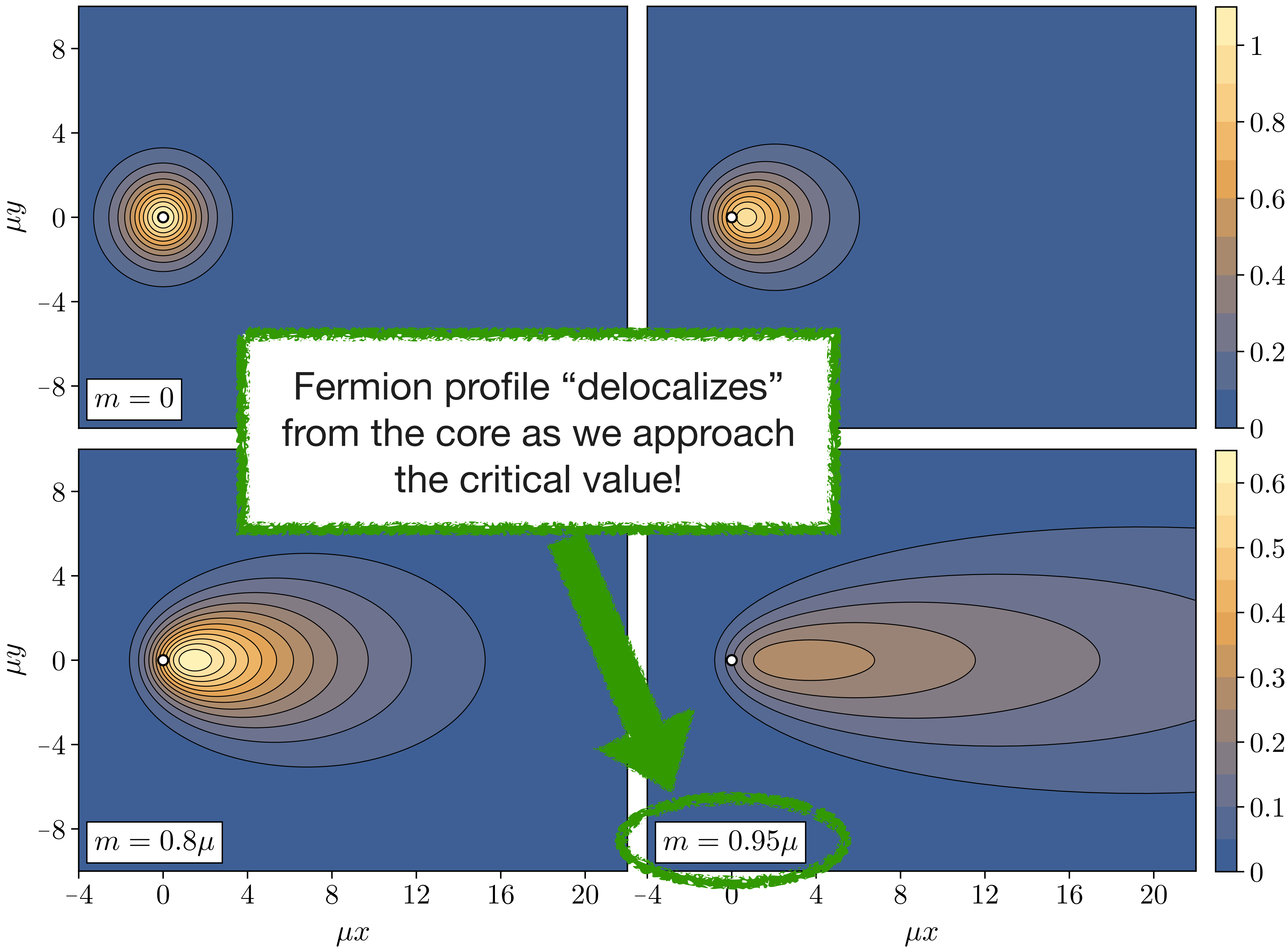


The Critical Case: $m = \mu$

We can study the critical case analytically. Far from the string, the "mass" term in the path integral approaches:

$$\lim_{r\to\infty} M(r,\varphi) = \sqrt{(m-\mu\cos\varphi)^2 + \mu^2\sin^2\varphi}$$

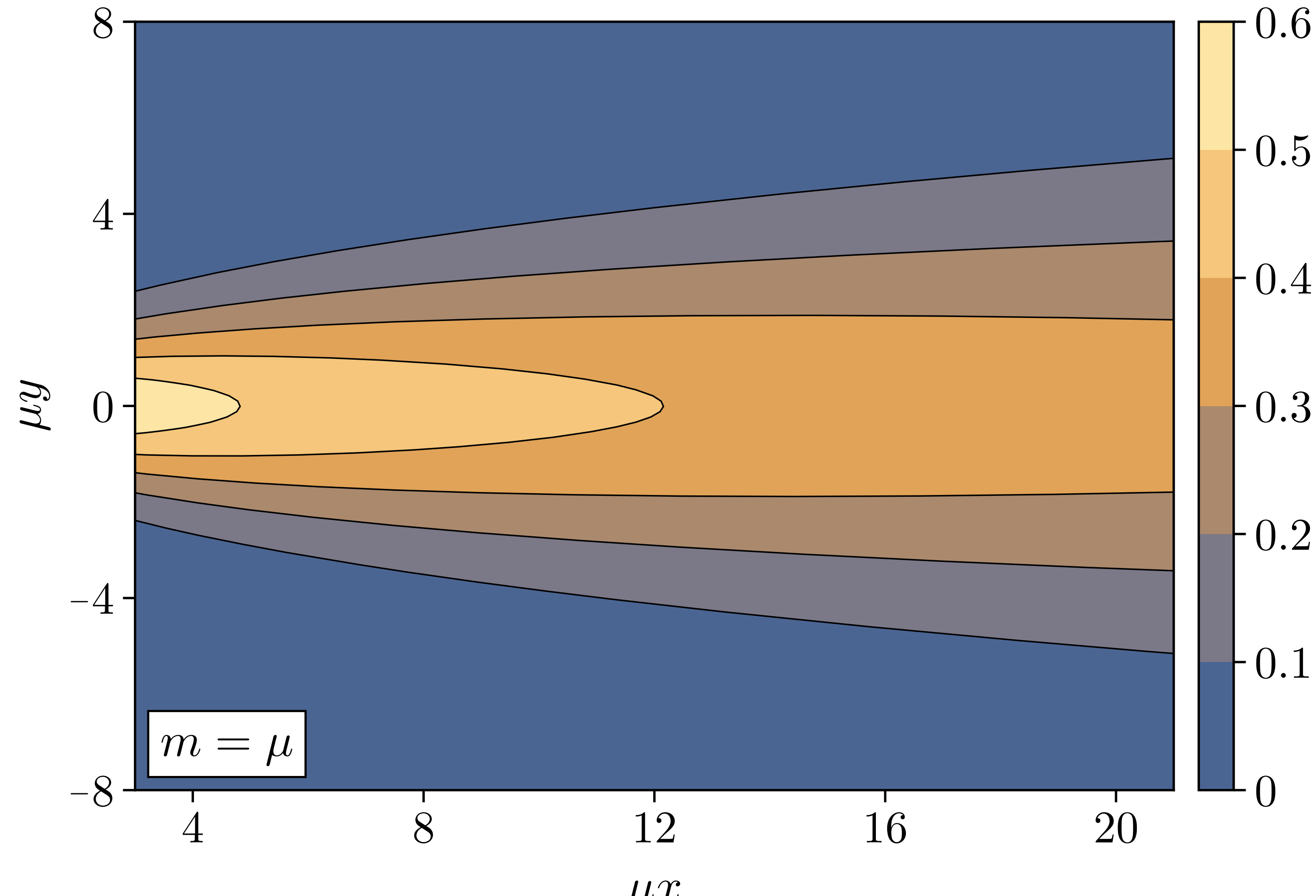
$\Longrightarrow$ Fermions can propagate freely along the z ***and*** x directions!

$$\psi(x) = \psi(x,y)e^{-i\omega t + ip_x x + ip_z z}$$

$$\psi(x,y) \approx \sqrt{\frac{\omega}{2}}\begin{pmatrix}\cos\alpha \\ \sin\alpha \\ i\sin\alpha \\ -i\cos\alpha\end{pmatrix} F(x,y)$$

The e.o.m can be solved for $x \gg |y|$ to find:

$$F(x,y) \sim \left(\frac{\mu}{\pi x}\right)^{\frac{1}{4}} \exp\left(-\frac{\mu y^2}{2x}\right)$$

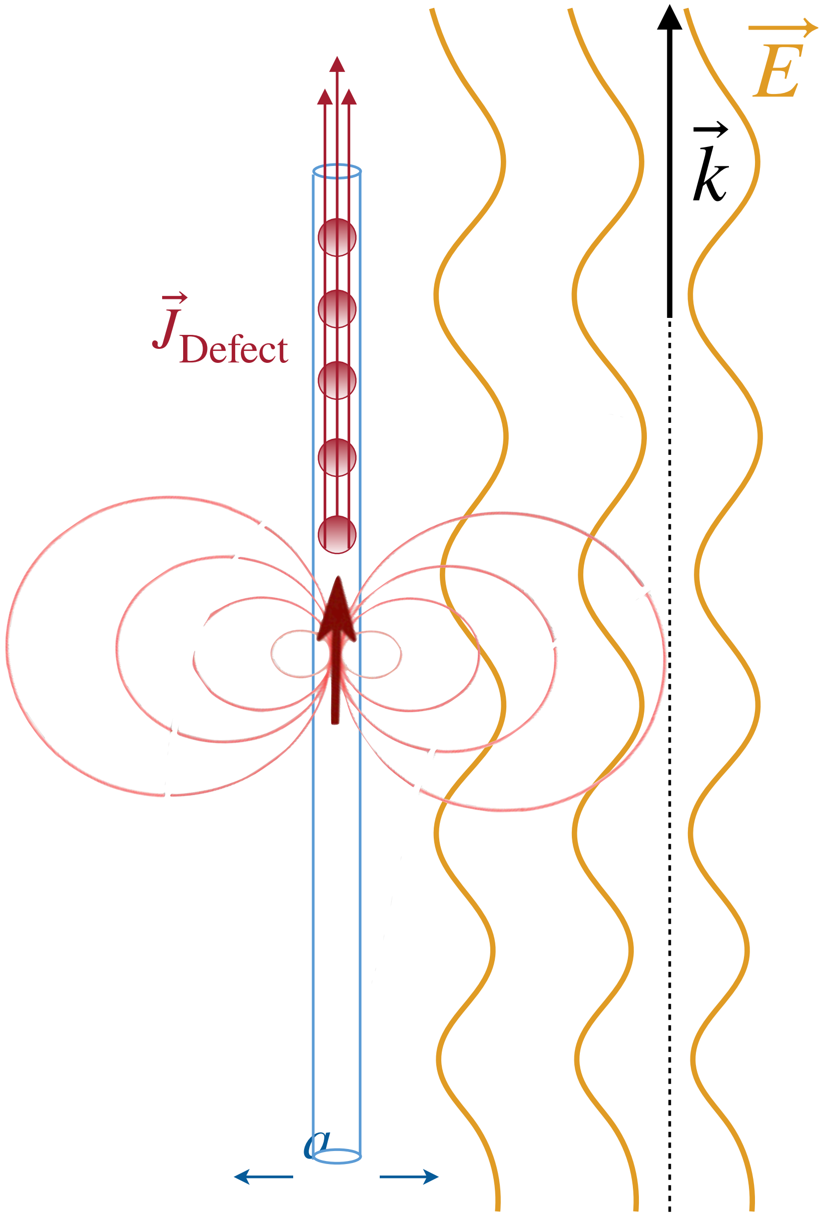


The Worldsheet Effective Theory

In the four-dimensional theory, the zero modes interact with gauge fields via $\mathcal{L} \supset j^\mu A_\mu$

$$j^\mu = \bar{\psi}\gamma^\mu\psi \propto |\mathcal{F}(r,\varphi)|^2\,(\bar{\chi}\gamma^a\chi)(t,z)A_\mu(x)$$

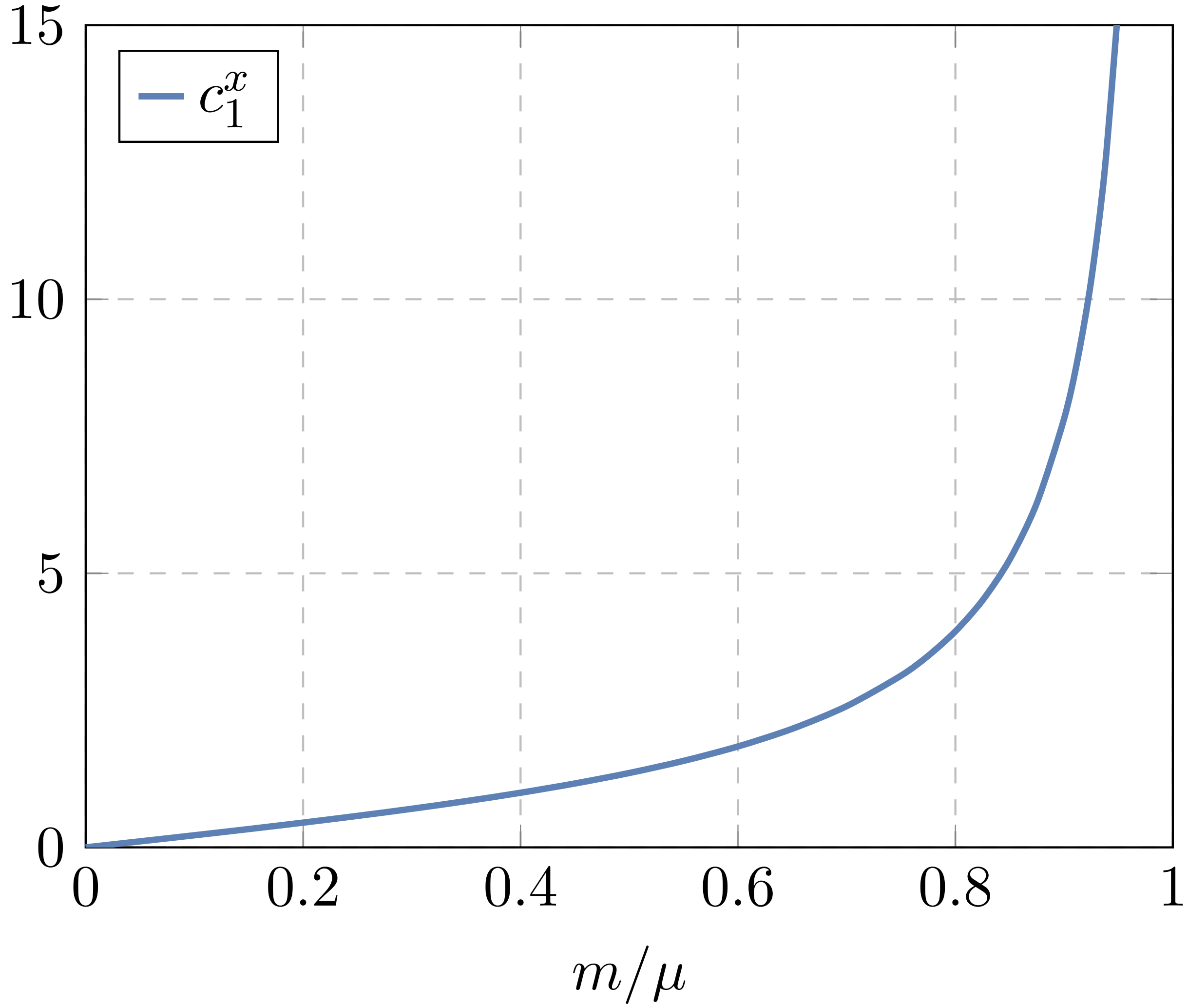
Fermion current localized to the string core

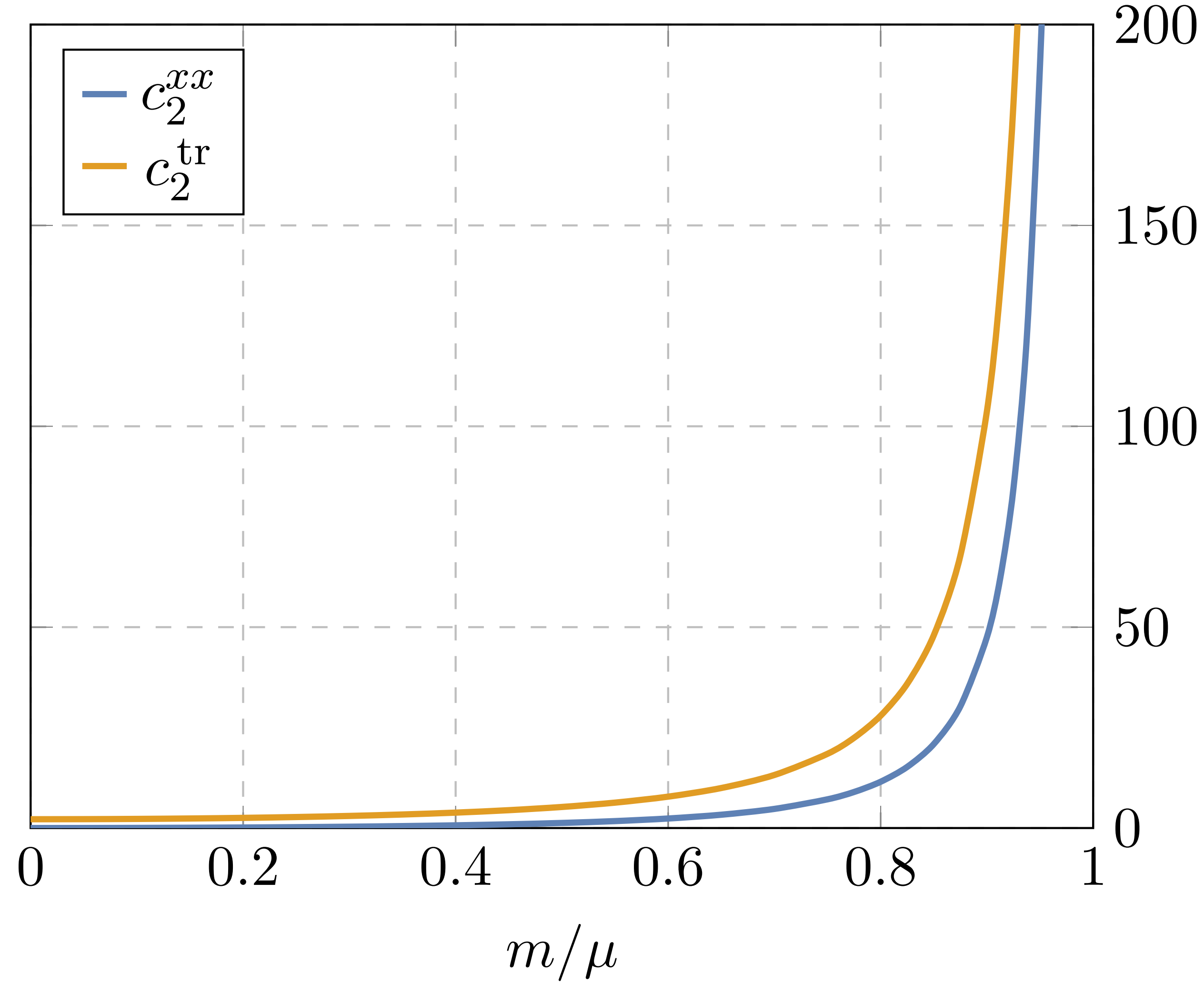


In the long-wavelength limit, $ka \ll 1$, we can Taylor expand the gauge field, and organize the effective theory in a multi-polar structure interacting with the 2D current, $j^a = \bar{\chi}\gamma^a\chi$

$$S_{\text{EM}} \supset \int \mathrm{d}^2\sigma \left[\frac{c_1^x}{\mu} j^a F_{ia} + \frac{1}{4}\frac{c_2^{ii}}{\mu^2} j^a \partial_{(i}F_{j)a} - \frac{1}{4}\frac{c_2^{\text{tr}}}{\mu^2} j^a \partial^b F_{ba} + \ldots\right]$$

The Worldsheet Effective Theory





$$\frac{c_1^x}{\mu} \propto \int \mathrm{d}^2 r\, x |F(r,\varphi)|^2$$

$$\frac{c_2^{ii}}{\mu^2} \propto \frac{1}{2} \int \mathrm{d}^2 r\, x^i x^i |F(r,\varphi)|^2$$

$$\frac{c_2^{\mathrm{tr}}}{\mu^2} \propto \int \mathrm{d}^2 r\, r^2 |F(r,\varphi)|^2$$

The Wilson coefficients blow up as $m \to \mu$!

Conclusions

- Superconductivity of axion strings persists even with an explicit, PQ-breaking mass term, up to a critical value of the mass.
- At the critical value, fermion zero modes delocalize from the string, in the direction of the PQ-breaking domain wall.
- How does this story look in less minimal scenarios?
  - Can this “transition” be realized at finite temperature, via the Debye mass of the fermions?
  - Does similar behavior arise in models with electroweak axion strings?
  - Are there important / observable consequences in Cosmology?

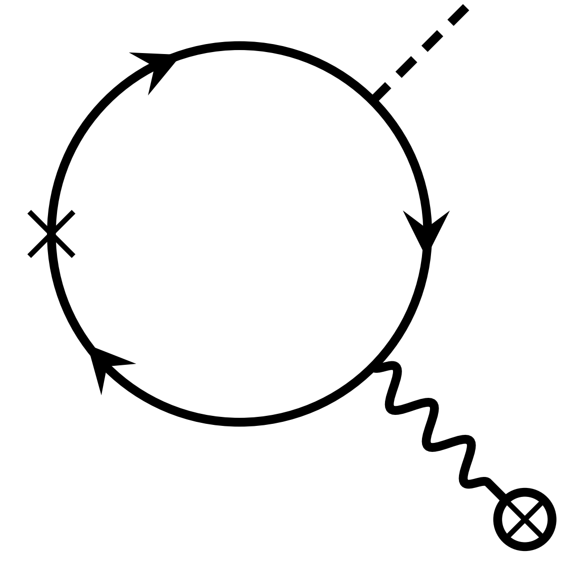
Thank you for your attention!

Backup

Backup: The Goldstone–Wilczek Current

Another way of deriving the effective action and seeing anomaly inflow is to derive the *Goldstone–Wilczek* current:

$$\langle J^\mu(z)\rangle = \frac{e^2}{8\pi}\frac{\mu}{\mu - m}\epsilon^{\mu\nu\kappa\lambda}\partial_\nu\theta(x_0)F_{\kappa\lambda}$$



This current arises from an *effective action*:

$$S_{\text{eff}} = \frac{1}{8\pi^2}\frac{\mu}{\mu - m}\int \mathrm{d}\theta \wedge A \wedge F$$

For $m = 0$, this is the usual effective action, and leads to anomaly inflow due to the string winding. For nonzero m, however, this effective action appears to violate the quantization condition for the axion-gauge field coupling!

$\Longrightarrow$ there is an ambiguity in what we call the "axion" — instead, consider α, and note that for small field values,

$$\mathrm{d}\alpha(\theta) = \left[\frac{\mu}{\mu - m} + \mathcal{O}(\theta^2)\right]\mathrm{d}\theta$$

This is the correct prefactor for the "charge-quantized" field, α! (Somewhat analogous to the violation of coupling quantization due to π^0 mixing for the QCD axion)

The PQ "Domain Wall"

Some of these results can be demystified by considering the scalar potential more carefully.

Integrating out the massive fermions leads to an effective potential, with the leading Coleman-Weinberg estimate:

$$V \propto M^4(r,\varphi)\log\frac{M(r,\varphi)}{\mu}$$

This potential is minimized along the x-axis ($\varphi = 0$) and leads to a domain wall solution for the scalar field whose phase is the axion.

We've assumed the CW potential term is "fine-tuned" away, but it's clear that some remnant of it survives in the zero mode profiles.