Unexciting Classical Backgrounds

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Motivation

Quantum effects in classical backgrounds —

★ Hawking radiation during gravitational collapse.

★ Schwinger pair creation.

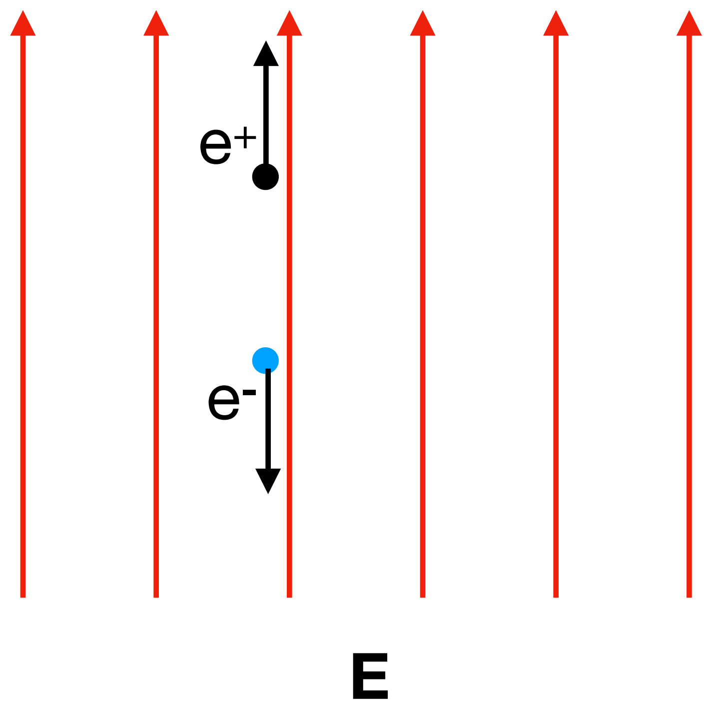
★Coupling of classical inflaton to other quantum fields.

★ etc.

Particle production implies a quantum instability of the background.
(Black holes evaporate, capacitors discharge, universe reheats, etc..)

Are there backgrounds that are protected from such dissipation?

Schwinger pair production in QED



Schwinger, 1951

$$\dot{n} \propto e^2 E^2 \exp(-\pi m_e^2 / eE)$$

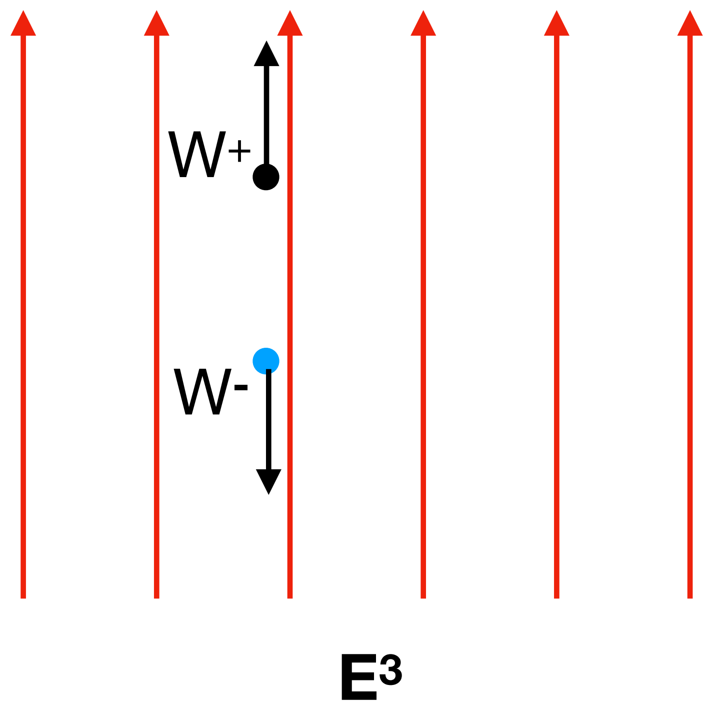
Usually thought of as a tunneling process.

Use temporal gauge. Then the background gauge field is time-dependent.

$$\mathbf{A}(t) = \mathbf{E}\, t$$

Schwinger pair production in non-Abelian gauge theory

Carlos Cardona & TV



Yildiz & Cox
Ambjorn & Hughes
Cooper & Nayak
Nayak & Nieuwenhuizen
Ragsdale & Singleton
Cooper, Dawson & Mihaila
Matinyan & Savvidy
Brown & Weisberger
Nair & Yelnikov

Are there electric field configurations, *e.g.* flux tubes, that don't lead to Schwinger pair production?

Example: massless QED$_{1+1}$

Yi-Zen Chu & TV

Gold, McGady, Patil & Vardanyan

$$L = \bar{\psi}\gamma^\mu(i\partial_\mu + eA_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

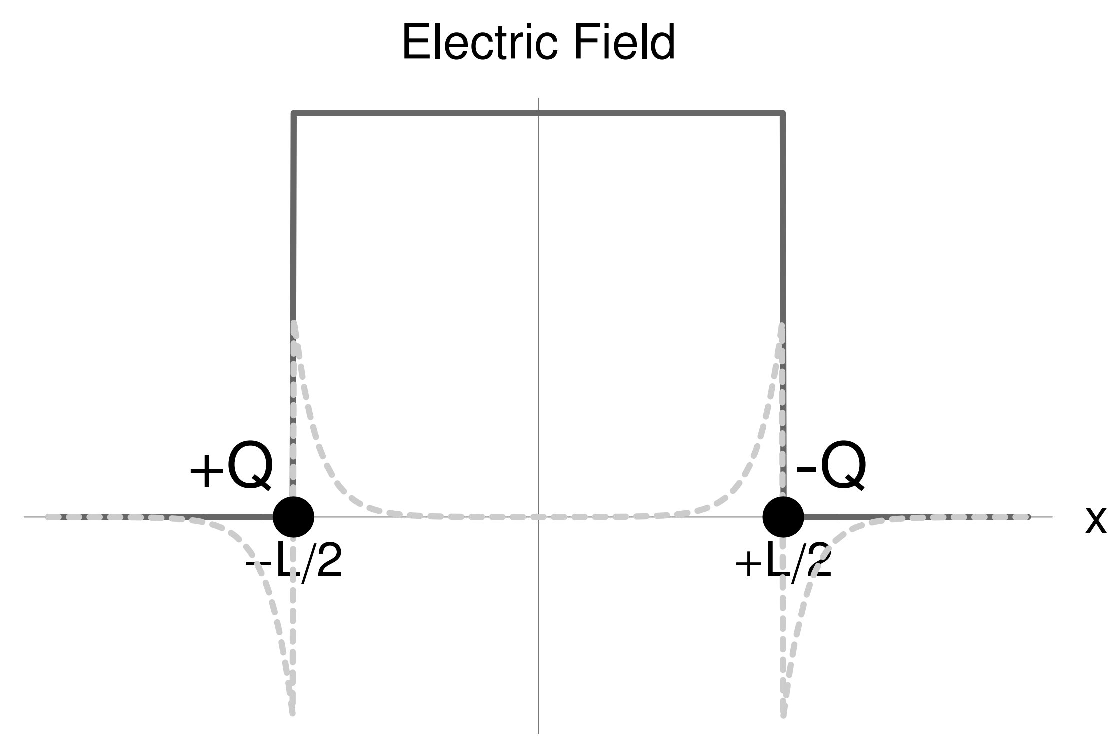
Bosonize: $:\bar{\psi}\gamma^\mu\psi: \Longleftrightarrow \epsilon^{\mu\nu}\partial_\nu\phi$

$$L' = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{g}{2}\phi\epsilon^{\mu\nu}F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Unexciting electric field:

$$E = F_{01} = Q[\Theta(x+L/2) - \Theta(x-L/2)] + g[f(x+L/2) - f(x-L/2)]$$

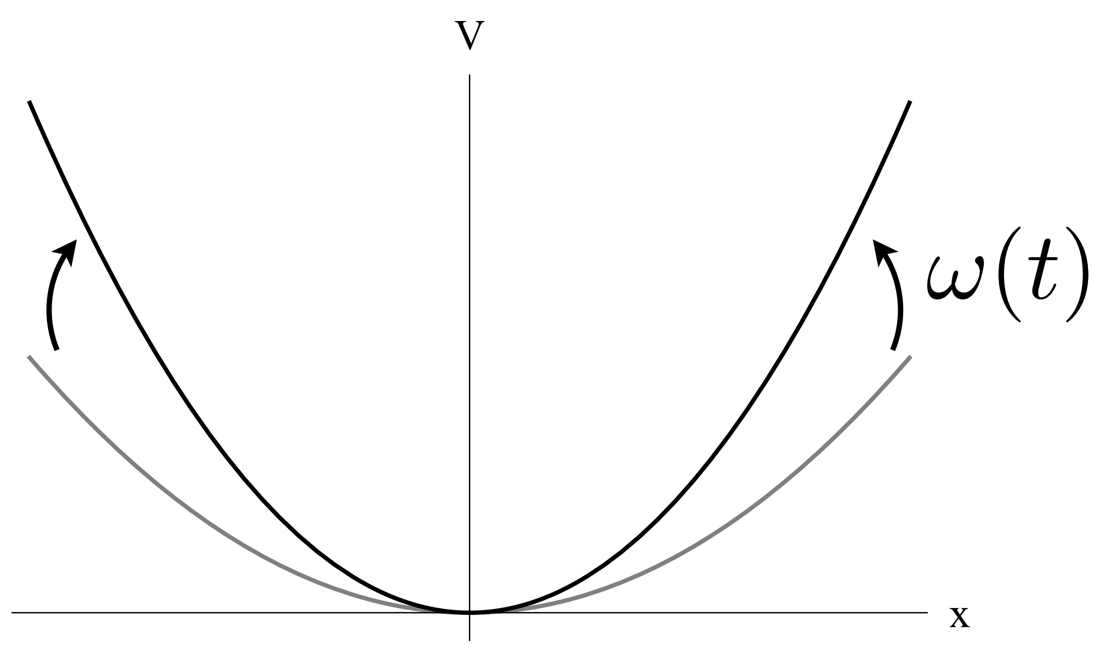
$$f(x) = -\frac{Q}{2g}\mathrm{sgn}(x)[1 - e^{-g|x|}]$$



Q's are **fixed, external** charges

Simple Harmonic Oscillator

Are there time-dependent frequencies that don't excite the SHO?



Lewis
Lewis & Riesenfeld
Parker
Popov & Perelomov
…

TV & Zahariade

ω(t) is assumed to be **externally driven.**

Work with ladder operators:

$$\hat{a} = \frac{\hat{p} - im\omega(t)\hat{x}}{\sqrt{2m\omega(t)}}, \quad \hat{a}^\dagger = \frac{\hat{p} + im\omega(t)\hat{x}}{\sqrt{2m\omega(t)}}$$

Classical-Quantum Correspondence for the SHO

Heisenberg equations: $$\frac{d\hat{a}}{dt} = -i[\hat{a}, H] + \frac{\partial \hat{a}}{\partial t}$$ $$[\hat{a}, \hat{a}^\dagger] = 1$$

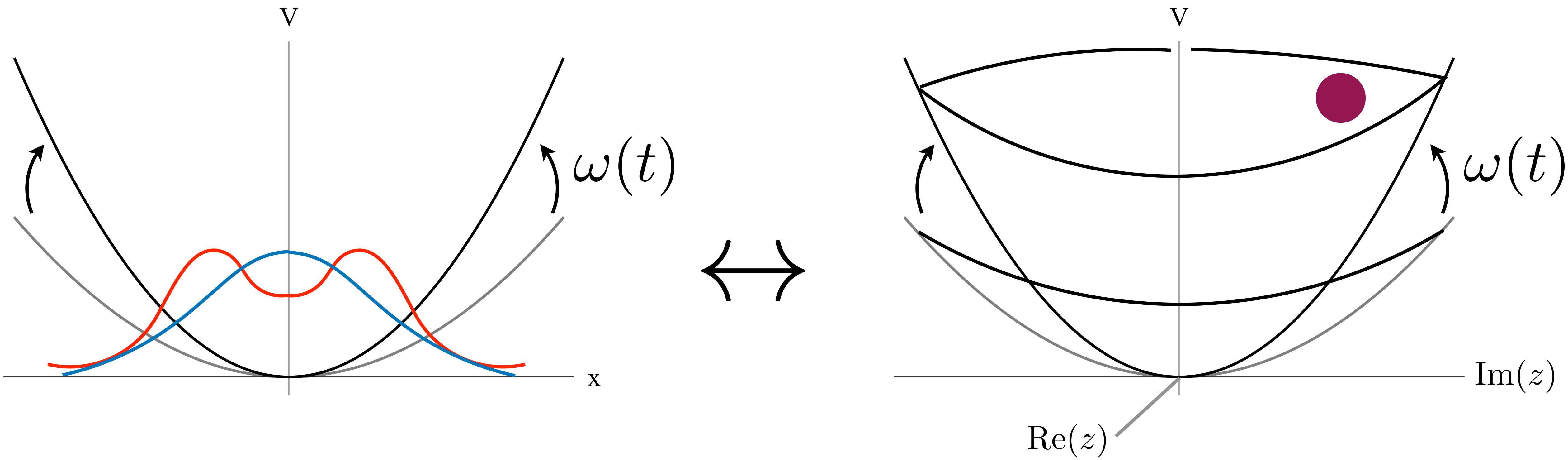
Solution: $$\hat{a}(t) = \frac{(p_z^* - im\omega z^*)}{\sqrt{2m\omega}}\hat{a}_0 + \frac{(p_z - im\omega z)}{\sqrt{2m\omega}}\hat{a}_0^\dagger$$

"Bogolyubov coefficients"

where, $$\ddot{z} + \omega^2(t)z = 0$$ z is complex!

with initial conditions $$z(0) = \frac{-i}{\sqrt{2m\omega_0}}, \quad \dot{z}(0) = \sqrt{\frac{\omega_0}{2m}}$$

Classical-Quantum Correspondence (CQC)



Quantum **Classical**

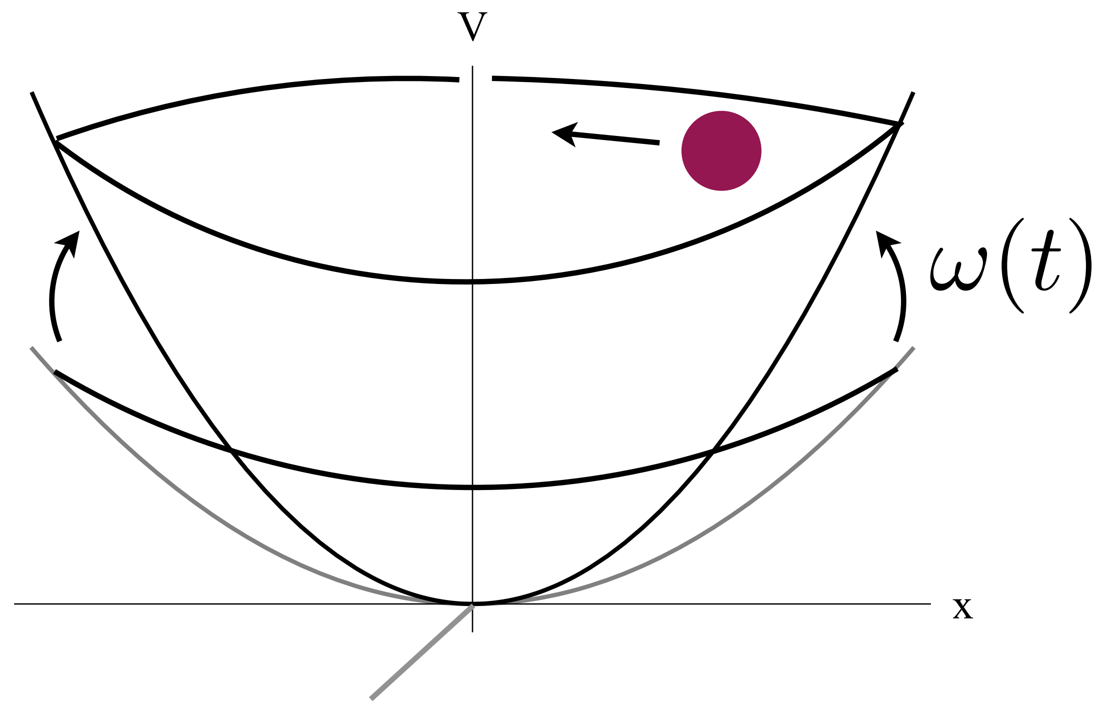
$$\hat{x} \leftrightarrow z$$

$$|0\rangle \leftrightarrow \{z_0, \dot{z}_0\}$$

Note 1: for any ω(t).

Note 2: all quantum operators can be written in terms of the classical variable z(t) and the initial values of the operators.

Initial Conditions



Quantum ground state implies classical system must have:

- zero point energy = $\omega/2$
- angular momentum = 1/2

Energy can increase with time but angular momentum stays constant.

$$E(t) = \frac{1}{2}|z' - i\omega z|^2 \qquad J = -i(z'z^* - zz^{*\prime})/2 = 1/2$$

Unexciting SHO backgrounds

$$E(t_f) = \frac{1}{2}|z'_f - i\omega_f z_f|^2 = 0$$

Note: Only require final (asymptotic). excitation energy to vanish. Stricter requirement with zero excitation energy at all times gives trivial background (discussed later).

Write: $z(t) = \rho(t)e^{i\theta(t)}$

Angular momentum constraint gives: $\theta' = -\frac{1}{2\rho^2}$

Equation of motion gives: $\rho'' + \omega^2\rho = \frac{1}{4\rho^3}$

Write as solution for background: $\omega^2 = \frac{1}{4\rho^4} - \frac{\rho''}{\rho}$

Unexciting SHO solution

$$E(t) = \frac{\rho'^2}{2} + \frac{\rho^2}{2}\left(\frac{1}{2\rho^2} - \omega\right)^2$$

Therefore unexciting if: $\rho'_f = 0 = \rho''_f$

since ρ_f”=0 ensures vanishing angular part of energy (second term).

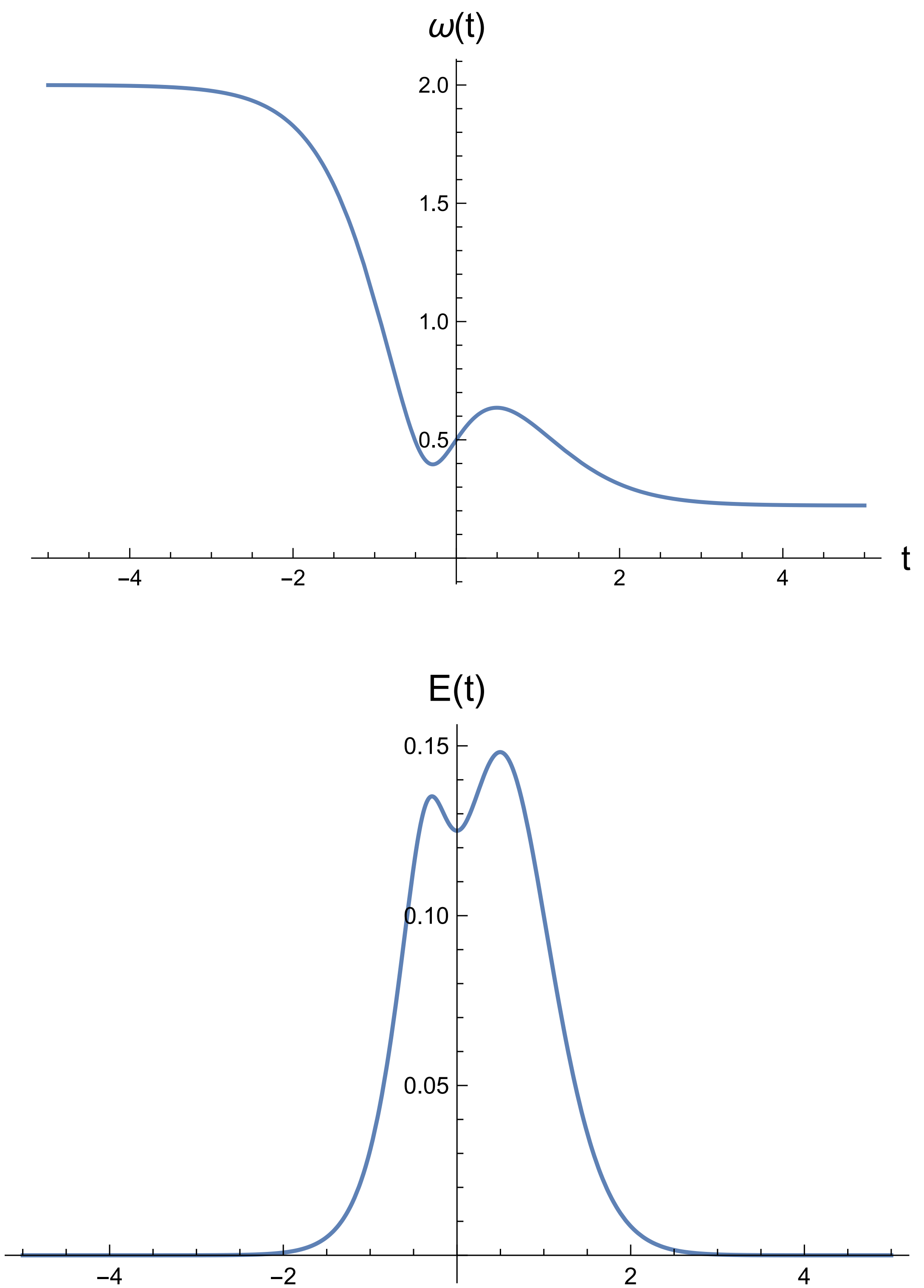
Unexciting solution: $$\boxed{\omega(t) = \sqrt{\frac{1}{4\rho^4} - \frac{\rho''}{\rho}}}$$

for any real function ρ(t) satisfying: $\rho'_i = \rho''_i = \rho'_f = \rho''_f = 0$

Additionally, we choose ρ(t) so that ω(t) is real.

An Explicit Example

$$\rho(t) = 1 + \frac{1}{2}\tanh(t)$$



Phase of the wavefunction

Even if the background is unexciting and the final frequency equals the initial frequency, the wavefunction obtains an additional phase.

$$\psi(t,x) = \frac{e^{i\gamma(t)}}{(2\pi\rho^2)^{1/4}} \exp\left[\frac{i}{2}\left(\frac{\dot{\rho}}{\rho} + \frac{i}{2\rho^2}\right)x^2\right]$$

$$\gamma(t) = -\int_{t_i}^{t} \frac{dt'}{4\rho^2(t')}$$

$$\Delta\gamma = -\frac{1}{4}\int_{t_i}^{t_f} dt' \left(\frac{1}{\rho^2(t')} - \frac{1}{\rho_i^2}\right)$$

Field Theory: CQC

George Zahariade & TV

Now the “Bogolyubov coefficients” are matrices: $z(t) \to Z_{ij}(t)$

Equation of motion: $Z'' + \Omega^2 Z = 0$

$$\Omega^2 = -\nabla^2 + \text{interaction matrix (V)} \qquad \nabla^2 = \begin{cases} -2/a^2, & i = j \\ 1/a^2, & i = j \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

Initial conditions: $Z_i = -\frac{i}{\sqrt{2}} \left(\sqrt{\Omega_i}\right)^{-1}, \quad Z_i' = \frac{1}{\sqrt{2}} \sqrt{\Omega_i}$

Constraint equations: $Z^\dagger Z' - Z^{\dagger\prime} Z = i, \quad Z^\dagger Z^{*\prime} - Z^{\dagger\prime} Z^* = 0$

Energy: $E = \frac{1}{2} \mathrm{Tr} |Z' - i\Omega Z|^2$

Homogeneous backgrounds

For *spatially homogeneous* backgrounds, the quantum excitations can be decomposed into modes. Each mode is a SHO with a time-dependent frequency.

$$z''_{\mathbf{k}} + \omega^2_{\mathbf{k}} z_{\mathbf{k}} = 0$$

$$\omega^2_{\mathbf{k}}(t) = \frac{1}{4\rho^4_{\mathbf{k}}} - \frac{\rho''_{\mathbf{k}}}{\rho_{\mathbf{k}}}$$

$$= -\mathbf{k}^2 + V(\cancel{\mathbf{k}}, t)$$ since the background is homogeneous

Homogeneous backgrounds can be chosen (just as for an SHO) so that any chosen mode remains unexcited but there will always be some modes that are excited.

Homogeneous example

$$\lambda\phi^2(t)\hat{\psi}^2$$

Consider **k**=**k**$_*$ mode.

$$\begin{aligned}\omega^2_{\mathbf{k}_*}(t) &= \frac{1}{4\rho^4_{\mathbf{k}_*}} - \frac{\rho''_{\mathbf{k}_*}}{\rho_{\mathbf{k}_*}} \\ &= \mathbf{k}^2_* + m^2_\psi + \lambda\phi^2(t)\end{aligned}$$

k$_*$ mode fixes background: $\lambda\phi^2(t) = \frac{1}{4\rho^4_{\mathbf{k}_*}} - \frac{\rho''_{\mathbf{k}_*}}{\rho_{\mathbf{k}_*}} - \mathbf{k}^2_* - m^2_\psi$

Consider a different mode, say **k**=**p**.

Then: $z''_{\mathbf{p}} + [(\mathbf{p}^2 - \mathbf{k}^2_*) + \omega^2_{\mathbf{k}_*}]z_{\mathbf{p}} = 0$

Unexciting boundary conditions imply: $\rho_{\mathbf{p}}(t_i),\ \rho'_{\mathbf{p}}(t_i) = 0,\ \rho'_{\mathbf{p}}(t_f) = 0 = \rho''_{\mathbf{p}}(t_i)$

Too many boundary conditions to satisfy in general.

General backgrounds

Equation of motion: $$Z'' + \Omega^2 Z = 0$$

Constraint equations: $$Z^\dagger Z' - Z^{\dagger\prime} Z = i, \quad Z^\dagger Z^{*\prime} - Z^{\dagger\prime} Z^* = 0$$

Define: $$\rho^2 = Z Z^\dagger$$

$$Z = \rho U$$ U is unitary

Solve the constraints: $$[\rho, \rho'] = 0, \quad \{\rho^2, U' U^\dagger\} = i$$

$$U' U^\dagger = \frac{i}{2}\rho^{-2}$$

Unexciting background: $$\Omega^2 = -\rho''\rho^{-1} + \frac{1}{4}\rho^{-4}$$

with: $$[\rho, \rho'] = 0, \quad \rho'_i = \rho''_i = \rho'_f = \rho''_f = 0$$

For example: $$\rho(t) = A + \frac{1}{2}\tanh(t) B, \quad [A, B] = 0$$

Physical backgrounds?

Unexciting background: $\Omega^2 = -\rho''\rho^{-1} + \frac{1}{4}\rho^{-4}$

Example: $\frac{\lambda}{2}\phi^2\psi^2$ interaction

Then: $\Omega^2 = -\nabla^2 + m^2 + \lambda\phi^2$

$$\lambda\phi^2 = -\left((\partial_t^2 - \nabla^2)\rho + m^2\rho - \frac{1}{4}\rho^{-3}\right)\frac{1}{\rho}$$

and the right-hand side should be diagonal for this to be a valid background.

How can we choose ρ(t) such that the interaction is “physical”?

Answer will depend on the physical system of interest.

Unexciting for all times

Energy: $$E = \frac{1}{2}\mathrm{Tr}|Z' - i\Omega Z|^2$$

$$E = 0 \quad \Rightarrow \quad Z' = i\Omega Z$$

$$Z'' + \Omega^2 Z = i\Omega' Z$$

$$\Omega' Z = 0$$

If Z is invertible: $$\Omega' = 0$$

If Z is not invertible, write: $Z = \rho U$ U is unitary

Therefore, $\Omega' \rho = 0, \quad \rho = \text{constant matrix}$ "always unexciting"

ρ(t) is set by initial conditions: $$\rho(t) = \frac{1}{2}\Omega_i^{-1}$$

Always Unexciting

In certain cases Z may not exist and the background may be unexciting for all times.

Write: $Z = \rho U$ U is unitary

If a background is unexciting for all times, time derivatives of ρ(t) have to vanish and ρ is constant.

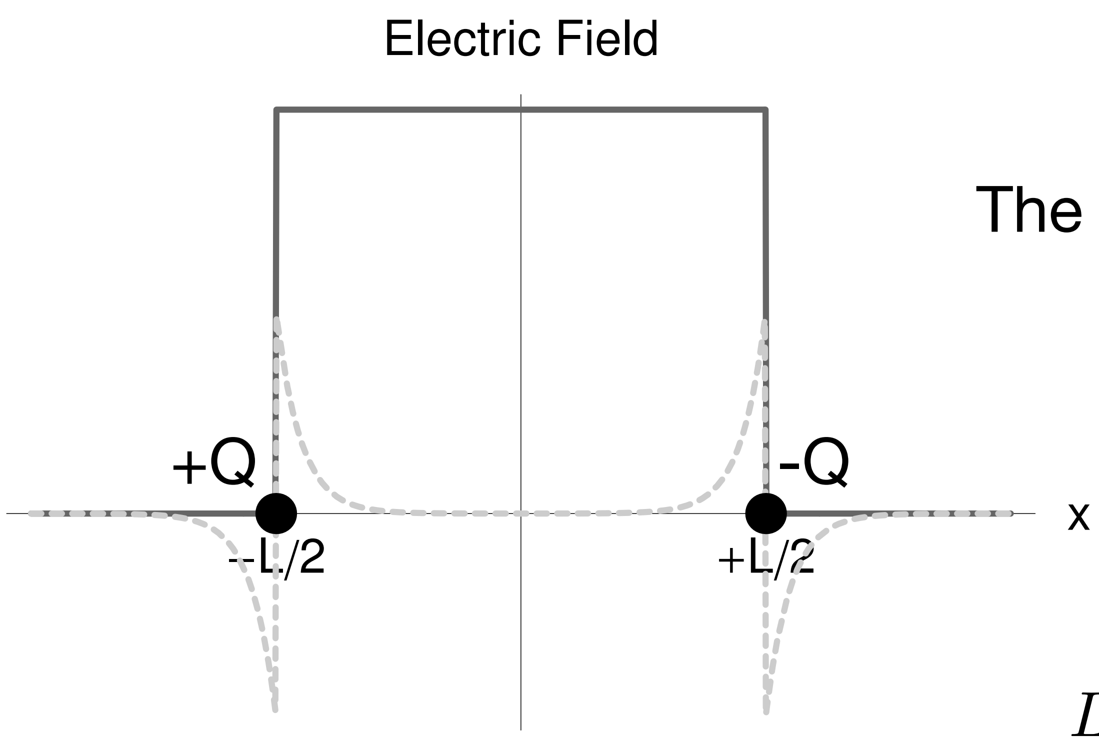
Therefore, $\rho^2 = \frac{1}{2}\Omega_i^{-1}$

which does not exist if the background has a zero mode.

Example: Quantum excitations in the background of a boosted soliton (also pp-waves) have a zero mode because of translational invariance.
Boosted solitons do not produce particles and are "always unexciting".

Lesson: An "always unexciting" background should have symmetries that lead to zero modes of the quantum excitations.

Always-unexciting E in QED$_{1+1}$



$$L = \bar{\psi}\gamma^{\mu}(i\partial_{\mu} + e A_{\mu})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The gauge potential, A_x, is a time-dependent background for the fermions.

CQC only applies to bosons.

$$L' = \frac{1}{2}(\partial_{\mu}\phi)^2 + \frac{g}{2}\phi\,\epsilon^{\mu\nu}F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The electric field is a time-independent background for the bosons.

Conclusions

- Unexciting classical backgrounds are protected from quantum decay.
- Huge space of unexciting backgrounds for the simple harmonic oscillator.
- Homogeneous field backgrounds can be unexciting for some quantum field modes but not for all modes.
- We are able to construct unexciting inhomogeneous backgrounds but whether these can be “physical” depends on the system of interest and remains an open question.
- “Always unexciting” backgrounds require symmetries of the background and existence of corresponding zero modes of the quantum excitations.