Lattice calculations in muon $g - 2$

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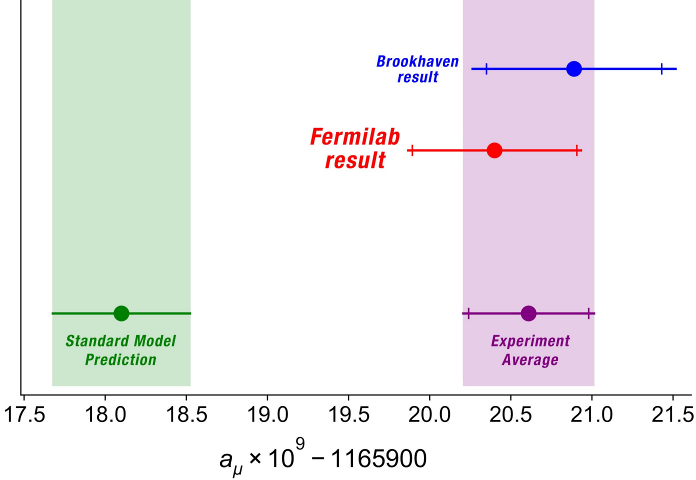
May 3, 2021

Theory Seminar

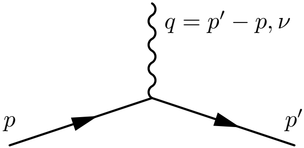
Department of Physics and Astronomy

University of California, Davis

Outline



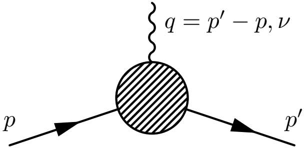
- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g-2$ experiment.



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$



$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

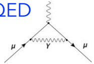
$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.

- Theory Initiative Whiterpaper posted 10 June 2020:
  arXiv:2006.04822 [Phys. Rept. 887 (2020) 1-166]
  (132 authors, 82 institutions, 21 countries)

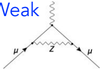
$$a_\mu(\text{SM}) = a_\mu(\text{QED}) + a_\mu(\text{Weak}) + a_\mu(\text{Hadronic})$$

QED



+… $116\,584\,718.9\,(1)\times 10^{-11}$ 0.001 ppm

Weak

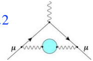


+… $153.6\,(1.0)\times 10^{-11}$ 0.01 ppm

Hadronic…

…Vacuum Polarization (HVP)

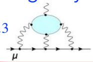
α^2



+… $6845\,(40)\times 10^{-11}$ [0.6%] 0.37 ppm

…Light-by-Light (HLbL)

α^3

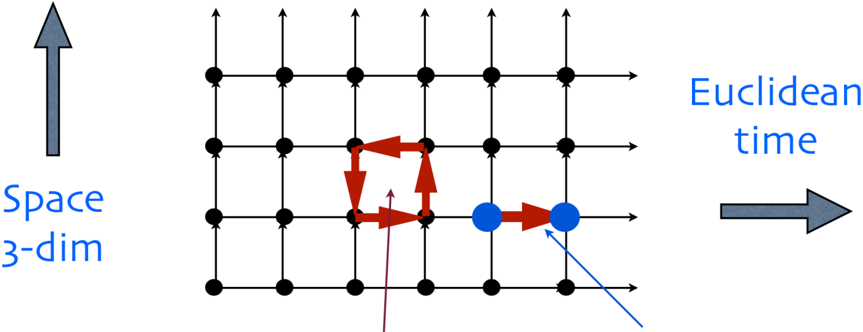


+… $92\,(18)\times 10^{-11}$ [20%] 0.15 ppm

Slide from Christoph Lehner's talk at https://indico.cern.ch/event/1019685

Outline

1. Introduction
2. **Lattice QCD**
3. Hadronic Vacuum Polarization contribution
4. Hadronic Light-by-Light contribution
5. Summary



$$S_E^{\rm latt} = -\sum_{\Box} \frac{6}{g^2} \, \mathrm{Re}\, \mathrm{tr}_N (U_{\Box,\mu\nu}) - \sum_q \overline{q}(D_\mu^{\rm lat}\gamma_\mu + a m_q) q$$

Wilson gauge action   Lattice fermion action

Operator:

$$\langle \mathcal{O}(U,q,\bar{q}) \rangle = \frac{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}} \mathcal{O}(U,q,\bar{q})}{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}}}$$

$$= \frac{\int [\mathcal{D}U] e^{-S_{\text{glue}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right) \tilde{\mathcal{O}}(U)}{\int [\mathcal{D}U] e^{-S_{\text{glue}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right)}$$

Monta Carlo:

- The integration is performed for all the link variables: U. Dimension is $L^3 \times T \times 4 \times 8$.
- Sample points the following distribution:

$$e^{-S_{\text{glue}}^{\text{latt}}(U)} \prod_q \det\left(D_\mu^{\text{latt}}(U)\gamma_\mu + am_q\right)$$

- Therefore:

$$\langle \mathcal{O}(U,q,\bar{q}) \rangle = \frac{1}{N_{\text{conf}}} \sum_{k=1}^{N_{\text{conf}}} \tilde{\mathcal{O}}(U^{(k)})$$

- How many parameters?

$$g \qquad am_l \qquad am_s$$

isospin symmetric ($m_u = m_d = m_l$) and three flavor u, d, s theory.

- We are supposed to take $a \to 0$ limit, how?

$$g \to 0$$

For different g, as long as it is small, the lattice calculation is describe the same physics, just with different a.

$$a \approx a_0 \exp\left(-\frac{1}{11 - \frac{2}{3}N_f}\frac{8\pi^2}{g^2}\right)$$

This is the the renormalization equation.

- Why do we need three inputs m_π, m_K, m_Ω ?

- How many parameters?

$$g \qquad am_l \qquad am_s$$

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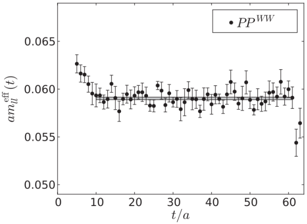
- Why do we need three inputs m_π, m_K, m_Ω ?

One of them, m_Ω, is used determine the overall scale in the unit of GeV.
Or, we actually only need two parameters: m_π/m_Ω and m_K/m_Ω.

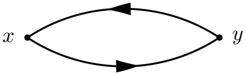
64I $a^{-1} = 2.359\text{GeV}$ $am_\pi = 0.059$ RBC-UKQCD



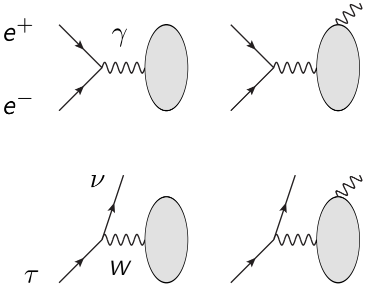
Correlation function:

$$
\begin{aligned}
C(t) &= \langle \pi^-(\vec{x}, t) \sum_{\vec{y}} \pi^+(\vec{y}, 0) \rangle \\
&= \langle \bar{d}(\vec{x}, t) i\gamma_5 u(\vec{x}, t) \sum_{\vec{y}} \bar{u}(\vec{y}, 0) i\gamma_5 d(\vec{y}, 0) \rangle \\
&= \frac{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}} \bar{d}(\vec{x}, t) i\gamma_5 u(\vec{x}, t) \sum_{\vec{y}} \bar{u}(\vec{y}, 0) i\gamma_5 d(\vec{y}, 0)}{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}}} \\
&= \frac{\int [\mathcal{D}U] e^{-S_{\text{glue}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right) \times \sum_{\vec{y}} \text{Tr}\left[\left(D_\mu^{\text{latt}}\gamma_\mu + am_u\right)^{-1}_{(\vec{x},t;\vec{y},0)} \gamma_5 \left(D_\mu^{\text{latt}}\gamma_\mu + am_d\right)^{-1}_{(\vec{y},0;\vec{x},t)} \gamma_5\right]}{\int [\mathcal{D}U] e^{-S_{\text{glue}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right)} \\
&\propto e^{-m_\pi t}
\end{aligned}
$$

$$
m_\pi^{\text{eff}}(t) = ln\left(\frac{C(t)}{C(t+1)}\right)
$$



Outline



$$e^+e^- \to \text{hadrons}(\gamma)$$

$$J_\mu = V_\mu^{I=1,I_3=0} + V_\mu^{I=0,I_3=0}$$

$$\tau \to \nu\text{hadrons}(\gamma)$$

$$J_\mu = V_\mu^{I=1,I_3=\pm1} - A_\mu^{I=1,I_3=\pm1}$$

Knowledge of isospin-breaking corrections and separation of vector and axial-vector components needed to use τ decay data. Can do this from LQCD+QED (Bruno, Izubuchi, CL, Meyer, 1811.00508)!

Can have both energy-scan and ISR setup.

Lattice Calculation of the Lowest-Order Hadronic Contribution to the Muon Anomalous Magnetic Moment

T. Blum
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(Received 18 December 2002; published 30 July 2003)

We present a quenched lattice calculation of the lowest order [$\mathcal{O}(\alpha^2)$] hadronic contribution to the anomalous magnetic moment of the muon which arises from the hadronic vacuum polarization. A general method is presented for computing entirely in Euclidean space, obviating the need for the usual dispersive treatment which relies on experimental data for e^+e^- annihilation to hadrons. While the result is not yet of comparable precision to those state-of-the-art calculations, systematic improvement of the quenched lattice computation to this level is straightforward and well within the reach of present computers. Including the effects of dynamical quarks is conceptually trivial; the computer resources required are not.

DOI: 10.1103/PhysRevLett.91.052001 PACS numbers: 12.38.Gc, 13.40.Em, 14.60.Ef, 14.65.Bt

The magnetic moment of the muon is defined by the $q^2 \to 0$ (static) limit of the vertex function which describes the interaction of the electrically charged muon with the photon,

$$\Gamma_\rho(p_2, p_1) = \gamma_\rho F_1(q^2) - \frac{i}{4m_\mu}(\gamma_\rho \not{q} - \not{q}\gamma_\rho)F_2(q^2), \quad (1)$$

where m_μ is the muon mass, $q = p_2 - p_1$ is the photon momentum, and p_1, p_2 are the incoming and outgoing momentum of the muon. Lorentz invariance and current conservation have been used in obtaining Eq. (1). Form factors $F_1(q^2)$ and $F_2(q^2)$ contain all information about the muon's interaction with the electromagnetic field. In particular, $F_1(0) = 1$ is the electric charge of the muon in

though a discrepancy with a calculation that uses τ decay data may indicate a theory error as large as 5% [2] and reduces the disagreement with experiment to roughly 1.6 standard deviations. A purely theoretical, first principles, calculation has been lacking and is desirable, and also has several advantages over the conventional approach. For instance, the separation of QED effects from hadronic corrections is automatic, as is the treatment of isospin corrections if different quark masses are used in the simulation. Thus, it is possible that lattice calculations may eventually help to settle the above-mentioned discrepancy between $e^+ e^-$ annihilation and τ decay.

The method described here is simple and direct. We begin with Ref. [5] which describes the computation of

PHYSICAL REVIEW LETTERS **121,** 022003 (2018)

Editors' Suggestion

Calculation of the Hadronic Vacuum Polarization Contribution to the Muon Anomalous Magnetic Moment

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(RBC and UKQCD Collaborations)

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(Received 25 January 2018; published 12 July 2018)

We present a first-principles lattice QCD + QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_\mu^{\mathrm{HVP\ LO}} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with R-ratio data, we significantly improve the precision to $a_\mu^{\mathrm{HVP\ LO}} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_\mu^{\mathrm{HVP\ LO}}$.

Article

Leading hadronic contribution to the muon magnetic moment from lattice QCD

https://doi.org/10.1038/s41586-021-03418-1

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Check for updates

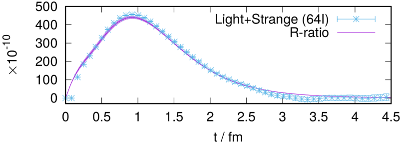
Sz. Borsanyi[1], Z. Fodor[1,2,3,4,5] ✉, J. N. Guenther[6,10], C. Hoelbling[1], S. D. Katz[4], L. Lellouch[7], T. Lippert[1,2], K. Miura[7,8,9], L. Parato[7], K. K. Szabo[1,2], F. Stokes[2], B. C. Toth[1], Cs. Torok[2] & L. Varnhorst[1,10]

The standard model of particle physics describes the vast majority of experiments and observations involving elementary particles. Any deviation from its predictions would be a sign of new, fundamental physics. One long-standing discrepancy concerns the anomalous magnetic moment of the muon, a measure of the magnetic field surrounding that particle. Standard-model predictions[1] exhibit disagreement with measurements[2] that is tightly scattered around 3.7 standard deviations. Today, theoretical and measurement errors are comparable; however, ongoing and planned experiments aim to reduce the measurement error by a factor of four. Theoretically, the dominant source of error is the leading-order hadronic vacuum polarization (LO-HVP) contribution. For the upcoming measurements, it is essential to evaluate the prediction for this contribution with independent methods and to reduce its uncertainties. The most precise, model-independent determinations so far rely on dispersive techniques, combined with measurements of the cross-section of electron–positron annihilation into hadrons[3–6]. To eliminate our reliance on these experiments, here we use ab initio quantum chromodynamics (QCD) and quantum electrodynamics simulations to compute the LO-HVP contribution. We reach sufficient precision to discriminate between the measurement of the anomalous magnetic moment of the muon and the predictions of dispersive methods. Our result favours the experimentally measured value over those obtained using the dispersion relation. Moreover, the methods used and developed in this work will enable further increased precision as more powerful computers become available.

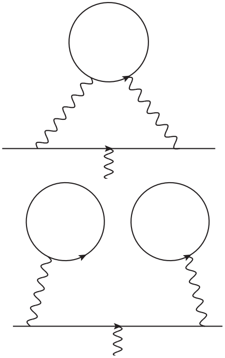
T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle$$

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t) C(t)$$



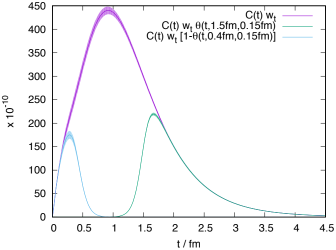
- In Euclidean space-time, $C(t)$ decreases exponentially as t increases.
  For $t \sim 1$ fm, $C(t) \sim e^{-m_\rho t}$.
  For $t \to \infty$, $C(t) \sim e^{-2m_\pi t}$.
  Lattice statistical error: $\delta C(t) \sim e^{-m_\pi t}$.
- For $t \lesssim 1$ fm, $w(t) \sim t^4$.
  For $t \to \infty$ $(m_\mu t \gg 1)$, $w(t) \sim t^2$.

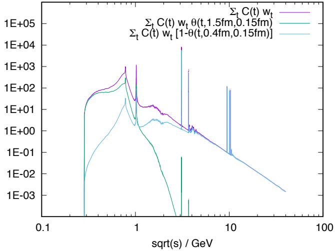


HVP: hybrid (lattice & dispersive)

RBC-UKQCD PRL 121, 022003 (2018)

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t) \qquad w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$





Most of $\pi\pi$ peak is captured by window from $t_0 = 0.4$ fm to $t_1 = 1.5$ fm, so replacing this region with lattice data reduces the dependence on BaBar versus KLOE data sets.

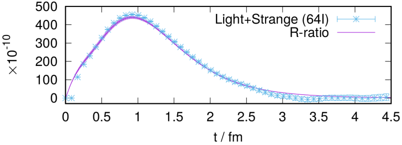
Babar & KLOE tension => syst error $5.6/2 = 2.8$ in unit of 10^{-10}.

The window method is much less affected by this tension.

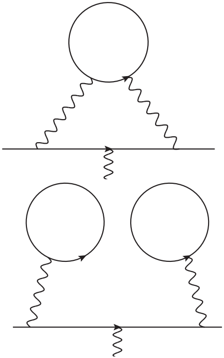
arXiv:1402.0244, arXiv:hep-lat/0409056.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle$$

$$= \frac{1}{N_y}\sum_{y}\frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(y)\rangle$$



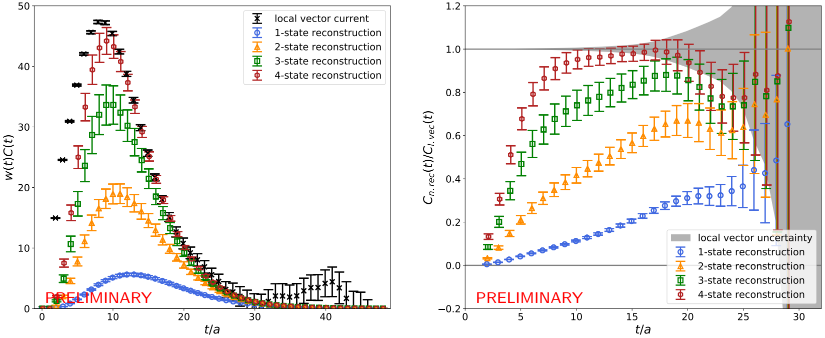
- All modes averaging (AMA): Use approximate quark propagator to calculate the correlation function to increase N_y and then selectively calculate the correlation function accruately to correct the bias.
- Low modes averaging (LMA): Only use low modes to calculate quark propagator to further increase N_y. Then, use the above AMA method to calculate the bias.



- Main idea is that: one does not have to calculate the long distance part of the correlation function directly.

$$
\begin{aligned}
C(t) &= \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle \\
&= \sum_n \frac{V}{3}\sum_{j=0,1,2}\langle 0|J_j(0)|n\rangle\langle n|J_j(0)|0\rangle e^{-E_n t}
\end{aligned}
$$

- The summation over n is limited to zero momentum states and states are normalized to "1".
- At large t, only lowest few states contribute. We only need the matrix elements $\langle n|J_j(0)|0\rangle$ and the corresponding energy E_n.
- Need to study the spectrum of the $\pi\pi$ system!
- Can reduce the statistical error beyond the gauge noise limit!



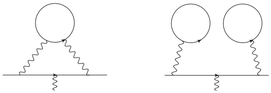
GEVP results to reconstruct long-distance behavior of
local vector correlation function needed to compute connected HVP

Explicit reconstruction good estimate of correlation function at long-distance,
missing excited states at short-distance

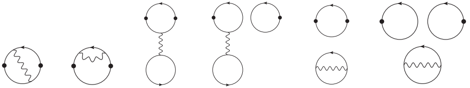
More states $\implies$ better reconstruction, can replace $C(t)$ at shorter distances

RBC-UKQCD by Aaron Meyer and Christoph Lehner
Preliminary

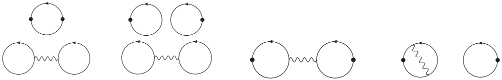
Isospin limit



QED corrections

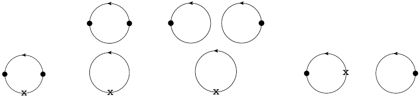


(a) V (b) S (c) T (d) T_d (e) D1 (f) $D1_d$



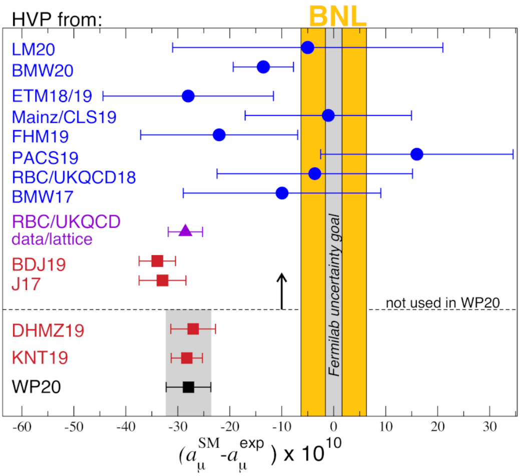
(g) D2 (h) $D2_d$ (i) F (j) D3

Strong isospin breaking



(a) M (b) R (c) R_d (d) O

Status and impact of hadronic vacuum polarization contribution



Ab-initio lattice QCD(+QED) calculations are maturing

Difficult problem: scales from $2m_\pi$ to several GeV enter; cross-checks needed at high precision

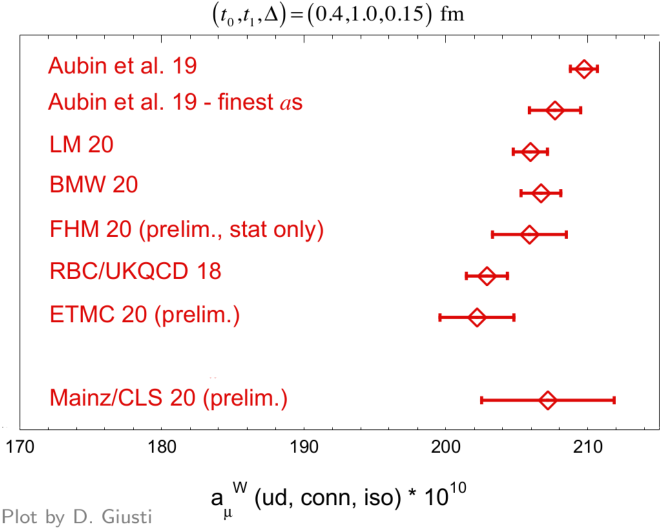
Hybrid window method restricts scales that enter from lattice/dispersive data

Dispersive, $e^+e^- \to$ hadrons (20+ years of experiments)

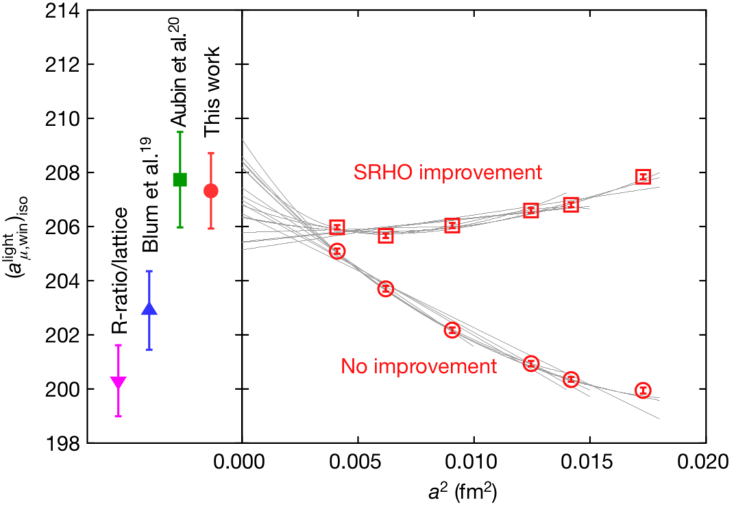
Now first published lattice result with sub-percent precision available (BMW20), cross-checks are crucial to establish or refute high-precision lattice methodology (same situation as for HLbL) $\Rightarrow$ Theory Initiative as a platform to do this

An important internal cross-check: Euclidean time windows

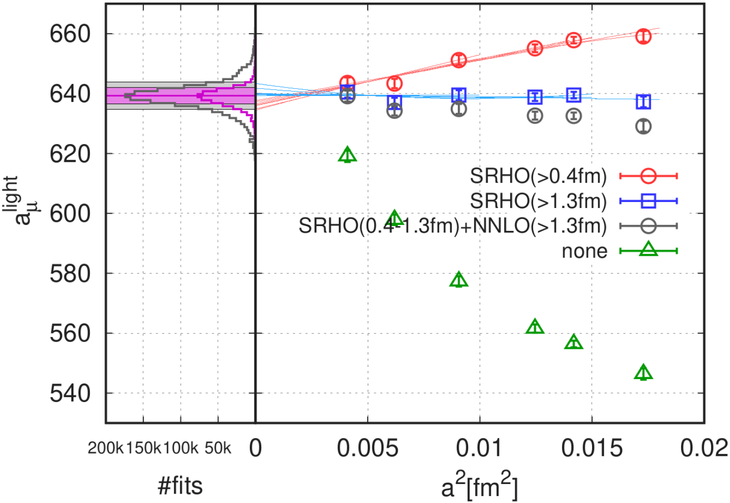
Defined in RBC/UKQCD 2018, related to HVP with suppression of very high and low energies



Plot by D. Giusti



- Staggered fermion has a special lattice artifacts: taste breaking effects.
- Curves show different treatments of correcting the taste breaking effects.



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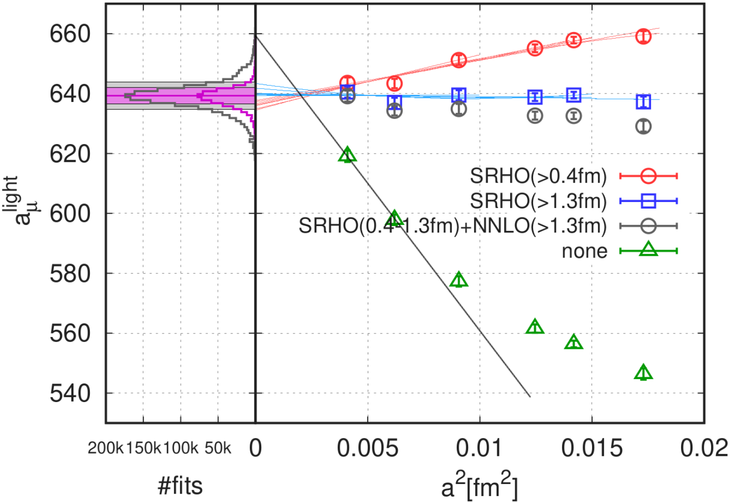

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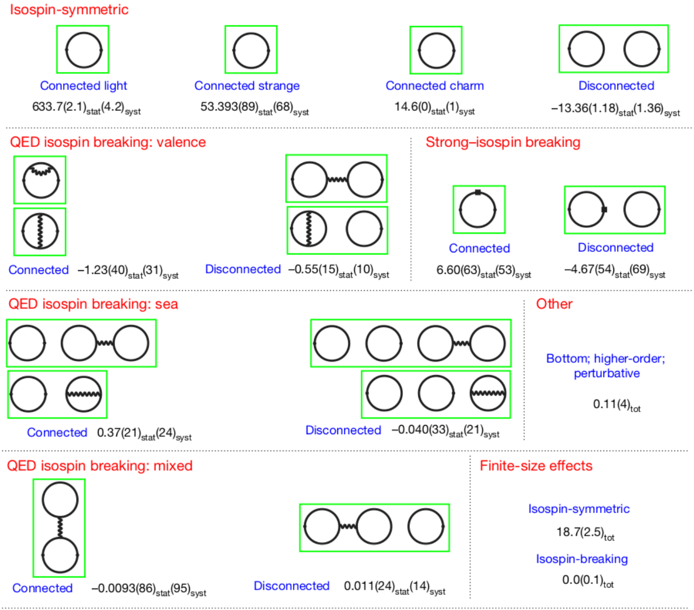
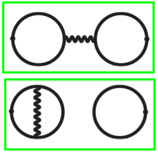
HVP: BMW 2021



$$a_\mu^{\text{LO-HVP}}\ (\times 10^{10}) = 707.5(2.3)_{\text{stat}}(5.0)_{\text{syst}}(5.5)_{\text{tot}}$$

HVP: Compare BMW 2021 and RBC-UKQCD 2018

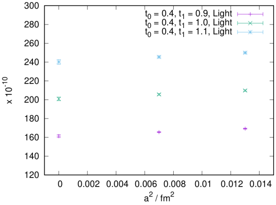
$a_\mu^{\text{ud, conn, isospin}}$	$649.7(14.2)_S(2.8)_C(3.7)_V(1.5)_A(0.4)_Z(0.1)_{E48}(0.1)_{E64}$
$a_\mu^{\text{s, conn, isospin}}$	$53.2(0.4)_S(0.0)_C(0.3)_A(0.0)_Z$
$a_\mu^{\text{c, conn, isospin}}$	$14.3(0.0)_S(0.7)_C(0.1)_Z(0.0)_M$
$a_\mu^{\text{uds, disc, isospin}}$	$-11.2(3.3)_S(0.4)_V(2.3)_L$
$a_\mu^{\text{QED, conn}}$	$5.9(5.7)_S(0.3)_C(1.2)_V(0.0)_A(0.0)_Z(1.1)_E$
$a_\mu^{\text{QED, disc}}$	$-6.9(2.1)_S(0.4)_C(1.4)_V(0.0)_A(0.0)_Z(1.3)_E$
a_μ^{SIB}	$10.6(4.3)_S(0.6)_C(6.6)_V(0.1)_A(0.0)_Z(1.3)_{E48}$
$a_\mu^{\text{udsc, isospin}}$	$705.9(14.6)_S(2.9)_C(3.7)_V(1.8)_A(0.4)_Z(2.3)_L(0.1)_{E48}(0.1)_{E64}(0.0)_M$
$a_\mu^{\text{QED, SIB}}$	$9.5(7.4)_S(0.7)_C(6.9)_V(0.1)_A(0.0)_Z(1.7)_E(1.3)_{E48}$
$a_\mu^{\text{R-ratio}}$	
a_μ	$715.4(16.3)_S(3.0)_C(7.8)_V(1.9)_A(0.4)_Z(1.7)_E(2.3)_L(1.5)_{E48}(0.1)_{E64}(0.3)_b(0.2)_c(1.1)_{\overline{S}}(0.3)_{\overline{Q}}(0.0)_M$

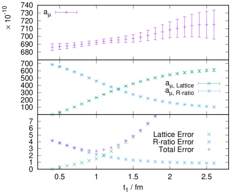


Disconnected $-0.55(15)_{\text{stat}}(10)_{\text{syst}}$

- The left table shows result from RBC-UKQCD 2018. The right figure shows the result from BMW 2021.
- This discrepancy needs further study and more cross checks.

HVP: Computational cost for RBC-UKQCD calculation





- Cost 0.25 billion BG/Q core hours ($\sim$ 3 million V100 card hours).
- Next RBC-UKQCD paper (in a few months) will focus on the window contrib.
  - Four times the statistics (half the statistical error).
  - Include a third lattice spacing $1/a \approx 2.8$ GeV.
    $\sim$ 4 million V100 card hours used so far.
    $\sim$ 10 million V100 card hours used to generate these configurations.
  - The new results from the three different lattice spacings will have similar statistical error.
- Then, next work will use the spectrum study to reduce the long distance noise.

HVP: Some comments

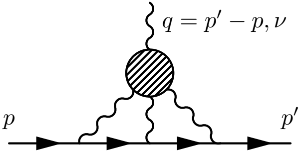
- Lattice calculation is completely based on the first principle.
  The only inputs are: α_{QED}, m_μ, $m_\pi^\pm$, m_K^0, $m_K^\pm$, m_{η_c}, m_Ω.
- The bottleneck is the leading contribution from the light quark connectd diagram.
- Continuum extrapolation is very hard with taste breaking effects of the Staggered fermion.
  Cross check with independent lattice calculations by other collaborations using different fermion formulations is needed.
- Cross checks on the subleading diagrams, finite volume effects, (which can be defined and calculated individually) are also needed.

Outline

Contribution	PdRV(09) [471]	N/JN(09) [472, 573]	J(17) [27]	Our estimate
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} − 1(3)
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

Table 15: Comparison of two frequently used compilations for HLbL in units of 10^{-11} from 2009 and a recent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus"); N/JN = Nyffeler / Jegerlehner, Nyffeler; J = Jegerlehner.

- Values in the table is in unit of 10^{-11}.
- The total HLbL contribution is on the order of 10×10^{-10}.
- Uncertainty of the analytically approach mostly come from the short distance part.



HLbL: RBC-UKQCD 2020

PHYSICAL REVIEW LETTERS **124,** 132002 (2020)

Editors' Suggestion | Featured in Physics

Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD

Thomas Blum,[1,2] Norman Christ,[3] Masashi Hayakawa,[4,5] Taku Izubuchi,[6,2]
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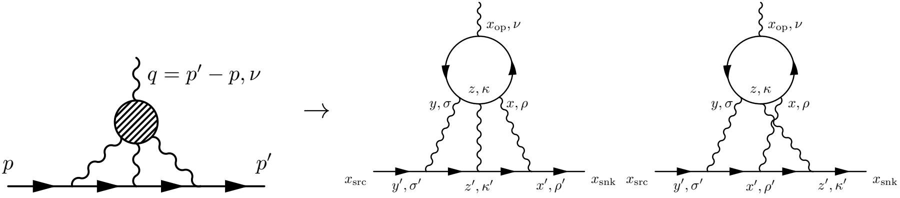
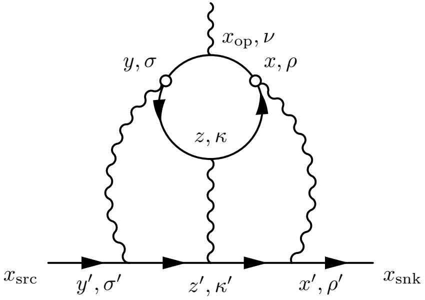
(Received 18 December 2019; accepted 27 February 2020; published 1 April 2020)

- First lattice result for the hadronic light-by-light scattering contribution to the muon $g-2$ with all errors systematically controlled.
- Lattice calculation directly at the physical pion mass and no Chiral extrapolation is needed.



- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional four different permutations of photons not shown.
- The photons can be connected to different quark loops. These are referred to as the disconnected diagrams. They will be discussed later.
- First results are obtained by T. Blum et al. 2015 (PRL 114, 012001).



- Two point sources at x, y: randomly sample x and y.
- Importance sampling: focus on small $|x-y|$.
- Complete sampling for $|x-y| \leq 5a$ upto discrete symmetry.

$$\frac{a_\mu}{m_\mu} \bar{u}_{s'}(\vec{0}) \frac{\Sigma}{2} u_s(\vec{0}) = \sum_{r=x-y} \sum_{z} \sum_{x_{\rm op}} \frac{1}{2} (\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \bar{u}_{s'}(\vec{0}) i \vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\rm op}) u_s(\vec{0})$$

$$\vec{\mu} = \sum_{\vec{x}_{\rm op}} \frac{1}{2} (\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \vec{J}(\vec{x}_{\rm op})$$

Reorder summation

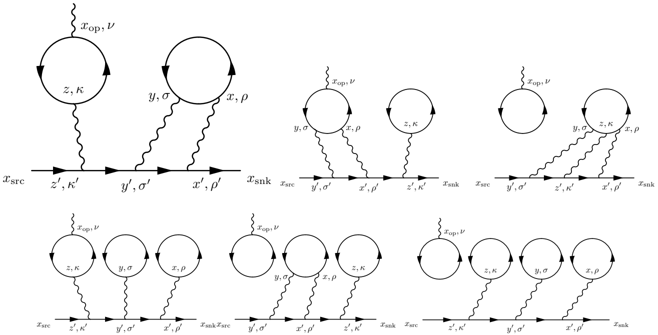
$$|x-y| \leq \min(|y-z|, |x-z|)$$

- Muon is plane wave, $x_{\rm ref} = (x+y)/2$.
- Sum over time component for $x_{\rm op}$.
- Only sum over $r = x - y$.

T. Blum et al 2016. (PRD 93, 1, 014503)

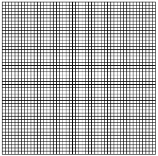
HLbL: disconnected diagrams

- One diagram (the biggest diagram below) do not vanish even in the $\mathrm{SU}(3)$ limit.
- We extend the method and computed this leading disconnected diagram as well.

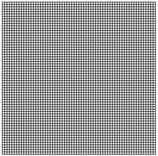


- Permutations of the three internal photons are not shown.
- **Gluons exchange between and within the quark loops are not drawn.**
- We need to make sure that the loops are connected by gluons by "vacuum" subtraction. So the diagrams are 1-particle irreducible.

HLbL: RBC-UKQCD lattices

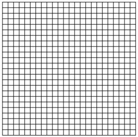


48I: $48^3 \times 96$, $5.5\,\text{fm}$ box

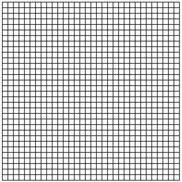


64I: $64^3 \times 128$, $5.5\,\text{fm}$ box

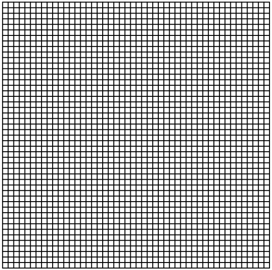
Phys. Rev. D 93, 074505 (2016)



24D: $24^3 \times 64$, $4.8\,\text{fm}$ box



32D: $32^3 \times 64$, $6.4\,\text{fm}$ box

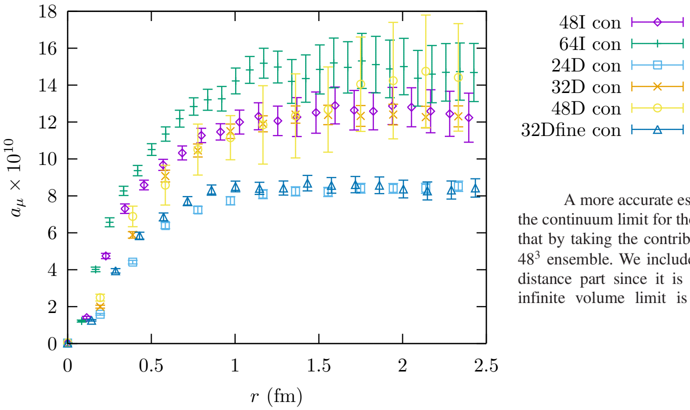


48D: $48^3 \times 64$, $9.6\,\text{fm}$ box

32Dfine: $32^3 \times 64$, $4.8\,\text{fm}$ box

T. Blum et al 2020. (PRL 124, 13, 132002)

$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\text{op}}}\frac{1}{2}(\vec{x}_{\text{op}}-\vec{x}_{\text{ref}})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\text{op}})u_s(\vec{0})$$

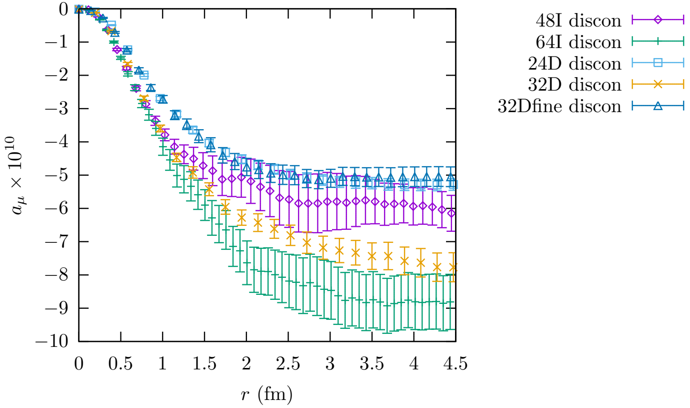


A more accurate estimate can be obtained by taking the continuum limit for the sum up to $r = 1$ fm, and above that by taking the contribution from the relatively precise 48^3 ensemble. We include a systematic error on this long distance part since it is not extrapolated to $a = 0$. The infinite volume limit is taken as before.

Partial sum is plotted above. Full sum is the right most data point.

T. Blum et al 2020. (PRL 124, 13, 132002)

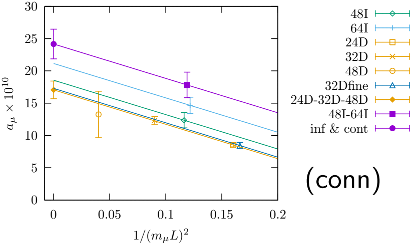
$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\text{op}}}\frac{1}{2}(\vec{x}_{\text{op}}-\vec{x}_{\text{ref}})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\text{op}})u_s(\vec{0})$$



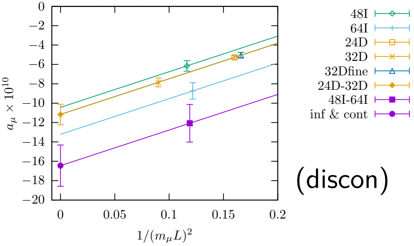
Partial sum is plotted above. Full sum is the right most data point.

T. Blum et al 2020. (PRL 124, 13, 132002)

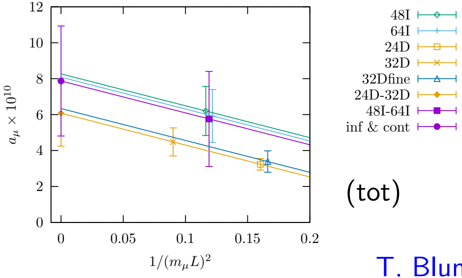
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\right)$$



(conn)



(discon)



(tot)

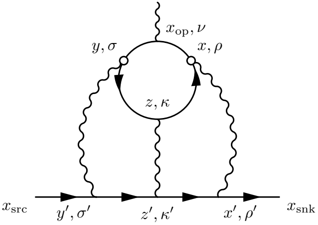
T. Blum et al 2020. (PRL 124, 13, 132002)

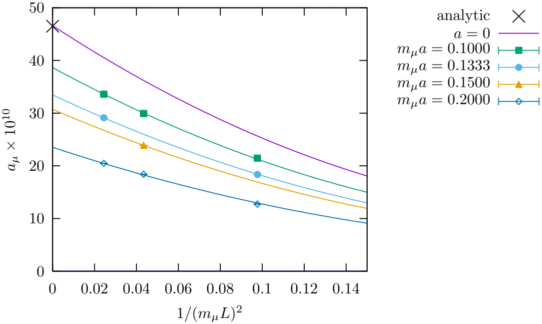
	con	discon	tot
a_μ	24.16(2.30)	-16.45(2.13)	7.87(3.06)
sys hybrid $\mathcal{O}(a^2)$	0.20(0.45)	0	0.20(0.45)
sys $\mathcal{O}(1/L^3)$	2.34(0.41)	1.72(0.32)	0.83(0.56)
sys $\mathcal{O}(a^4)$	0.88(0.31)	0.71(0.28)	0.95(0.92)
sys $\mathcal{O}(a^2\log(a^2))$	0.23(0.08)	0.25(0.09)	0.02(0.11)
sys $\mathcal{O}(a^2/L)$	4.43(1.38)	3.49(1.37)	1.08(1.57)
sys strange con	0.30	0	0.30
sys sub-discon	0	0.50	0.50
sys all	5.11(1.32)	3.99(1.29)	1.77(1.13)

- Same method is used for esimating the systematic error of individual and total contribution.
- Systematic error has some cancellation between the connected and disconnected diagrams.

T. Blum et al 2020. (PRL 124, 13, 132002)

- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.





$$F_2(a, L) = F_2 \left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4} \right) (1 - c_2 a^2 + c_2' a^4) \;\rightarrow\; F_2 = 46.6(2) \times 10^{-10} \quad (19)$$

- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

Conclusion and outlook

- $a_\mu = 7.87(3.06)_{\text{stat}}(1.77)_{\text{sys}} \times 10^{-10}$.
  T. Blum et al 2020. (PRL 124, 13, 132002)
- Consistent with analytical approach (hadronic model & dispersion relations):
  $9.2(1.9) \times 10^{-10}$ (White paper 2020).
- Leaves little room for the HLbL contribution to explain the difference between the Standard Model and the BNL/Fermilab experiment.
- Better accuracy is desired to compare with the final Fermilab muon $g-2$ experimental result.
- Working on the infinite volume QED approach.

Calculation costs about 1 billion BG/Q core hours ($\sim$ 13 million V100 card hours) with 5 consecutive ALCC allocations.

Hadronic light-by-light contribution to $(g-2)_\mu$ from lattice QCD: a complete calculation

En-Hung Chao,[1] Renwick J. Hudspith,[1] Antoine Gérardin,[2]
Jeremy R. Green,[3] Harvey B. Meyer,[1,4,5] and Konstantin Ottnad[1]

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[2] *Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France*
[3] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland*
[4] *Helmholtz Institut Mainz, Staudingerweg 18, D-55128 Mainz, Germany*
[5] *GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany*

(Dated: April 7, 2021)

- The method is overall similar as is used in the RBC/UKQCD calculation. It is developed to a very large degree independently, but with some healthy public/private communications.
- Mainz pioneered in using the infinite volume QED method in HLbL. The QED weighting function can be saved to disk and reuse.
- Use the subtraction method for the QED weighting funcion invented by RBC-UKQCD based on the QED_∞: T. Blum et al 2017. PRD 96, 3, 034515
- Use 4D rotational symmetry in combining the hadronic 4-point function with the QED weighting function.

HLbL: Mainz 2021

For the connected and disconnected diagrams' contributions individually:

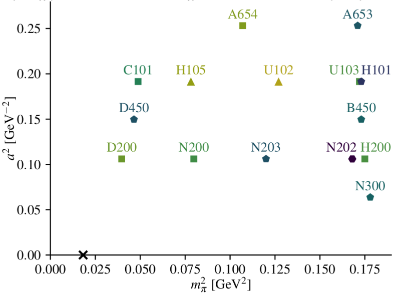
$$a_\mu(m_\pi^2, m_\pi L, a^2) = A\, e^{-m_\pi L/2} + B\, a^2 + C\ S(m_\pi^2) + D + E\, m_\pi^2\ , \qquad (21)$$

$$\text{Pole} :: \ \frac{1}{m_\pi^2}$$

$$\text{Log} :: \ \log m_\pi^2$$

$$\text{Log2} :: \ \log^2\left(m_\pi^2\right)$$

$$\text{m2Log} :: \ m_\pi^2 \log\left(m_\pi^2\right)\ .$$

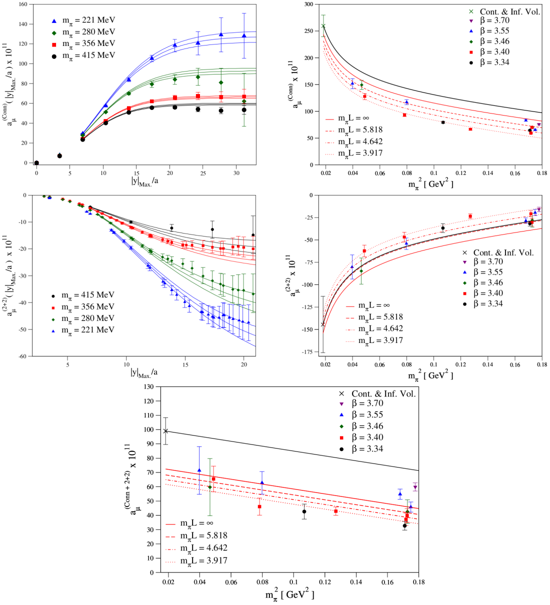


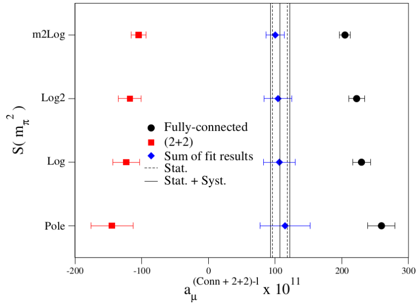
For the total contribution:

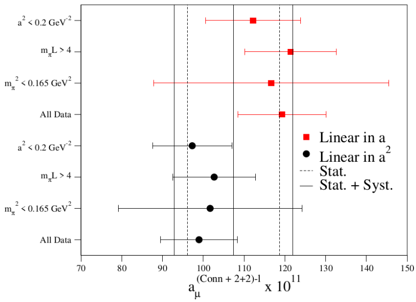
$$a_\mu(m_\pi^2, m_\pi L, a^2) = a_\mu(0, \infty, 0)(1 + Am_\pi^2 + Be^{-m_\pi L/2} + Ca^2), \qquad (23)$$

- Pion masses are heavier than physical value and Chiral extrapolation is used. Result depends on the form of Chiral extrapolation.
- Long distance contribution obtained by fitting an ansatz: $f(|y|) = |y|^3 A e^{-B|y|}$. Result depends on the form of the ansatz.

HLbL: Mainz 2021

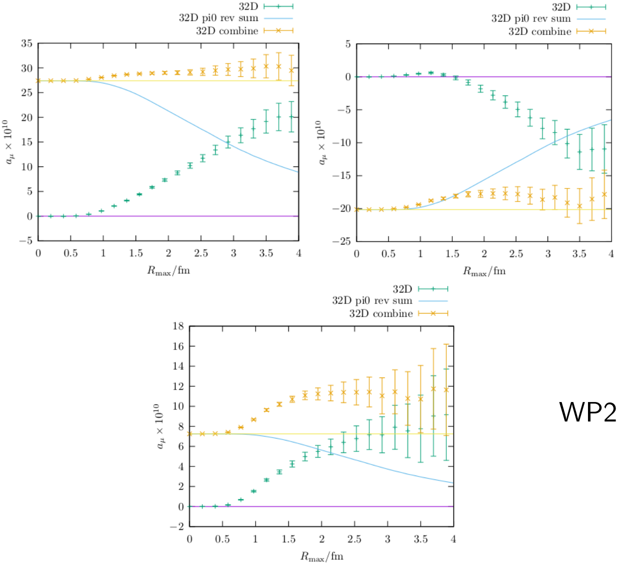






As an estimate of the systematic error, we compute the root-mean-squared deviation of the fit results y_i compared to the average result $\bar{y}$, i.e. $(\sum_{i=1}^{N}(y_i - \bar{y})^2/N)^{1/2}$. We finally end up with a value of

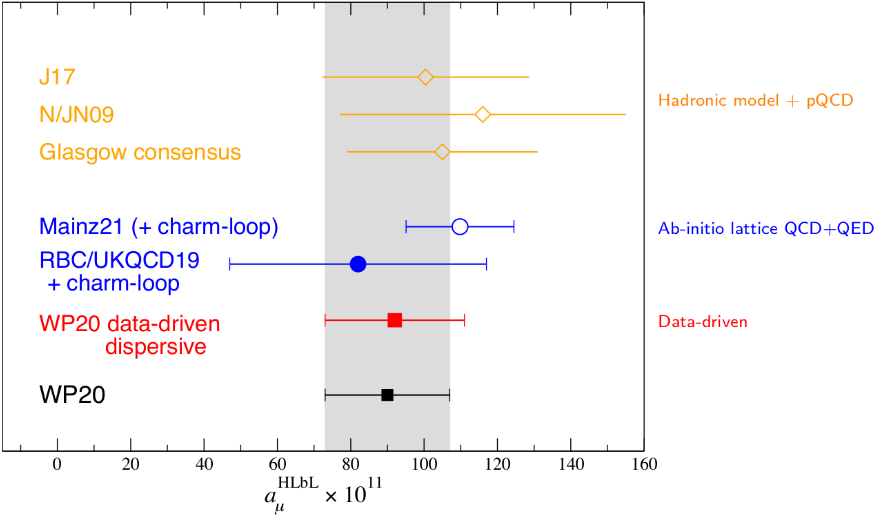
$$a_\mu^{(\text{Conn.}+(2+2))\text{-}l} = 107.4(11.3)(9.2) \times 10^{-11}, \tag{28}$$



WP2020

Figure 91: Combined lattice and pion-pole contributions to the HLbL scattering part of the muon anomaly. Partial sums for the hadronic contributions, connected (top-left), leading disconnected (top-right), and total (bottom), computed with QED$_\infty$. a^{-1} = 1 GeV, L = 6.4 fm, and M_π = 142 MeV. Lines denote the π^0-pole contribution computed from the LMD model and are summed right-to-left.

Status of hadronic light-by-light contribution



Outline

Summary

- The errors of lattice QCD calculations comes from:
  1. finite statistics $\rightarrow$ statistical error
  2. non-zero lattice spacing $\rightarrow$ discretization error
  3. finite lattice size $\rightarrow$ finite volume error
  4. non-physical pion mass $\rightarrow$ chiral extrapolation
     Many lattice calculations are now performed with physical pion mass, eliminating this source of the systematic errors.
- Lattice QCD calculation is playing important role in determining the hadronic contribution to muon $g - 2$ and many other physical observables.
- More accurate lattice results are expected when Fermilab releases their final result.

Thank You!