Amplitudes for Monopoles

Ofri Telem (UC Berkeley)
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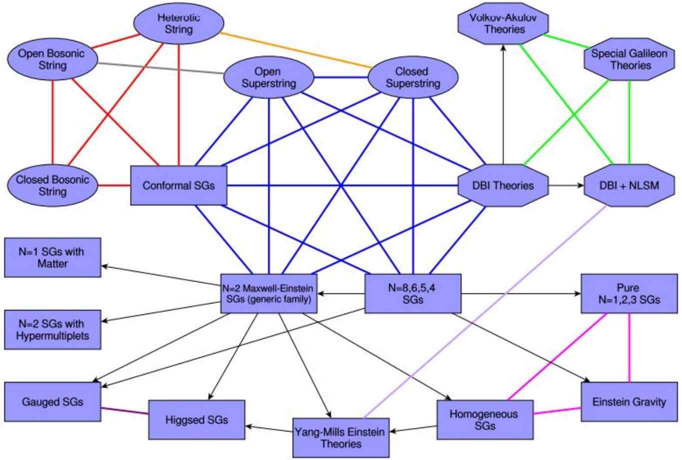
Collaborators: C. Csáki, S. Hong, Y. Shirman, J. Terning, M. Waterbury

Motivation:
On-Shell Success Where Field Theory Fails



* Image taken from Bern et al. arXiv 1909.01358

Success of the On-Shell Program

- The on-shell program addresses relativistic quantum physics without referring to an action
- Many recent cutting edge results, for example:
  - Six gluon planar N=4 SYM @ 6 and 7 Loops   Caron-Hout, Dixon, et al ‘19
  - Non-renormalization and operator mixing in SMEFT   Bern, Parra-Martinez, Sawyer ‘20
  - Black Hole Binary Dynamics   Bern, Cheung, et al ‘19, ...
  - Cosmological bootstrap   Arkani-Hamed, Baumann, et al ‘18
  - Massless amplitudes beyond polylogarithms   Bourjaily, McLeod, et al ‘18

  … and many more

The On-Shell Program - Faster, Stronger or also *Deeper?*

- A key question is if the on-shell program allows for a *deeper* understanding of nature, which cannot be seen in conventional Field Theory

- Some very suggestive hints:
    - Color-Kinematics duality and the Double copy — Bern, Carrasco, Johansson '08

      (Gravity = YM^2 and other relations) — Bern, Carrasco, et al. '19 .... many more

      Classical Double Copy — Monteiro, O'connell, White '14 ....
    - Dual conformal invariance — Drummond, Henn et al. '08
    - Amplituhedra — Arkani-Hamed, Trnka '13 ...
    - "Hard" S-matrix — Hannesdottir, Schwartz '19

Monopoles: Where "No" Lagrangian Exists

- Since the days of Dirac, no clear way to write a local, Lorentz invariant Lagrangian for Monopoles & electric charges
  - Schwinger approach: non-local Lagrangian — Schwinger '66
  - Zwanziger approach: local Lagrangian, — Zwanziger '71

    loss of manifest Lorentz by introducing Dirac string

- Weinberg's Paradox:
  - Amplitude for charge monopole 1-photon exchange explicitly breaks Lorentz! — Weinberg '65
  - Resolution: in EM soft corrections to a hard scattering, Lorentz violation exponentiates. For closed trajectories: is an integer & drops out — Terning, Verhaaren '19

Monopoles: an On-Shell Opportunity

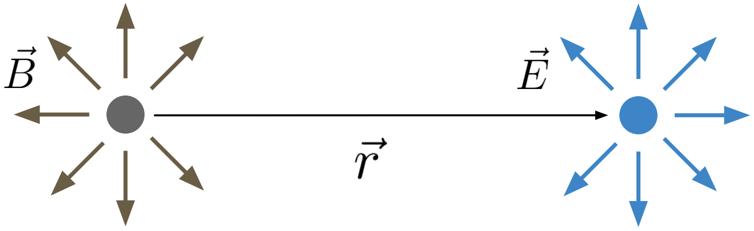
- The S-matrix for charge-monopole scattering is local and Lorentz invariant, but we cannot see this in the field theory language

- The S-matrix has to be “special” in some way, otherwise why no Lagrangian?

- Dirac quantization should play a leading role
  - $q \equiv e\,g$ is half integer. Other half integers for the S-matrix? - Spins and helicities!
  - Helcities & spins are associated with 1 particle states
  - $q \equiv e\,g$ associated with charge-monopole pairs

“*pairwise*” helicity?

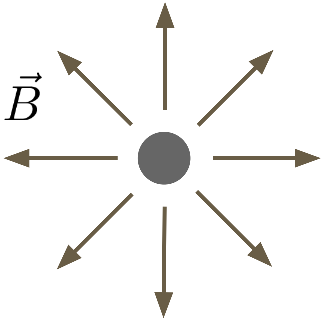
Charge - Monopole Scattering: A Non-Relativistic Prelude



Magnetic Monopoles

Sources of U(1) field* with non-trivial winding number $\pi_1[\mathrm{U}(1)] = \mathbb{Z}$

$$\vec{B}_{\mathrm{U}(1)} = \frac{g}{r^2}\,\hat{r}$$



- At $r \gg m^{-1}$ effectively abelian Dirac '31
- At $r \sim m^{-1}$ have non-abelian cores 't Hooft / Polyakov '74

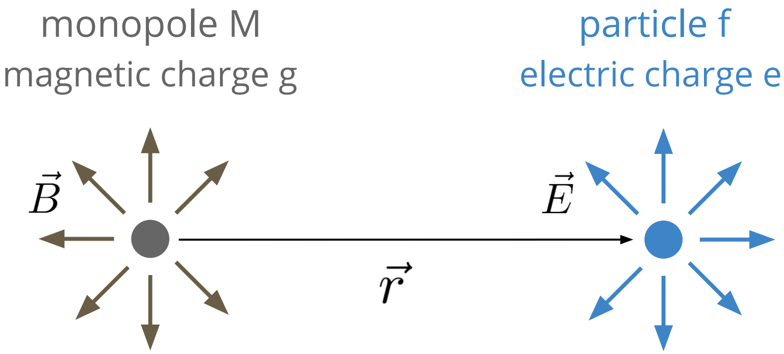
We won't care.
For us they are just scattering particles.

- Lead to charge quantization Dirac '31, Wu & Yang '76

* In this talk we only consider these

Classically: An Extra Angular Momentum

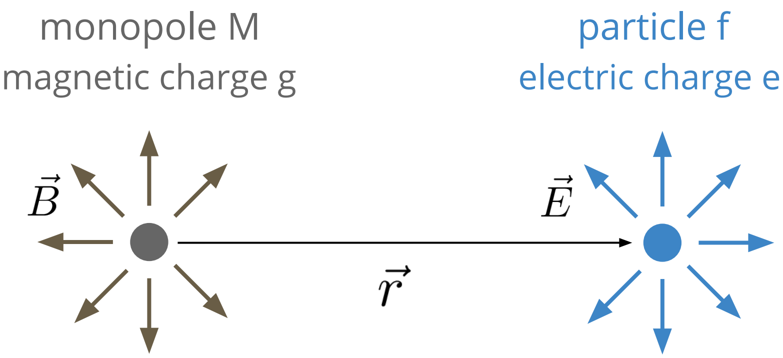
- In the presence of electrically and magnetically charged particles there's a *catch*



- The E&M field has angular momentum, even at infinite separation!
- Have to include this extra angular momentum in the quantum theory

Classically: An Extra Angular Momentum

Thomson 1904



$$\vec{J}_{\text{field}} = \frac{1}{4\pi}\int \vec{r}\,' \times \left(\vec{E}\times\vec{B}\right) d^3r\,' = -\frac{g}{4\pi}\int \left(\vec{\nabla}\,'\cdot\vec{E}\right)\hat{r}\,' d^3r' = -eg\hat{r}$$

Distance independent!

In the quantum theory $\vec{J}_{\text{field}}$ quantized $\longrightarrow$ $eg = \frac{n}{2}$ Dirac quantization

Non-Relativistic Quantum Theory

$$H = -\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2 + V(r) = -\frac{1}{2m}\vec{D}^2 + V(r)$$

where $\vec{D} = \vec{\nabla} - ie\vec{A}$ and A is the vector potential from a monopole at r=0

Need two patches to define A : $A_\phi = \frac{\pm g}{r\sin\theta}(1 \mp \cos\theta)$

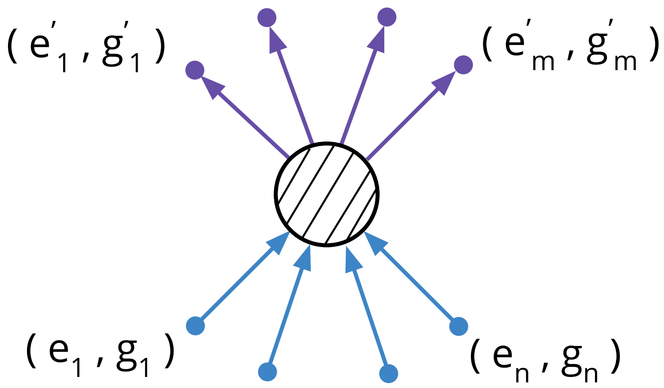
- Naive $\vec{L} = -i\vec{r} \times \vec{D}$ no longer satisfies angular momentum algebra, instead Lipkin et al. '69

$$\vec{L} = -i\vec{r} \times \vec{D} - eg\hat{r} = m\vec{r} \times \dot{\vec{r}} - eg\hat{r}$$

is the conserved angular momentum operator $\longrightarrow eg = \frac{n}{2}$ Dirac quantization

- For dyons, trivial generalization: $e_1 g_2 - e_2 g_1 = \frac{n}{2}$ Zwanziger '68, Schwinger '69

The S-Matrix for Charges, Monopoles and Dyons*



* will use the words charge, monopole and dyon interchangeably = a particle with electric and/or magnetic charges

Main Idea

- Charge-monopole scattering inherently non-perturbative: eg=n/2
  - The physical scattering amplitude involves all photon exchanges between the charge and the monopole
  - The sum of all exchanges should mimic the angular momentum in the classical EM field
- Our approach:
  - Instead of summing over all photon exchanges, re-define the Hilbert space to “integrate out” the classical EM
  - In the modified Hilbert space, the extra angular momentum is associated with charge monopole pairs
  - The S-matrix for the modified Hilbert space captures quantum scattering with nontrivial classical “magnetic” angular momentum

Plan

- The manifestly relativistic, electric-magnetic S-matrix
  - Pairwise little group and pairwise helicity
  - The extra LG phase of the magnetic S-matrix
  - Pairwise spinor-helicity variables
  - Electric Magnetic amplitudes: a cheat sheet

- Results
  - All 3-pt electric-magnetic amplitudes. Novel selection rules.
  - LG covariant partial wave decomposition
  - Charge-monopole scattering:

    Helicity-flip selection rule at lowest partial wave

    Higher partial waves: monopole spherical harmonics

Defining Relativistic Quantum States

Wigner '37
Bargmann & Wigner '48

- Relativistic Quantum states are defined via their irreducible representations under Poincaré

- Hard to explore irreps. of non-compact Poincaré group

- Single particle solution: Poincaré irreps. ≃ little group irreps.

Little group = compact subgroup of Lorentz which leaves a reference momentum invariant

Massive irreps. : $k = (m, 0, 0, 0) \longrightarrow$ little group SU(2), particles labeled by spin

Massless irreps. : $k = (E, 0, 0, E) \longrightarrow$ little group U(1)*, particles labeled by helicity

$U(\Lambda)\,|p;\sigma\rangle$ induced from $D(W)_{\sigma'\sigma}\,|k;\sigma'\rangle$

Lorentz irrep. Little group irrep.

- Mutiparticle states? Usually tensor products of single particle states

The Quantum State of Scalar Monopole & Charge

Zwanziger '72

- How does Lorentz act on a 2-particle state with a scalar monopole and a scalar charge?
  - Naively, because they are scalars:

$$U(\Lambda)\,|p_1,\, p_2\rangle = |\Lambda p_1,\, \Lambda p_2\rangle$$

    can't be true because that implies no $q_{12} \equiv e_1 g_2 - e_2 g_1$ contribution to the angular momentum

  - Instead:

$$U(\Lambda)\,|p_1, p_2; q_{12}\rangle = e^{i\, q_{12}\, \phi(p_1, p_2, \Lambda)}\,|\Lambda p_1, \Lambda p_2; q_{12}\rangle$$

    where φ is a *pairwise* little group phase associated with *both* momenta

- This is clearly the right definition as it assigns an extra angular momentum associated with the *half-integer* q_{12}, but we can also derive it by generalizing Wigner's method of induced representations

Wigner's Method for Scalar Charge-Monopole States

Zwanziger '72

- Define the reference momenta in the COM frame

$$(k_1)_\mu = (E_1^c, 0, 0, +p_c)$$
$$(k_2)_\mu = (E_2^c, 0, 0, -p_c)$$

with

$$p_c = \sqrt{\frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{s}}$$
$$E_{1,2}^c = \sqrt{m_{1,2}^2 + p_c^2}$$

Definition: Pairwise Little Group (LG) - All Lorentz transformations which leave both $k_{1,2}$ invariant

- Always just a U(1) - rotations around the z-axis
- We label charge-monopole pairs by their pairwise LG charge q_{12}
- $q_{12} \equiv e_1 g_2 - e_2 g_1$ by matching to NR limit

$$U\left[R_z(\phi)\right] \left|k_1, k_2\ ;\ q_{12}\right\rangle \equiv e^{iq_{12}\,\phi} \left|k_1, k_2\ ;\ q_{12}\right\rangle$$

Wigner's Method for Scalar Charge-Monopole States

Zwanziger '72

- Define canonical Lorentz transformation L_p as the COM → Lab transformation

$$p_1 = L_p k_1 \qquad p_2 = L_p k_2$$

- Wigner's trick:

$$\begin{aligned} U(\Lambda)\,|p_1, p_2\,;\,q_{12}\rangle &= U\left(L_{\Lambda p}\right)\,U\left(L^{-1}_{\Lambda p}\Lambda L_p\right)|k_1, k_2\,;\,q_{12}\rangle \\ &= U\left(L_{\Lambda p}\right)\underbrace{U\left(W_{k_1,k_2}\right)}_{\text{Pairwise LG rotation}}|k_1, k_2\,;\,q_{12}\rangle \end{aligned}$$

So that:

$$\boxed{U(\Lambda)\,|p_1, p_2; q_{12}\rangle = e^{i\,q_{12}\,\phi(p_1,p_2,\Lambda)}\,|\Lambda p_1, \Lambda p_2\,;\,q_{12}\rangle}$$

Where $R_z(\phi) \equiv L^{-1}_{\Lambda p}\,\Lambda\,L_p\,.$ This is the *electric-magnetic two scalar state*

Electric-Magnetic Multiparticle States

hep-th/2010.13794

- We can easily generalize the two scalar state to arbitrary *electric-magnetic multiparticle states*

$$U(\Lambda)\,|p_1,\dots,p_n\,;\,\sigma_1,\dots,\sigma_n\,;\,q_{12},q_{13},\dots q_{n-1,n}\rangle =$$

$$\underbrace{e^{i\sum_{i<j} q_{ij}\phi(p_i,p_j,\Lambda)}}_{\text{Pairwise LG}}\prod_{i=1}^{n}\underbrace{\mathcal{D}^i_{\sigma_i'\sigma_i}}_{\text{Single particle LG}}\,|\Lambda p_1,\dots,\Lambda p_n\,;\,\underbrace{\sigma_1',\dots,\sigma_n'}_{\text{Spins / helicities}}\,;\,\underbrace{q_{12},q_{13},\dots,q_{n-1,n}}_{\text{Pairwise helicities}}\rangle$$

where $\mathcal{D}^i_{\sigma_i'\sigma_i}$ are the matrices (phases) for each single particle massive (massless) LG

- Electric-magnetic multiparticle states are *not* direct products of single particle states!
- This is just the right amount of “non-locality” to explain the absence of a Lagrangian description

Electric-Magnetic Multiparticle States: Derivation

hep-th/2010.13794

- Consider n single-particle Hilbert spaces H_i + ½n(n-1) two-particle Hilbert spaces H_{ij}
- Each H_i and H_{ij} carry a representation of a *different copy* of Poincaré [½n(n+1) copies]
  - The H_i carry single-particle representations $U(\Lambda)\,|p_i\,;\,\sigma_i\rangle = \mathcal{D}_{\sigma_i'\sigma_i}\,|\Lambda p_i\,;\,\sigma_i'\rangle$
  - The H_{ij} carry two-scalar-dyon representations $U(\Lambda)\,|\tilde{p}_i,\,\tilde{p}_j;\,q_{ij}\rangle = e^{iq_{ij}\phi_{ij}}\,|\tilde{p}_i,\,\tilde{p}_j;\,q_{ij}\rangle$
- Define the direct sum Hilbert space:

$$\mathcal{H}_\oplus \equiv \bigoplus_{i=1}^{n} \mathcal{H}_i \oplus \bigoplus_{i<j}^{n} \mathcal{H}_{ij} = \{|p_1, \ldots p_n\,;\,(\tilde{p}_1, \tilde{p}_2), \ldots, (\tilde{p}_l, \tilde{p}_k)\,;\,\sigma_i\,;\,q_{ij}\rangle\}$$

- For a general state in $\mathcal{H}_\oplus$, $\tilde{p}_i \neq p_i$

Electric-Magnetic Multiparticle States: Derivation

- Define the physical subspace

invariant under diagonal Poincaré

$$\mathcal{H} = \{|p_1, \ldots p_n \, ; \, (\tilde{p}_1, \tilde{p}_2), \ldots, (\tilde{p}_l, \tilde{p}_k) \, ; \, \sigma_i \, ; \, q_{ij}\rangle \;|\; \tilde{p}_i = p_i\} \subset \mathcal{H}_{\oplus}$$

Electric-Magnetic Multiparticle States: Derivation

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- Define the physical subspace

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$$\mathcal{H} = \{|p_1, \ldots p_n\,;\, (\tilde{p}_1, \tilde{p}_2), \ldots, (\tilde{p}_l, \tilde{p}_k)\,;\, \sigma_i\,;\, q_{ij}\rangle \;|\; \tilde{p}_i = p_i\} \subset \mathcal{H}_\oplus$$

rename

$$|p_1, \ldots, p_n\,;\, \sigma_1, \ldots, \sigma_n\,;\, q_{12}, q_{13}, \ldots q_{n-1,n}\rangle$$

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$$\mathcal{H} = \{|p_1, \ldots p_n\,;\, (\tilde{p}_1, \tilde{p}_2), \ldots, (\tilde{p}_l, \tilde{p}_k)\,;\, \sigma_i\,;\, q_{ij}\rangle \;|\, \tilde{p}_i = p_i\} \subset \mathcal{H}_\oplus$$

rename

$$|p_1, \ldots, p_n\,;\, \sigma_1, \ldots, \sigma_n\,;\, q_{12}, q_{13}, \ldots q_{n-1,n}\rangle$$

- $\mathcal{H}$ naturally carries a "magnetic" multiparticle representation under the diagonal Poincaré group

$$U(\Lambda)\,|p_1, \ldots, p_n\,;\, \sigma_1, \ldots, \sigma_n\,;\, q_{12}, q_{13}, \ldots q_{n-1,n}\rangle =$$

$$\underbrace{e^{i\sum_{i<j} q_{ij}\phi(p_i,p_j,\Lambda)}}_{\text{Pairwise LG}} \prod_{i=1}^{n} \underbrace{\mathcal{D}^i_{\sigma'_i\sigma_i}}_{\text{Single particle LG}} |\Lambda p_1, \ldots, \Lambda p_n\,;\, \underbrace{\sigma'_1, \ldots, \sigma'_n}_{\text{Spins / helicities}}\,;\, \underbrace{q_{12}, q_{13}, \ldots, q_{n-1,n}}_{\text{Pairwise helicities}}\rangle$$

The Electric-Magnetic S-Matrix

- To define the S-matrix, we define electric-magnetic in- and out- states as

\+ for 'in' - for 'out'

$$U(\Lambda)\;|p_1,\ldots,p_n\,;\,\pm\rangle = \prod_i \mathcal{D}\,(W_i)\;|\Lambda p_1,\ldots,\Lambda p_n\,;\,\pm\rangle\;e^{\pm i\Sigma}$$

Where $\Sigma \equiv \sum_{i>j}^{n} q_{ij}\,\phi\,(p_i,p_j,\Lambda).$

note the ±

- The ± for the pairwise LG phase of the in / out state is very important!

- Has to be there to reproduce the angular momentum in the E&M field in the classical limit:

$$M^{\nu\rho}_{\text{field }\,;\pm} \;=\; \pm\sum_{i>j} q_{ij}\,\frac{\epsilon^{\nu\rho\alpha\beta}p_{i\alpha}p_{j\beta}}{\sqrt{(p_i\cdot p_j)^2 - m_i^2m_j^2}}$$

Zwanziger '72

The Electric-Magnetic S-Matrix

- The S-matrix then transforms as:

$$S\left(p_1',\ldots,p_m' \mid p_1,\ldots,p_n\right) \equiv \langle p_1',\ldots,p_m';- \mid p_1,\ldots,p_n;+\rangle$$

$$= \langle p_1',\ldots,p_m';- \left| U(\Lambda)^\dagger U(\Lambda) \right| p_1,\ldots,p_n;+\rangle$$

$$= e^{i(\Sigma_+ + \Sigma_-)} \prod_{i=1}^{m} \mathcal{D}\left(W_i\right)^\dagger \prod_{j=1}^{n} \mathcal{D}\left(W_j\right)\ S\left(\Lambda p_1',\ldots,\Lambda p_m' \mid \Lambda p_1,\ldots,\Lambda p_n\right)$$

with $$\Sigma_+ \equiv \sum_{i>j}^{n} q_{ij}\,\phi(p_i,p_j,\Lambda) \quad , \quad \Sigma_- \equiv \sum_{i>j}^{m} q_{ij}\,\phi(p_i',p_j',\Lambda)\,.$$

- The extra *pairwise LG phase* is the key element in our formalism
- Every electric-magnetic S-matrix must transform with this phase by construction!

Plan

- The manifestly relativistic, electric-magnetic S-matrix
  - ✓ Pairwise little group and pairwise helicity
  - ✓ The extra LG phase of the magnetic S-matrix
  - Pairwise spinor-helicity variables
  - Electric Magnetic amplitudes: a cheat sheet

- Results
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  - Charge-monopole scattering:

    Helicity-flip selection rule at lowest partial wave

    Higher partial waves: monopole spherical harmonics

The Standard Spinor-Helicity Formalism

De Causmaecker et al. '82
Parke, Taylor '86
....
Arkani-Hamed at al. '17

- In the standard massless/massive spinor-helicity formalism, scattering amplitudes are formed from spinor helicity variables transforming covariantly under the single particle LGs

Massless:

$$\underbrace{\Lambda_\alpha^{\ \beta}}_{\text{Lorentz trans.}} |p_i\rangle_\beta = \underbrace{e^{+\frac{i}{2}\phi(p_i,\Lambda)}}_{\text{LG phase}} |\Lambda p_i\rangle_\alpha \,, \qquad [p_i|_{\dot\beta} \underbrace{\tilde\Lambda^{\dot\beta}_{\ \dot\alpha}}_{\text{Lorentz trans.}} = \underbrace{e^{-\frac{i}{2}\phi(p_i,\Lambda)}}_{\text{LG phase}} [\Lambda p_i|_{\dot\alpha}$$

Massive:

$$\underbrace{\Lambda_\alpha^{\ \beta}}_{\text{Lorentz trans.}} |\mathbf{p_i}\rangle_\beta^I = \underbrace{\mathcal{D}^I_J(W_i)}_{\text{LG SU(2)}} |\mathbf{\Lambda p_i}\rangle_\alpha^J \,, \qquad [\mathbf{p_i}|_{I\dot\beta} \underbrace{\tilde\Lambda^{\dot\beta}_{\ \dot\alpha}}_{\text{Lorentz trans.}} = \underbrace{\mathcal{D}^{\dagger J}_I(W_i)}_{\text{LG SU(2)}} [\mathbf{\Lambda p_i}|_{J\dot\alpha}$$

New Building Blocks for the S-Matrix: Pairwise Spinors

- Can't saturate the S-matrix pairwise LG phase with the standard spinors

- Need new *pairwise* spinors transforming covariantly under pairwise LG
    - Associated with pairs of momenta
    - Have U(1) phase even if momenta are massive

- Idea: define null linear combinations of every pair (p_i , p_j) and decompose into massless spinors

New Building Blocks for the S-Matrix: Pairwise Spinors

The particles could be massive!

- In the COM frame for every pair, define *null* reference momenta:

$$\left(k_{ij}^{\flat\pm}\right)_\mu = p_c\,(1,0,0,\pm 1) \qquad p_c = \sqrt{\frac{p_i\cdot p_j - m_i^2\,m_j^2}{s}}$$

COM momentum

- We can boost $k^\flat_{ij}$ to get $p^\flat_{ij}$ in the lab frame, which are null linear combinations of p_i and p_j

$$p_{ij}^{\flat+} = \frac{1}{E_i^c + E_j^c}\left[\left(E_j^c + p_c\right)p_i - \left(E_i^c - p_c\right)p_j\right]$$

$$p_{ij}^{\flat-} = \frac{1}{E_i^c + E_j^c}\left[\left(E_i^c + p_c\right)p_j - \left(E_j^c - p_c\right)p_i\right]$$

- By linearity, $L_p\,k_{ij}^{\flat\pm} = p_{ij}^{\flat\pm}$ where L_p is the same canonical transformation which takes $k_i \to p_i$, $k_j \to p_j$. Our pairwise spinors will have the same LG phase as the S-matrix

New Building Blocks for the S-Matrix: Pairwise Spinors

We can now define reference pairwise spinors as the “square roots” of the reference pairwise momenta

$$\left|k_{ij}^{\flat +}\right\rangle_{\alpha} = \sqrt{2p_c}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \left|k_{ij}^{\flat -}\right\rangle_{\alpha} = \sqrt{2p_c}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left[k_{ij}^{\flat +}\right|_{\dot{\alpha}} = \sqrt{2p_c}\,(1 \quad 0) \quad , \quad \left[k_{ij}^{\flat -}\right|_{\dot{\alpha}} = \sqrt{2p_c}\,(0 \quad 1)$$

so that $$k_{ij}^{\flat \pm} \cdot \sigma_{\alpha\dot{\alpha}} = |k_{ij}^{\flat \pm}\rangle_{\alpha} \left[k_{ij}^{\flat \pm}\right|_{\dot{\alpha}}$$

This mirrors the definition of regular spinor-Helicity variables, only with pairwise momenta.

New Building Blocks for the S-Matrix: Pairwise Spinors

- In the lab frame, we define

$$\left|p_{ij}^{\flat\pm}\right\rangle_\alpha = \underbrace{(\mathcal{L}_p)_\alpha^{\ \beta}}_{\text{Canonical Lorentz}} \left|k_{ij}^{\flat\pm}\right\rangle_\beta \quad , \quad \left[p_{ij}^{\flat\pm}\right|_{\dot\alpha} = \left[k_{ij}^{\flat\pm}\right|_{\dot\beta} \underbrace{\left(\tilde{\mathcal{L}}_p\right)^{\dot\beta}_{\ \dot\alpha}}_{\text{Canonical Lorentz}}$$

- By another “Wigner trick” we get

$$\Lambda_\alpha^{\ \beta}\left|p_{ij}^{\flat\pm}\right\rangle_\beta = e^{\pm\frac{i}{2}\phi(p_i,p_j,\Lambda)}\left|\Lambda p_{ij}^{\flat\pm}\right\rangle_\alpha, \quad \left[p_{ij}^{\flat\pm}\right|_{\dot\beta}\tilde{\Lambda}^{\dot\beta}_{\ \dot\alpha} = e^{\mp\frac{i}{2}\phi(p_i,p_j,\Lambda)}\left[\Lambda p_{ij}^{\flat\pm}\right|_{\dot\alpha}$$

2 pairs of spinors transforming covariantly under pairwise LG, with opposite weights

- Now we have everything we need to construct electric-magnetic amplitudes!

New Building Blocks for the S-Matrix: Pairwise Spinors

- By definition, in the $m_i \to 0$ limit, the pairwise spinors approach the regular spinors,

$$\left|p_{ij}^{\flat +}\right\rangle_\alpha = |i\rangle_\alpha \quad , \quad \left[p_{ij}^{\flat +}\right|_{\dot\alpha} = [i|_{\dot\alpha}$$

$$\left|p_{ij}^{\flat -}\right\rangle_\alpha = \underbrace{\sqrt{2p_c}\,|\hat\eta_i\rangle_\alpha}_{\text{“P-conjugate” of } |i\rangle} \quad , \quad \left[p_{ij}^{\flat -}\right|_{\dot\alpha} = \underbrace{\sqrt{2p_c}\,[\hat\eta_i|_{\dot\alpha}}_{\text{“P-conjugate” of } [i|}$$

- This will imply extra selection rules in the $m_i \to 0$ limit, since

$$\left[p_{ij}^{\flat +} i\right] = \left\langle i p_{ij}^{\flat +}\right\rangle = \left[\hat\eta_i p_{ij}^{\flat -}\right] = \left\langle p_{ij}^{\flat -}\hat\eta_i\right\rangle = 0$$

$$\left[p_{ij}^{\flat -} i\right] = \left\langle i p_{ij}^{\flat -}\right\rangle = \left[\hat\eta_i p_{ij}^{\flat +}\right] = \left\langle p_{ij}^{\flat +}\hat\eta_i\right\rangle = 2p_c$$

In particular, it will impose a mandatory helicity-flip in the lowest partial wave for charge-monopole scattering. Stay tuned!

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    Higher partial waves: monopole spherical harmonics

Constructing Electric-Magnetic Amplitudes

- We showed that the electric-magnetic S-matrix transforms as

$$S\left(\Lambda p_1', \ldots, \Lambda p_m' \mid \Lambda p_1, \ldots, \Lambda p_n\right) =$$

$$e^{-i(\Sigma_- + \Sigma_+)} \prod_{i=1}^{m} \mathcal{D}\left(W_i\right) \prod_{j=1}^{n} \mathcal{D}\left(W_j\right)^\dagger \; S\left(p_1', \ldots, p_m' \mid p_1, \ldots, p_n\right)$$

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In practice we work in the *all-outgoing* convention:
Have to flip helicity, but not pairwise helicity!

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- 1st surprise: remember the beginning of every QFT textbook?

$$S_{\alpha\beta} = \delta(\alpha - \beta) - 2i\pi\delta^{(4)}\left(p_\alpha - p_\beta\right)\mathcal{A}_{\alpha\beta}$$

Constructing Electric-Magnetic Amplitudes

- We showed that the electric-magnetic S-matrix transforms as*

$$S\left(\Lambda p_1', \ldots, \Lambda p_m' \mid \Lambda p_1, \ldots, \Lambda p_n\right) =$$

$$e^{-i(\Sigma_- + \Sigma_+)} \prod_{i=1}^{m} \mathcal{D}\left(W_i\right) \prod_{j=1}^{n} \mathcal{D}\left(W_j\right)^{\dagger} \, S\left(p_1', \ldots, p_m' \mid p_1, \ldots, p_n\right)$$

In practice we work in the *all-outgoing* convention:
Have to flip helicity, but not pairwise helicity!

- 1st surprise: remember the beginning of every QFT textbook?

$$S_{\alpha\beta} = \cancel{\delta(\alpha - \beta)} - 2i\pi\delta^{(4)}\left(p_\alpha - p_\beta\right) \mathcal{A}_{\alpha\beta}$$

doesn't transform with the pairwise LG phase!

Forward scattering (i.e. no scattering) not an option for the electric-magnetic S-matrix!

Electric-Magnetic Amplitudes: a Cheat-Sheet

- To construct electric-magnetic amplitudes, contract standard and pairwise spinors to get the right overall LG transformation. The rules are:

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\lvert i\rangle_\alpha$, $[i\rvert_{\dot\alpha}$	$-\frac{1}{2}$, $\frac{1}{2}$	—	—
$\langle\mathbf{i}\rvert^{I;\alpha}$	—	$\square$	—
$\left\lvert p_{ij}^{\flat+}\right\rangle_\alpha$, $\left[p_{ij}^{\flat+}\right\rvert_{\dot\alpha}$	—	—	$-\frac{1}{2}$, $\frac{1}{2}$
$\left\lvert p_{ij}^{\flat-}\right\rangle_\alpha$, $\left[p_{ij}^{\flat-}\right\rvert_{\dot\alpha}$	—	—	$\frac{1}{2}$, $-\frac{1}{2}$

- This will enable us to completely fix the angular dependence of amplitudes from LG and pairwise LG considerations. The dynamical information left unfixed is just like phase shifts in QM.

- Our results are fully *non-perturbative*, as we never rely on a perturbative expansion

Electric-Magnetic Amplitudes: Examples

- To construct electric-magnetic amplitudes, contract standard and pairwise spinors to get the right overall LG transformation

- 1st example: Massive fermion decaying to massive fermion + massless scalar, q = e g = -1

$$S\left(\mathbf{1}^{s=1/2}\,|\,\mathbf{2}^{s=1/2},3^0\right)_{q_{23}=-1} \sim \left\langle p_{23}^{\flat -}\,\mathbf{1}\right\rangle\left\langle p_{23}^{\flat -}\,\mathbf{2}\right\rangle$$

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\|i\rangle_\alpha\,,\ [i\|_{\dot\alpha}$	$-\frac{1}{2}\,,\ \frac{1}{2}$	–	–
$\langle\mathbf{i}\|^{I;\alpha}$	–	$\square$	–
$\left\|p_{ij}^{\flat +}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat +}\right\|_{\dot\alpha}$	–	–	$-\frac{1}{2}\,,\ \frac{1}{2}$
$\left\|p_{ij}^{\flat -}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat -}\right\|_{\dot\alpha}$	–	–	$\frac{1}{2}\,,\ -\frac{1}{2}$

Electric-Magnetic Amplitudes: Examples

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\lvert i\rangle_\alpha\,,\ [i\rvert_{\dot\alpha}$	$-\frac{1}{2}\,,\ \frac{1}{2}$	–	–
$\langle \mathbf{i}\rvert^{I;\alpha}$	–	$\square$	–
$\lvert p_{ij}^{\flat+}\rangle_\alpha\,,\ [p_{ij}^{\flat+}\rvert_{\dot\alpha}$	–	–	$-\frac{1}{2}\,,\ \frac{1}{2}$
$\lvert p_{ij}^{\flat-}\rangle_\alpha\,,\ [p_{ij}^{\flat-}\rvert_{\dot\alpha}$	–	–	$\frac{1}{2}\,,\ -\frac{1}{2}$

- 2nd example: Massive fermion decaying to massive scalar + massless vector, q = e g = -1

$$\boxed{S\left(\mathbf{1}^{s=0}\,|\,\mathbf{2}^{s=0},3^{+1}\right)_{q_{23}=-1} \sim \left[p_{23}^{\flat+}\,3\right]^2 \sim \left\langle p_{23}^{\flat-}|2|3\right]^2}$$

what about the -1 helicity case for the vector?

- No way to write a LG covariant expression, since $\left\langle p_{23}^{\flat-}\,3\right\rangle = \left[p_{23}^{\flat+}\,2\right] = 0\,.$
- Our first encounter with a *pairwise LG selection rule*

Electric-Magnetic Amplitudes: Examples

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\lvert i\rangle_\alpha$, $[i\rvert_{\dot\alpha}$	$-\frac{1}{2}$, $\frac{1}{2}$	–	–
$\langle \mathbf{i}\rvert^{I;\alpha}$	–	$\square$	–
$\lvert p_{ij}^{\flat +}\rangle_\alpha$, $[p_{ij}^{\flat +}\rvert_{\dot\alpha}$	–	–	$-\frac{1}{2}$, $\frac{1}{2}$
$\lvert p_{ij}^{\flat -}\rangle_\alpha$, $[p_{ij}^{\flat -}\rvert_{\dot\alpha}$	–	–	$\frac{1}{2}$, $-\frac{1}{2}$

- 3rd example: Massive vector decaying to to different massless fermions, q = e g = -1

$$S\left(\mathbf{1}^{s=1}\,|\,2^{-1/2},3^{-1/2}\right)_{q_{23}=-1} \sim \left\langle 2p_{23}^{\flat-}\right\rangle\left\langle p_{23}^{\flat+}\,3\right\rangle\left\langle \mathbf{1}\,p_{23}^{\flat-}\right\rangle^2$$

- Here the number of pairwise spinors is not -2q
- We need 4 pairwise spinors to contract with 4 standard spinors
- We use 3 pairwise spinors with (pairwise) LG weight ½ and on with -½
- $h_2 = - h_3 = ½$ case forbidden by selection rule

Can we systematize this? Yes!

Plan

- The manifestly relativistic, electric-magnetic S-matrix
  - ✓ Pairwise little group and pairwise helicity
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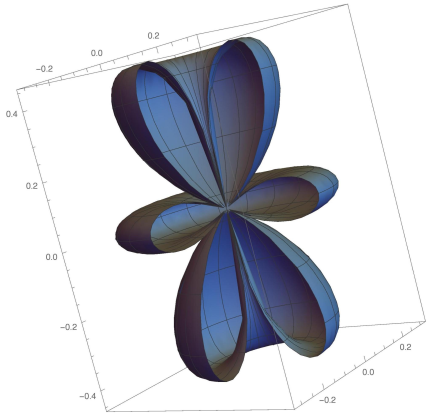
- Results
  - All 3-pt electric-magnetic amplitudes. Novel selection rules.
  - LG covariant partial wave decomposition
  - Charge-monopole scattering:

    Helicity-flip selection rule at lowest partial wave

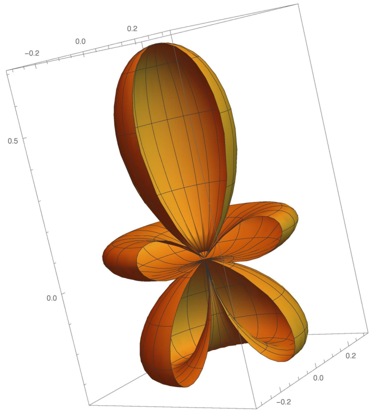
    Higher partial waves: monopole spherical harmonics

Results



$$Y_{\frac{5}{2},-\frac{1}{2}}(\theta,\phi)$$

Spherical Harmonics



$${}_{\frac{1}{2}}Y_{\frac{5}{2},-\frac{1}{2}}(\theta,\phi)$$

Monopole - Spherical Harmonics

All 3-pt Electric-Magnetic Amplitudes

- Pairwise LG + individual LGs allow us to classify all 3-pt amplitudes
    - This generalizes the massive amplitude formalism by Arkani-Hamed at al. '17
    - Our amplitudes & selection rules reduce to theirs for $q = 0$

All 3-pt Electric-Magnetic Amplitudes

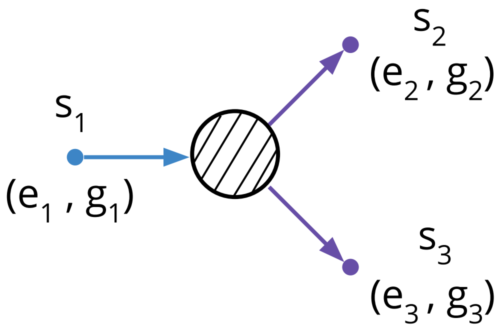
- Pairwise LG + individual LGs allow us to classify all 3-pt amplitudes
  - This generalizes the massive amplitude formalism by Arkani-Hamed at al. ‘17
  - Our amplitudes & selection rules reduce to theirs for q = 0

1. *Incoming massive particle, two outgoing massive particles*

To saturate the individual SU(2) LG for each particle, need

$$\underbrace{\left(\langle \mathbf{1}|^{2s_1}\right)}^{\{\alpha_1\ldots\alpha_{2s_1}\}} \underbrace{\left(\langle \mathbf{2}|^{2s_2}\right)}^{\{\beta_1\ldots\beta_{2s_2}\}} \underbrace{\left(\langle \mathbf{3}|^{2s_3}\right)}^{\{\gamma_1\ldots\gamma_{2s_3}\}}$$

S_i symmetrized insertions of the massive spinor for particle i



$q_{23} \equiv e_2\, g_3 - e_3\, g_2$

These need to be contracted with pairwise spinors for a Lorentz invariant amp. with overall $-q_{23}$ pairwise LG weight

All 3-pt Electric-Magnetic Amplitudes

1. *Incoming massive particle, two outgoing massive particles*

Define: $|w\rangle_\alpha \equiv \left|p_{23}^{\flat -}\right\rangle_\alpha$ and $|r\rangle_\alpha \equiv \left|p_{23}^{\flat +}\right\rangle_\alpha$

Most general term with pairwise LG weight -q and 2 $\hat{s} \equiv 2\,(s_1+s_2+s_3)$ spinor indices:

$$S^q_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}\{\gamma_1,\ldots,\gamma_{2s_3}\}} = \sum_{i=1}^{C} a_i \left(\underbrace{|w\rangle^{\hat{s}-q}|r\rangle^{\hat{s}+q}}_{\frac{1}{2}(\hat{s}-q)\,-\,(-\frac{1}{2}(\hat{s}+q))\,=\,-q}\right)_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}\{\gamma_1,\ldots,\gamma_{2s_3}\}}$$

The sum is over all different ways to assign α, β, γ indices (2 $\hat{s}$ elements in 3 bins)

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Most general term with pairwise LG weight -q and 2 ŝ ≡ 2 ($s_1+s_2+s_3$) spinor indices:

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The sum is over all different ways to assign α, β, γ indices (2 ŝ elements in 3 bins)

ŝ±q non-negative integers $\longrightarrow$ $\boxed{\textit{Selection rule: } |q| \leq \hat{s}}$

In particular a *massive scalar dyon* cannot decay to *two massive scalar dyons*

All 3-pt Electric-Magnetic Amplitudes

2. *Incoming massive particle, outgoing massive particle + massless particle, unequal mass*

All 3-pt Electric-Magnetic Amplitudes

2. *Incoming massive particle, outgoing massive particle + massless particle, unequal mass*

This time, the massive part is $\left(\left\langle \mathbf{1}\right|^{2s_1}\right)^{\{\alpha_1\ldots\alpha_{2s_1}\}}\left(\left\langle \mathbf{2}\right|^{2s_2}\right)^{\{\beta_1\ldots\beta_{2s_2}\}}$

Need to contract with standard & pairwise spinors for LG weight h_3 and pairwise LG weight $-q$

Define:

$$(|u\rangle_\alpha, |v\rangle_\alpha) = (\underbrace{|3\rangle_\alpha}_{-½}, \underbrace{|2|3]_\alpha}_{½}) \qquad (|w\rangle_\alpha, |r\rangle_\alpha) = \left(\underbrace{\left|p_{23}^{\flat -}\right\rangle_\alpha}_{½}, \underbrace{\left|p_{23}^{\flat +}\right\rangle_\alpha}_{-½}\right)$$

Most general massless part:

$$S^{h,q,\text{ unequal}}_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}} = \sum_{i=1}^{C}\sum_{j,k} a_{ijk}\,\langle ur\rangle^{\max(j+k,0)}\langle vw\rangle^{\max(-j-k,0)} \left(|u\rangle^{\frac{\hat{s}}{2}-h-j}|v\rangle^{\frac{\hat{s}}{2}+h+k}|w\rangle^{\frac{\hat{s}}{2}-q+j}|r\rangle^{\frac{\hat{3}}{2}+q-k}\right)_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}}$$

All 3-pt Electric-Magnetic Amplitudes

2. *Incoming massive particle, outgoing massive particle + massless particle, unequal mass*

$$S^{h,q,\text{ unequal}}_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}} = \sum_{i=1}^{C}\sum_{j,k} a_{ijk}\,\langle ur\rangle^{\max(j+k,0)}\langle vw\rangle^{\max(-j-k,0)} \left(|u\rangle^{\frac{\hat{s}}{2}-h-j}|v\rangle^{\frac{\hat{s}}{2}+h+k}|w\rangle^{\frac{\hat{s}}{2}-q+j}|r\rangle^{\frac{\hat{s}}{2}+q-k}\right)_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}}$$

The j and k sums are over values that give non-negative integer powers, i.e.

$$-\frac{\hat{s}}{2}+q \le j \le \frac{\hat{s}}{2}-h \qquad -\frac{\hat{s}}{2}-h \le k \le \frac{\hat{s}}{2}+q$$

→ *Selection rule:* $|h+q| \le \hat{s}$

In particular $s_1 = s_2 = 0 \;\rightarrow\; h = -q$

All 3-pt Electric-Magnetic Amplitudes

3. *Incoming massive particle, outgoing massive particle + massless particle, equal mass*

All 3-pt Electric-Magnetic Amplitudes

3. *Incoming massive particle, outgoing massive particle + massless particle, equal mass*

For equal masses, we have $|u\rangle \sim |v\rangle$ as well as $|w\rangle \sim |r\rangle$,

and we can define the famous “x-factor” from Arkani-Hamed at al. ‘17 :

$$m\,x\,|u\rangle = |v\rangle \qquad \textit{and} \qquad \langle ur\rangle^2\,x\,|w\rangle \sim |r\rangle$$

the x-factor has LG weight 1, and pairwise LG weight 0

$$S^{h,q,\text{ equal}}_{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}} = \sum_{i=1}^{C}\sum_{j}\sum_{k=-j}^{j} x^{h+q+j}\langle ur\rangle^{\max[2q+j-k,0]}\langle vw\rangle^{\max[-2q-j+k,0]} \left(|u\rangle^{j+k}|w\rangle^{j-k}\epsilon^{\hat{s}-j}\right)_{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}}$$

In this case there is no selection rule.

All 3-pt Electric-Magnetic Amplitudes

4. *Incoming massive particle, two outgoing massless particles*

All 3-pt Electric-Magnetic Amplitudes

4. *Incoming massive particle, two outgoing massless particles*

The massive part is just $\left(\left\langle \mathbf{1}\right|^{2s}\right)^{\{\alpha_1\ldots\alpha_{2s}\}}$

The massless part has helicity weights h_2 and h_3 under individual LGs, and a -q pairwise LG weight

Defining $|u\rangle_\alpha = |2\rangle_\alpha \ , \ |v\rangle_\alpha = |3\rangle_\alpha$, we have

$$S^q_{\{\alpha_1,\ldots,\alpha_{2s}\}} = \sum_{ij} a_{ij} \left(|u\rangle^{s/2-i-\Delta} |v\rangle^{s/2-j+\Delta} |w\rangle^{s/2+j-q} |r\rangle^{s/2+i+q}\right)_{\{\alpha_1,\ldots,\alpha_{2s}\}}$$

$$[uv]^{\max[\Sigma+(s-i-j)/2,0]} \langle uv\rangle^{\max[-\Sigma-(s+i+j)/2,0]} \left(\langle uw\rangle [vr]\right)^{\frac{1}{2}\max[i-j,0]} \left([uw]\langle vr\rangle\right)^{\frac{1}{2}\max[j-i,0]}$$

With $\Sigma = h_2 + h_3 \ , \ \Delta = h_2 - h_3$, the i, j sum is over $-s/2 - q \leq i \leq s/2 - \Delta$

$$-s/2 + q \leq j \leq s/2 + \Delta$$

All 3-pt Electric-Magnetic Amplitudes

4. *Incoming massive particle, two outgoing massless particles*

$$-s/2 - q \leq i \leq s/2 - \Delta$$
$$-s/2 + q \leq j \leq s/2 + \Delta$$

$\longrightarrow$ $\boxed{\textit{Selection rule: } |\Delta - q| \leq s}$

For $q = \pm 1/2$:

$$s = 0 \;\rightarrow\; \text{forbidden}$$

$$s = 1 \;\rightarrow\; |h_2 - h_3 \mp 1/2| \leq 1 \;\rightarrow\; |h_2| = |h_3| = 0 \quad \text{or} \quad h_2 = -h_3 = \pm 1/2$$

$$s = 2 \;\rightarrow\; |h_2 - h_3 \mp 1/2| \leq 2 \;\rightarrow\; |h_2| = |h_3| \leq 1/2 \quad \text{or} \quad h_2 = -h_3 = \pm 1$$

For q = ±½, our selection rule is more restrictive than the non-magnetic case in Arkani-Hamed at al. '17

Plan

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2 → 2 Fermion-Monopole Scattering

- For 2 → 2 we cannot completely fix the amplitude and some dynamical information is needed
- However, just like scattering in NRQM, we can perform a partial wave decomposition
- Our PW decomposition will be fully Lorentz and LG covariant
- All of the dynamical information reduces to phase shifts, like in QM

$2 \to 2$ Fermion-Monopole Scattering

- For $2 \to 2$ we cannot completely fix the amplitude and some dynamical information is needed
- However, just like scattering in NRQM, we can perform a partial wave decomposition
- Our PW decomposition will be fully Lorentz and LG covariant
- All of the dynamical information reduces to phase shifts, like in QM
- At the lowest partial wave, selection rules + unitarity completely fix the amplitude, reproducing the counterintuitive helicity flip of the NRQM result Kazama, Yang, Goldhaber '77
- For higher partial waves, our spinors combine to yield Monopole-Spherical Harmonics

Angular Momentum in a Poincaré Invariant Theory

- In a Poincaré invariant theory, angular momentum (squared) is defined as a quadratic Casimir

- From the momentum generator P^{μ} and the Lorentz generator $M^{\mu\nu}$,

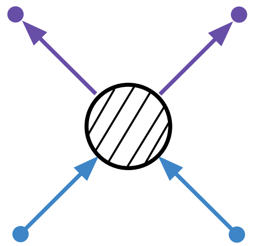
  form the Pauli-Lubański operator: $W_{\mu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P^{\nu} M^{\rho\sigma}$

- The operator W^2 is a quadratic Casimir of the poincare group, and its eigenvalues are given by:

$$W^2 = - P^2 J (J+1)$$

  where J is the total angular momentum

Lorentz and LG covariant Partial Wave Decomposition



- Consider the electric-magnetic S-matrix for $2 \to 2$ scattering

- We want to decompose the electric-magnetic S-matrix into partial waves

$$S = \sum_J S^J$$

so that J is associated with the total angular momentum of the incoming particles

including their spin and the "pairwise" angular momentum

- Formally, we need to represent the Lorentz group as *differential operators acting on spinors*

and then expand in a complete eigenbasis of the Pauli-Lubański Casimir operator W^2

Lorentz and LG covariant Partial Wave Decomposition

- The Lorentz generators in spinor space are well known: Witten '04

$$
\begin{aligned}
(\sigma_\mu)_{\alpha\dot\alpha}\, P^\mu &\equiv P_{\alpha\dot\alpha} = \sum_i |i\rangle_\alpha\, [i|_{\dot\alpha} \\
(\sigma_{\mu\nu})_{\alpha\beta}\, M^{\mu\nu} &\equiv M_{\alpha\beta} = i \sum_i |i\rangle_{\{\alpha}\, \frac{\partial}{\partial \langle i|^{\beta\}}} \\
(\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}\, M^{\mu\nu} &\equiv \tilde M_{\dot\alpha\dot\beta} = i \sum_i [i|_{\{\dot\alpha}\, \frac{\partial}{\partial\, |i]^{\dot\beta\}}} ,
\end{aligned}
$$

and they lead to the Casimir operator Jiang, Shu et al. '20

$$
W^2 = \frac{P^2}{8}\left[\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}\left(\tilde M^2\right)\right] - \frac{1}{4}\mathrm{Tr}\left(M P \tilde M P^{\mathrm{T}}\right)
$$

Lorentz and LG covariant Partial Wave Decomposition

- The generalization to electric-magnetic amplitudes is straightforward

$$(\sigma_{\mu\nu})_{\alpha\beta}\, M^{\mu\nu} \equiv M_{\alpha\beta} = i\left[\sum_i |i\rangle_{\{\alpha}\, \frac{\partial}{\partial\, \langle i|^{\beta\}}} + \sum_{i>j,\pm} \left|p_{ij}^{\flat\pm}\right\rangle_{\{\alpha}\, \frac{\partial}{\partial \left\langle p_{ij}^{\flat\pm}\right|^{\beta\}}}\right]$$

$$(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}\, M^{\mu\nu} \equiv \tilde{M}_{\dot{\alpha}\dot{\beta}} = i\left[\sum_i [i|_{\{\dot{\alpha}}\, \frac{\partial}{\partial\, |i]^{\dot{\beta}\}}} + \sum_{i>j,\pm} \left[p_{ij}^{\flat\pm}\right|_{\{\dot{\alpha}}\, \frac{\partial}{\partial \left|p_{ij}^{\flat\pm}\right]^{\dot{\beta}\}}}\right],$$

- The eigenfunctions of W^2 are symmetrized products of standard and pairwise spinors:

$$W^2\left(f\prod |s_k\rangle\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}} = -sJ(J+1)\left(f\prod |s_k\rangle\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}}$$

where $| s_k \rangle$ can be any standard / pairwise spinor, and the f is any contraction of spinors

Lorentz and LG covariant Partial Wave Decomposition

- For the PW decomposition, we expand in an eigenbasis of W^2 acting on the spinors / pairwise spinors associated with the incoming f and M:

$$S_{12\to 34} = \mathcal{N} \sum_J (2J+1)\, \mathcal{M}^J\,(p_c)\, \mathcal{B}^J$$

B^J are the *basis amplitudes*, $W^2 \mathcal{B}^J \;=\; -\, s\, J(J+1)\mathcal{B}^J$ ⟵ all angular dependence

M^J are *"reduced matrix elements"*, $W^2\, \mathcal{M}^J = 0$ ⟵ all dynamical info

$\mathcal{N} \equiv \sqrt{8\pi s}$ is a *Normalization factor*

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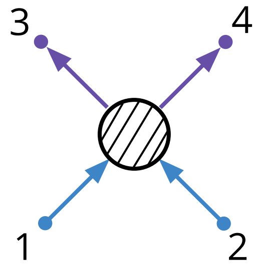

- The form of basis amplitudes B^J is constrained by their J eigenvalue

$$\mathcal{B}^J = \underbrace{C^{J;\,\mathrm{in}}_{\{\alpha_1,\,\ldots,\,\alpha_{2J}\}}}_{\text{eigenfunction of } W^2 \text{ for the incoming particles}} C^{J;\,\mathrm{out};\,\{\alpha_1,\,\ldots,\,\alpha_{2J}\}}$$

Jiang, Shu et al. ‘20

- The C^J are called “generalized Clebsch-Gordan coefficients” (more accurately “tensors”)
  - $C^{J\,\mathrm{in}}$ ($C^{J\,\mathrm{out}}$) only depend on the spinors for the incoming (outgoing) f and M
  - They saturate the LG and pairwise LG transformation of the S-matrix
  - We can extract them from the 3-pt amplitudes 1, 2 $\rightarrow$ spin J and spin J $\rightarrow$ 3, 4

Lorentz and LG covariant Partial Wave Decomposition

- As an example consider the C^J for a scalar charge + scalar monopole, q = -1

- The 3pt amplitude s + M $\rightarrow$ spin J is:

$$S\left(1^0, 2^0 \mid \mathbf{3}^J\right)_{q_{12}=-1} = a\left\langle \mathbf{3} p_{12}^{\flat -}\right\rangle^{J+1}\left\langle \mathbf{3} p_{12}^{\flat +}\right\rangle^{J-1}$$

- We get the Clebsch by stripping away the massive spinor $\langle \mathbf{3}|^{\alpha}$:

$$\left(C^{J;\,\mathrm{in}}_{0,0,-1}\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}} = \left(\left|p_{12}^{\flat -}\right\rangle^{J+1}\left|p_{12}^{\flat +}\right\rangle^{J-1}\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}}$$

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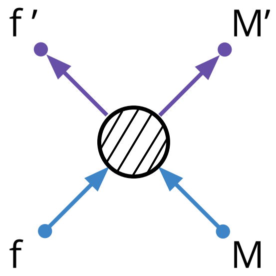
Fermion - Monopole Scattering

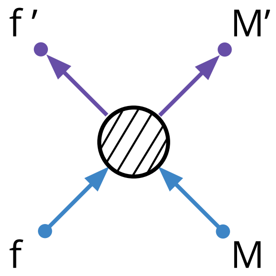


- Let's look at a massive fermionic charge and a massive scalar monopole

- The C^J is extracted from the "3-massive" 3-pt amplitude with selection rule $|q| \leq \hat{s}$
  - In this case $\hat{s} = \frac{1}{2} + 0 + J \geq |q| \longrightarrow J \geq |q| - \frac{1}{2}$
  - The J for lowest partial wave depends the pairwise helicity
  - This is the relativistic generalization of the NRQM modification of the angular momentum operator

- Let's focus on the lowest partial wave $J = |q| - \frac{1}{2}$ and extract C^J

Fermion - Monopole Scattering



- $C^{2|q|-1;\text{in}}$ for spin ½ + spin 0 $\rightarrow$ spin 2|q|-1 (for q>0) :

$$S^{3-\text{pt,in}}_{q>0} = a \left\langle \mathbf{f} p^{\flat +}_{fM} \right\rangle \left\langle \mathbf{J} p^{\flat +}_{fM} \right\rangle^{2|q|-1} \longrightarrow C^{|q|-1/2;\ \text{in}}_{q>0} = \left\langle \mathbf{f}\ p^{\flat +}_{fM} \right\rangle \left(\left| p^{\flat +}_{fM} \right\rangle^{2|q|-1} \right)_{\{\alpha_1,\ldots,\alpha_{2|q|-1}\}}$$

- Similarly: $$C^{|q|-1/2;\ \text{out}}_{q>0} = \left\langle \mathbf{f}'\ p^{\flat +}_{f'M} \right\rangle \left(\left| p^{\flat +}_{f'M} \right\rangle^{2|q|-1} \right)_{\{\alpha_1,\ldots,\alpha_{2|q|-1}\}}$$

- Contracting and repeating for q<0,

$$\boxed{\mathcal{B}^{|q|-1/2}_{q>0} = \frac{\left\langle \mathbf{f}\ p^{\flat +}_{fM} \right\rangle \left\langle \mathbf{f}'\ p^{\flat +}_{f'M'} \right\rangle}{4p_c^2} \left(\frac{\left\langle p^{\flat +}_{fM}\ p^{\flat +}_{f'M'} \right\rangle}{2p_c} \right)^{2|q|-1} \qquad \mathcal{B}^{|q|-1/2}_{q<0} = \frac{\left\langle \mathbf{f}\ p^{\flat -}_{fM} \right\rangle \left\langle \mathbf{f}'\ p^{\flat -}_{f'M'} \right\rangle}{4p_c^2} \left(\frac{\left\langle p^{\flat -}_{fM}\ p^{\flat -}_{f'M'} \right\rangle}{2p_c} \right)^{2|q|-1}}$$

Completely fixed the basis amplitude for the lower partial wave

Surprise at the Lowest PW: Helicity Flip!

- We derived the *basis amplitude* for the lowest partial wave
- But we know from NRQM that this amplitude should be very surprising
- In fact, Kazama et al. '77 show that at the lowest PW, the helicity of the fermion should flip between the initial state and the final state: e_L falling into a monopole comes out as e_R !

  can we reproduce this in our formalism?
- We take the $m_f \to 0$ limit to expose new selection rules

Surprise at the Lowest PW: Helicity Flip!

- As in Arkani-Hamed et al. '17, we take the $m_f \to 0$ limit by *unbolding* the massive spinors
  - Important: We have to make a choice of helicity when taking the massless limit

$$\langle \mathbf{1}|^\alpha$$

h_1 = -½ $\quad\quad$ h_1 = ½

$$\langle 1|^\alpha \qquad\qquad \sim \langle \hat{\eta}_1|^\alpha$$ P-conjugate of ⟨ 1 | $^\alpha$

  - In the $h_f = h_{f'}$ = -½ (helicity-flip)* case:

$$\mathcal{B}^{|q|-\frac{1}{2}} = \frac{\left\langle f\, p^{\flat\pm}_{fM}\right\rangle \left\langle f'\, p^{\flat\pm}_{f'M'}\right\rangle}{4p_c^2}\left(\frac{\left\langle p^{\flat\pm}_{fM}\, p^{\flat\pm}_{f'M'}\right\rangle}{2p_c}\right)^{2|q|-1} \quad \text{for } \operatorname{sgn}(q) = \pm 1$$

But in the massless limit $\left\langle f\, p^{\flat+}_{fM}\right\rangle = \left\langle f'\, p^{\flat+}_{f'M'}\right\rangle = 0$ and so the q>0 amplitude vanishes

*In the all-outgoing convention, h_f is minus the physical helicity of the fermion

Surprise at the Lowest PW: Helicity Flip!

- In the $h_f = h_{f'} = ½$ (helicity -flip) case:

$$\mathcal{B}^{|q|-\frac{1}{2}} = \frac{\left\langle \hat{\eta}_f \, p^{\flat\pm}_{fM} \right\rangle \left\langle \hat{\eta}_{f'} \, p^{\flat\pm}_{f'M'} \right\rangle}{4p_c^2} \left(\frac{\left\langle p^{\flat\pm}_{fM} \, p^{\flat\pm}_{f'M'} \right\rangle}{2p_c} \right)^{2|q|-1} \qquad \text{for } \operatorname{sgn}(q) = \pm 1$$

But in the massless limit $\left\langle \hat{\eta}_f \, p^{\flat-}_{fM} \right\rangle = \left\langle \hat{\eta}_{f'} \, p^{\flat-}_{f'M'} \right\rangle = 0$ and so the q<0 amplitude vanishes

- In the $h_f = -h_{f'} = \pm½$ (helicity non-flip) case, the amplitude vanishes for any q

- Conclusion: at the lowest PW, all helicity non-flip amplitude vanish!

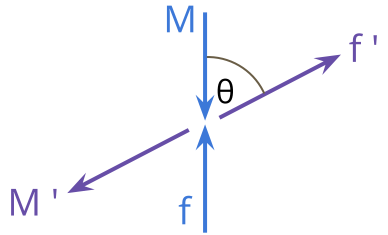
$$\mathcal{B}^{|q|-\frac{1}{2}}_{q<0} = \frac{\left\langle f \, p^{\flat-}_{fM} \right\rangle \left\langle f' \, p^{\flat-}_{f'M'} \right\rangle}{4p_c^2} \left(\frac{\left\langle p^{\flat-}_{fM} \, p^{\flat-}_{f'M'} \right\rangle}{2p_c} \right)^{2|q|-1}$$

q<0:   only RH fermion going to LH fermion

$$\mathcal{B}^{|q|-\frac{1}{2}}_{q>0} \sim \frac{\left[f \, p^{\flat-}_{fM} \right] \left[f' \, p^{\flat-}_{f'M'} \right]}{4p_c^2} \left(\frac{\left\langle p^{\flat+}_{fM} \, p^{\flat+}_{f'M'} \right\rangle}{2p_c} \right)^{2|q|-1}$$

q>0:   only LH fermion going to RH fermion

Surprise at the Lowest PW: Helicity Flip!



- In the COM frame: $\left|p_{ij}^{\flat\pm}\right\rangle_\alpha = \sqrt{2p_c}\,|\pm\hat{p}_c\rangle_\alpha$

  where $|\hat{n}\rangle_\alpha \equiv \begin{pmatrix} c_n \\ s_n \end{pmatrix}$ and $|-\hat{n}\rangle_\alpha \equiv \begin{pmatrix} -s_n^* \\ c_n \end{pmatrix}$, $s_n = e^{i\phi_n}\sin\left(\frac{\theta_n}{2}\right)$, $c_n = \cos\left(\frac{\theta_n}{2}\right)$

- Substituting in the lowest PW amplitude:

$$S^{|q|-\frac{1}{2}}_{f\to\bar{f}^\dagger} = \mathcal{N}\,2|q|\,\mathcal{M}^{|q|-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}\left[\sin\left(\frac{\theta}{2}\right)\right]^{2|q|-1} \qquad \text{for } q>0$$

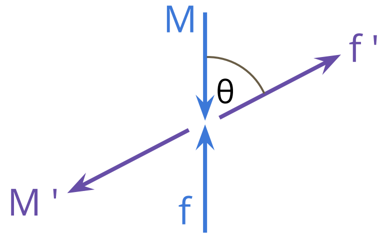
$$S^{|q|-\frac{1}{2}}_{\bar{f}^\dagger\to f} = \mathcal{N}\,2|q|\,\mathcal{M}^{|q|-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}\left[\sin\left(\frac{\theta}{2}\right)\right]^{2|q|-1} \qquad \text{for } q<0$$

remember:

$$S_{12\to34} = \mathcal{N}\sum_J (2J+1)\,\mathcal{M}^J(p_c)\,\mathcal{B}^J$$

$2J+1 = 2|q|$

Surprise at the Lowest PW: Helicity Flip!



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$$S^{|q|-\frac{1}{2}}_{\bar{f}^{\dagger}\to f} = \mathcal{N}\,2|q|\,\mathcal{M}^{|q|-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}\left[\sin\left(\frac{\theta}{2}\right)\right]^{2|q|-1} \quad \text{for } q<0$$

remember:

$$S_{12\to34} = \mathcal{N}\sum_J (2J+1)\,\mathcal{M}^J(p_c)\,\mathcal{B}^J$$

2 J +1 = 2 |q|

- In principle, the M are dynamics-dependent, however, at the lowest PW, unitarity implies:

$$\left|\mathcal{M}^{|q|-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}\right| = \left|\mathcal{M}^{|q|-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}\right| = 1 \xrightarrow[\text{only one of them nonzero, depending on q}]{\text{WLOG}} \mathcal{M}^{|q|-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} = -\mathcal{M}^{|q|-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} = 1$$

- is exactly the NRQM result from Kazma, Yang, Goldhaber '77

Higher PWs: Monopole-Spherical Harmonics

- For J > |q| - ½ we can use our general massive 3-pt amplitude to extract C^J and B^J:

$$\mathcal{B}^J \sim \sum_{\sigma}\sum_{\sigma'} a_\sigma a'_{\sigma'} \frac{\left\langle \mathbf{f} p^{\flat\sigma}_{fM} \right\rangle \left\langle \mathbf{f}' p^{\flat\sigma'}_{f'M'} \right\rangle}{4p_c^2} \tilde{\mathcal{B}}^J\left(-q_\sigma, -q_{\sigma'}\right) \qquad \begin{aligned} &\sigma, \sigma' \in \{+,-\} \\ &q_\pm = q \mp 1/2 \end{aligned}$$

and $$\tilde{\mathcal{B}}^J\left(\Delta, \Delta'\right) = \frac{1}{(2p_c)^{2J}} \left(\left\langle p^{\flat-}_{fM}\right|^{J+\Delta} \left\langle p^{\flat+}_{fM}\right|^{J-\Delta}\right)^{\{\alpha_1,\ldots,\alpha_{2J}\}} \left(\left|p^{\flat-}_{f'M'}\right\rangle^{J+\Delta'} \left|p^{\flat+}_{f'M'}\right\rangle^{J-\Delta'}\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}}$$

- The magic unfolds in the COM frame:

$$\tilde{\mathcal{B}}^J\left(\Delta, \Delta'\right) = (-1)^{J-\Delta'} \mathcal{D}^{J*}_{-\Delta,\Delta'}\left(\Omega_c\right)$$

where the D is the famous *Wigner D-matrix*: $\mathcal{D}^J_{-\Delta,\Delta'}(\Omega) \equiv \mathcal{D}^J_{-\Delta,\Delta'}(\phi,\theta,-\phi) = e^{i\phi\left(\Delta+\Delta'\right)} d^J_{-\Delta,\Delta'}(\theta)$

$$d^J_{m,m'}(\theta) = \langle J, m \,|\, \exp\left(-i\theta J_y\right) |\, J, m' \rangle$$

Higher PWs: Monopole-Spherical Harmonics

- In the massless limit, we can write the compact result ("magnetic Jacob-Wick"):

$$\boxed{S^J_{h_{\rm in}\to h_{\rm out}} = \mathcal{N}\,(2J+1)\,\mathcal{M}^J_{-h_{\rm in},h_{\rm out}}\,\mathcal{D}^{J*}_{q-h_{\rm in},-q+h_{\rm out}}\,(\Omega_c)}$$

in the *all-outgoing* convention, h_{in} = ½ (-½) for an incoming LH (RH) fermion

h_{out} = -½ (½) for an outgoing LH (RH) fermion

- This time the M are dynamics dependent, but they are only phase shifts:

$$\mathcal{M}^J_{\pm\frac{1}{2},\pm\frac{1}{2}} = e^{-i\pi\mu} \qquad \mu = \sqrt{\left(J+\frac{1}{2}\right)^2 - q^2}$$

Kazma, Yang, Goldhaber '77

obtained in NRQM by a tedious solution of the Dirac eq in monopole background

Higher PWs: Monopole-Spherical Harmonics

- PW unitarity implies:

$$\left|\mathcal{M}^J_{\pm\frac{1}{2},\mp\frac{1}{2}}\right|^2 = 1 - \left|\mathcal{M}^J_{\pm\frac{1}{2},\pm\frac{1}{2}}\right|^2 = 0$$

from NRQM:

$$\mathcal{M}^J_{\pm\frac{1}{2},\pm\frac{1}{2}} = e^{-i\pi\mu}$$

and so the helicity-flip amplitude for J > |q|-½ vanishes, consistently with the NRQM result

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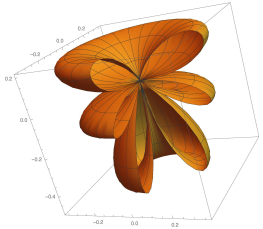
and so the helicity-flip amplitude for J > |q|-½ vanishes, consistently with the NRQM result

- Finally:

$$\mathcal{D}^{l*}_{q,m}(\Omega) = \sqrt{\frac{4\pi}{2l+1}}\,{}_qY_{l,m}(-\Omega)$$



Where the ${}_qY_{lm}$ are the *monopole-spherical harmonics* derived in Wu, Yang '76 as eigenfunctions of the magnetically modified J^2 and J_z

here they emerge from contracting pairwise spinors in a Lorentz and LG covariant way

Plan

- The manifestly relativistic, electric-magnetic S-matrix
  - ✓ Pairwise little group and pairwise helicity
  - ✓ The extra LG phase of the magnetic S-matrix
  - ✓ Pairwise spinor-helicity variables
  - ✓ Electric Magnetic amplitudes: a cheat sheet
- Results
  - ✓ All 3-pt electric-magnetic amplitudes. Novel selection rules.
  - ✓ LG covariant partial wave decomposition
  - ✓ Charge-monopole scattering:

    Helicity-flip selection rule at lowest partial wave

    Higher partial waves: monopole spherical harmonics

Conclusions

- Solved the problem of constructing Lorentz covariant electric-magnetic amplitudes
- Identified electric-magnetic multiparticle states that are not direct products
- Defined the pairwise LG, helicity and spinor-helicity variables
- Fixed all 3-pt amplitudes
- Fixed all angular dependence of $2 \to 2$ scattering and reproduced lowest PW helicity-flip

More applications to come...

Thank You!

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Backup

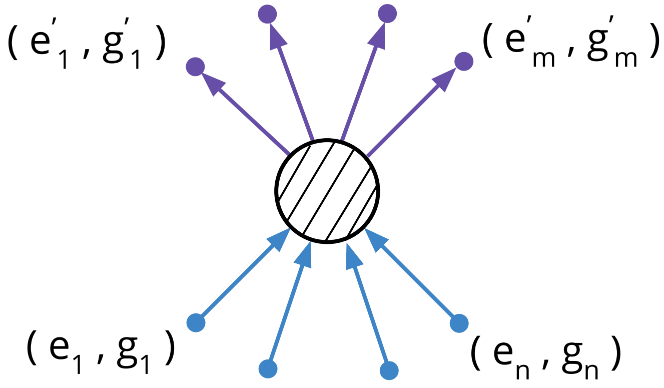
Zwanziger's Classical Relativistic Result

Zwanziger '71

Dyons $(e^-_1, g^-_1), \ldots, (e^-_n, g^-_n)$ scattering to $(e^+_1, g^+_1), \ldots, (e^+_m, g^+_m)$

What's the asymptotic $\vec{J}_{\text{field}}$ as $t \to \pm\infty$?

By Noether's theorem: $\left(\vec{J}^{\text{ field}}\right)_\ell = \frac{1}{2}\epsilon_{\ell mn} M^{mn}_{\text{field}}$



$$M^{\nu\rho}_{\text{field}} = \int x^{[\mu} T^{\nu]0}_{\text{field}} \, d^3x \qquad T^{\mu\nu}_{\text{field}} = \frac{1}{2}\left(F^{\mu}{}_{\lambda} F^{\lambda\nu} + F^{\mu}_{(\text{mag})\lambda} F^{\lambda\nu}_{(\text{mag})}\right)$$

Zwanziger's Classical Relativistic Result

Zwanziger '71

Dyons $(e^-_1,g^-_1), \ldots, (e^-_n,g^-_n)$ scattering to $(e^+_1,g^+_1), \ldots, (e^+_m,g^+_m)$

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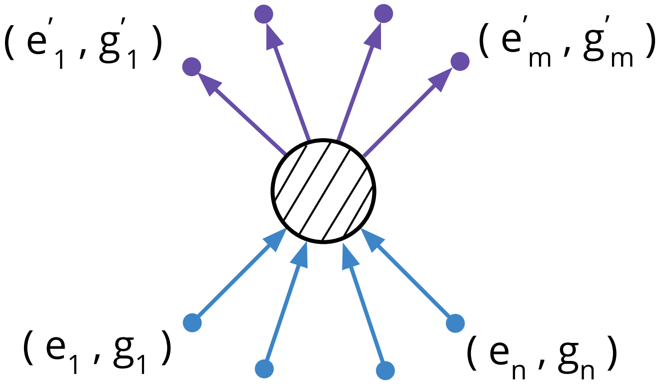


By Noether's theorem: $\left(\vec{J}^{\text{ field}}\right)_\ell = \frac{1}{2}\epsilon_{\ell mn} M^{mn}_{\text{field}}$

$$M^{\nu\rho}_{\text{field}} = \int x^{[\mu}\, T^{\nu]\,0}_{\text{field}}\; d^3x \qquad\qquad T^{\mu\nu}_{\text{field}} = \frac{1}{2}\left(F^{\mu}{}_{\lambda}F^{\lambda\nu} + F^{\mu}_{(\text{mag})\lambda}F^{\lambda\nu}_{(\text{mag})}\right)$$

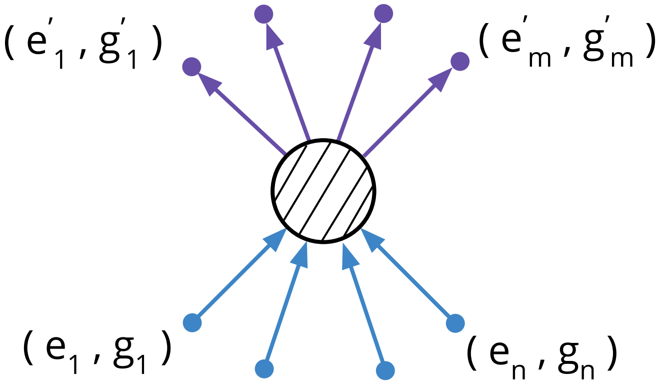
2 potential formalism
Schwinger '66
Zwanziger '68

Zwanziger's Classical Relativistic Result

Zwanziger '71

Dyons (e^-_1,g^-_1), … , (e^-_n,g^-_n) scattering to (e^+_1,g^+_1), … , (e^+_m,g^+_m)

What's the asymptotic $\vec{J}_{\text{field}}$ as t → ±∞ ?



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$$\boxed{\lim_{t\to\pm\infty} M^{\nu\rho}_{\text{field}} = \pm\sum_{i>j} q^{\pm}_{ij}\frac{\epsilon^{\nu\rho\alpha\beta}\,p_{i\alpha}\,p_{j\beta}}{\sqrt{(p_i\cdot p_j)^2 - m_i^2\,m_j^2}} \qquad q^{\pm}_{ij} = e^{\pm}_i g^{\pm}_j - e^{\pm}_j g^{\pm}_i}$$

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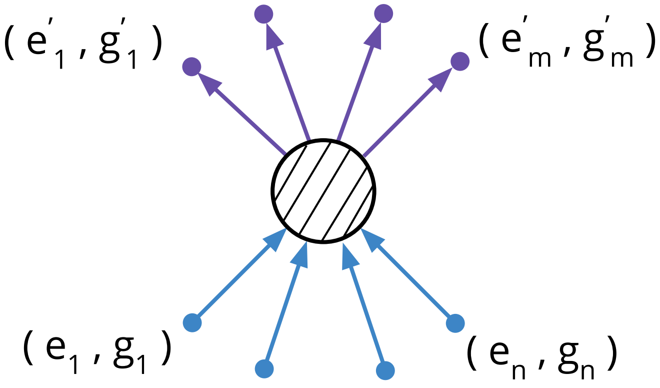

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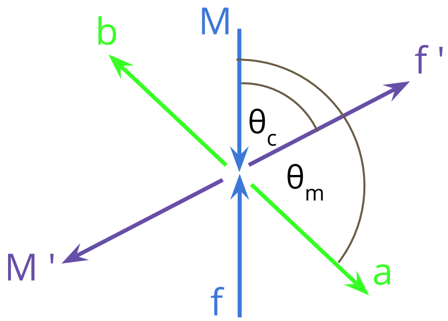
No crossing symmetry

half integer by Zwanziger-Schwinger condition

PW Unitarity for the Electric-Magnetic 2 → 2 S-matrix



$$S\,S^\dagger = I$$

Assuming only 2 particle intermediate states

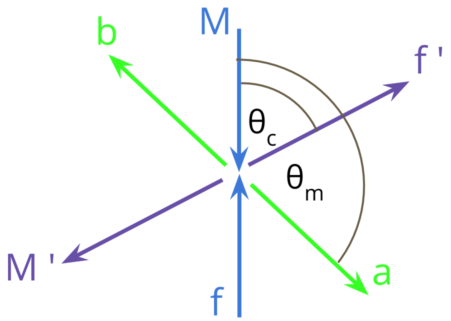
$$\frac{p_c}{16\pi^2\sqrt{s}}\int d\Omega_m \sum_{ab}\left(S_{(fM)_i\to ab}\,S^*_{(f^\dagger M)_f\to a^\dagger b^\dagger}\right) = \frac{16\pi^2\sqrt{s}}{p_c}\,\delta(\Omega_c)\,,$$

- The 2 → 2 S-matrices are:

$$S_{h_{\rm in}\to h_{\rm out}} = \mathcal{N}\sum_J (2J+1)\,\mathcal{M}^J_{-h_{\rm in},h_{\rm out}}\,\mathcal{D}^{J*}_{q-h_{\rm in},-q+h_{\rm out}}(\Omega_m)\,,$$

$$S_{h_{\rm in}\to h_{\rm out}} = \mathcal{N}\sum_J (2J+1)\,\mathcal{M}^J_{-h_{\rm in},h_{\rm out}}\sum_{p=-J}^{J}\mathcal{D}^J_{p,q-h_{\rm in}}(\Omega_c)\,\mathcal{D}^{J*}_{p,-q+h_{\rm out}}(\Omega_m)$$

PW Unitarity for the Electric-Magnetic $2 \to 2$ S-matrix



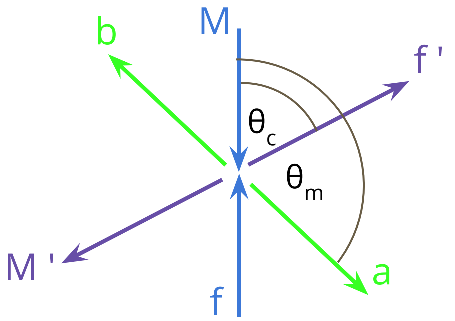
- Focusing on $(h_{in}, h_{out}) = (\frac{1}{2}, -\frac{1}{2})$:

$$\frac{1}{16\pi^2} \int d\Omega_m \sum_{J,J'} (2J+1)\,(2J'+1) \cdot$$
$$\Big\{ \mathcal{M}^{J}_{-\frac{1}{2},-\frac{1}{2}}\, \mathcal{M}^{J'\dagger}_{-\frac{1}{2},-\frac{1}{2}}\; \mathcal{D}^{J*}_{q-\frac{1}{2},-q-\frac{1}{2}}\,(\Omega_m) \sum_{p=-J'}^{J'} \mathcal{D}^{J'*}_{p,q+\frac{1}{2}}\,(\Omega_c)\, \mathcal{D}^{J'}_{p,-q-\frac{1}{2}}\,(\Omega_m)$$
$$+ \mathcal{M}^{J}_{-\frac{1}{2},\frac{1}{2}}\, \mathcal{M}^{J'\dagger}_{\frac{1}{2},-\frac{1}{2}}\; \mathcal{D}^{J*}_{q-\frac{1}{2},-q+\frac{1}{2}}\,(\Omega_m) \sum_{p=-J'}^{J'} \mathcal{D}^{J'*}_{p,q+\frac{1}{2}}\,(\Omega_c)\, \mathcal{D}^{J'}_{p,-q+\frac{1}{2}}\,(\Omega_m) \Big\} = \delta(\Omega_c)\,.$$

- Use the identity:

$$\int d\Omega_m\, \mathcal{D}^{J*}_{a,b}\,(\Omega_m)\, \mathcal{D}^{J'}_{a',b'}\,(\Omega_m) = \frac{4\pi}{2J+1}\, \delta_{aa'}\, \delta_{bb'}\, \delta_{JJ'}\,.$$

PW Unitarity for the Electric-Magnetic $2 \to 2$ S-matrix



- Everything simplifies,

$$\frac{1}{4\pi}\sum_J (2J+1)\left(\mathcal{M}^J\mathcal{M}^{J\dagger}\right)_{-\frac{1}{2},-\frac{1}{2}} \mathcal{D}^{J*}_{q-\frac{1}{2},q+\frac{1}{2}}(\Omega_c) \;=\; \delta(\Omega_c)$$

- Repeating for all h_{in}, h_{out}

$$\frac{1}{4\pi}\sum_J (2J+1)\left(\mathcal{M}^J\mathcal{M}^{J\dagger}\right)_{-h_{\rm in},h_{\rm out}} \mathcal{D}^{J*}_{q-h_{\rm in},q-h_{\rm out}}(\Omega_c) \;=\; \delta_{-h_{\rm in},h_{\rm out}}\,\delta(\Omega_c)$$

- Multiplying by $\mathcal{D}^{J}_{q-h_{\rm in},q-h_{\rm out}}(\Omega_c)$ and integrating,

$$\boxed{\mathcal{M}^J\mathcal{M}^{J\dagger} \;=\; I}$$

This is what happens in the non-magnetic case, and leads to the standard PW unitarity bound