Scale Anomalies in CFT

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Outline

- ➢ Scale anomalies in CFT
- ➢ Motivation
- ➢ Computing it in Euclidean position space
- ➢ Computing it in Minkowski momentum space

Scale anomalies in CFT

$$\left(-\sum_{i=1}^{n} x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}}\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle = n\Delta_{\mathcal{O}}\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle$$

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- Example: $\left\langle \mathcal{O}(x)\mathcal{O}(0)\right\rangle = \frac{1}{\left(x^2\right)^{\Delta_{\mathcal{O}}}}$

$$\left(x^{\mu}\frac{\partial}{\partial x^{\mu}} + 2\Delta_{\mathcal{O}}\right)\left\langle \mathcal{O}(x)\mathcal{O}(0)\right\rangle = 0$$

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- Caveat: $2\Delta_{\mathcal{O}} = D$

$$\left(x^{\mu}\frac{\partial}{\partial x^{\mu}} + 2\Delta_{\mathcal{O}}\right)\left\langle \mathcal{O}(x)\mathcal{O}(0)\right\rangle = \left(x^{\mu}\frac{\partial}{\partial x^{\mu}} + D\right)\frac{1}{\left(x^2\right)^{D/2}} = \Omega_D \delta^D(x)$$

Scale anomalies in CFT

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$$\to \frac{\partial}{\partial x^\mu} x^\mu$$

Scale anomalies in CFT

$$n\Delta_{\mathcal{O}} = (n-1)D$$

$$\left(\sum_{i=1}^{n} x_i^\mu \frac{\partial}{\partial x_i^\mu} + n\Delta_{\mathcal{O}}\right)\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\text{conn}} = c\,\delta^D(x_{12})\,\delta^D(x_{13})\cdots\delta^D(x_{1n})$$

Scale anomaly coefficient

Scale anomalies in CFT

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Scale anomaly coefficient

$$\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle = \left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}} + \frac{1}{\left(x_{12}^2 x_{34}^2\right)^{\Delta_{\mathcal{O}}}} + \frac{1}{\left(x_{13}^2 x_{24}^2\right)^{\Delta_{\mathcal{O}}}} + \frac{1}{\left(x_{14}^2 x_{23}^2\right)^{\Delta_{\mathcal{O}}}}$$

Scale anomalies in CFT

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$$n\Delta_{\mathcal{O}} = (n-1)D + 2r, \quad r = 0,1,2,\cdots$$

$$\left(\sum_{i=1}^{n} x_i^\mu \frac{\partial}{\partial x_i^\mu} + n\Delta_{\mathcal{O}}\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}} = c\,\partial^{2r}\delta^D(x_{12})\,\delta^D(x_{13})\cdots\delta^D(x_{1n})$$

Difference with the "scale anomaly" in QFT

$$\left(\sum_{i=1}^{n} x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}}+nd_{\mathcal{O}}\right)\left\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}}=\mu\frac{\partial}{\partial\mu}\left\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}}$$

Scale anomalies in CFT

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$$\left(\sum_{i=1}^{n} x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}}+nd_{\mathcal{O}}\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}}=\mu\frac{\partial}{\partial\mu}\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}}$$

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_a\beta_a(g)\frac{\partial}{\partial g_a}+n\gamma_{\mathcal{O}}(g)\right]\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}}=0$$

Difference with the "scale anomaly" in QFT

$$\left(\sum_{i=1}^{n} x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}} + n d_{\mathcal{O}}\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}} = \mu\frac{\partial}{\partial\mu}\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}}$$

$$\left[\mu\frac{\partial}{\partial\mu} + \sum_a \beta_a(g)\frac{\partial}{\partial g_a} + n\gamma_{\mathcal{O}}(g)\right]\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}} = 0$$

$$\left\{\sum_{i=1}^{n} x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}} + n\left[d_{\mathcal{O}} + \gamma_{\mathcal{O}}(g)\right] + \sum_a \beta_a(g)\frac{\partial}{\partial g_a}\right\}\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right\rangle_{\text{conn}} = 0$$

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At fixed point: $\beta_a(g_*) = 0, \quad \Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}(g_*)$

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$$n\Delta_{\mathcal{O}} = (n-1)D + 2r, \quad r = 0,1,2,\cdots$$

Motivation

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Why studying it?

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$$n\Delta_{\mathcal{O}} = (n-1)D + 2r, \quad r = 0,1,2,\cdots$$

Why studying it?

- Scaling dimension protected

$$T^{\mu\nu}: \quad \Delta_T = D, \qquad J^{\mu}: \quad \Delta_J = D-1$$

Motivation

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$$T^{\mu\nu}: \quad \Delta_T = D, \qquad J^{\mu}: \quad \Delta_J = D-1$$

- Computable from CFT data, physical meaning: “counts states”

$$c = \sum_{\Psi=\text{primary}} \lambda^2_{\mathcal{O}\mathcal{O}\Psi} c_\Psi, \quad c_\Psi \geq 0$$

Motivation

$$n\Delta_{\mathcal{O}} = (n-1)D + 2r, \quad r = 0,1,2,\cdots$$

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- At $D=4$

$$\left(\sum_{i=1}^{4} x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}} + 16\right) \left\langle T^{\mu_1\nu_1}(x_1)\cdots T^{\mu_4\nu_4}(x_4)\right\rangle_{\text{conn}} \sim c\,\partial^4 \delta^4(x_{12})\,\delta^4(x_{13})\,\delta^4(x_{14})$$

c counts states created by $T^{\mu\nu}$, $\quad \left\langle T^{\mu\nu} T^{\rho\sigma}\right\rangle \propto c$

Motivation

Focus of this talk: scalar operator $\mathcal{O}$, $n = 4$, $r = 0$

$$\left(\sum_{i=1}^{4} x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}} + 4\Delta_{\mathcal{O}} \right) \left\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_4) \right\rangle_{\text{conn}} = c\,\delta^D(x_{12})\,\delta^D(x_{13})\,\delta^D(x_{14})$$

Achievements:

Motivation

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Achievements:

- Computed c in Euclidean position space

  --- computable from CFT data

Motivation

Focus of this talk: scalar operator $\mathcal{O}$, $n=4$, $r=0$

$$\left(\sum_{i=1}^{4} x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}} + 4\Delta_{\mathcal{O}}\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}} = c\,\delta^D(x_{12})\,\delta^D(x_{13})\,\delta^D(x_{14})$$

Achievements:

- Computed c in Euclidean position space

  --- computable from CFT data

- Computed c in Minkowski momentum space

  --- formalism hopefully useful elsewhere

  --- positive definite sum:

$$c = \sum_{\Psi=\text{primary}} \lambda^2_{\mathcal{O}\mathcal{O}\Psi}\, c_{\Psi}, \quad c_{\Psi} \geq 0$$

Euclidean position space

$$\left(\sum_{i=1}^{4} x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}} + 3D\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}} = c\,\delta^D(x_{12})\,\delta^D(x_{23})\,\delta^D(x_{34}), \quad 4\Delta_{\mathcal{O}} = 3D$$

Euclidean position space

$$\left(\sum_{i=1}^{4} x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}}+3D\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}}=c\delta^{D}(x_{12})\delta^{D}(x_{23})\delta^{D}(x_{34}),\quad 4\Delta_{\mathcal{O}}=3D$$

$$\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}}=f\left(\vec{X}\right),\quad \vec{X}=(x_{12},x_{23},x_{34})\in\mathbb{R}^{3D}$$

Euclidean position space

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$$\vec{X}\cdot\frac{\partial}{\partial\vec{X}}$$

$$\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}} = f\left(\vec{X}\right), \quad \vec{X} = (x_{12}, x_{23}, x_{34}) \in \mathbb{R}^{3D}$$

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$$\frac{\partial}{\partial\vec{X}}\cdot\left[\vec{X} f\left(\vec{X}\right)\right] = c\delta^{3D}\left(\vec{X}\right)$$

Euclidean position space

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$$\frac{\partial}{\partial\vec{X}}\cdot\left[\vec{X} f(\vec{X})\right] = c\delta^{3D}(\vec{X})$$

$$c = \oint d\vec{S}\cdot\vec{X} f(\vec{X})$$

Euclidean position space

$$c = \oint d\vec{S} \cdot \vec{X} f\left(\vec{X}\right) \quad , \quad 3D-1 \text{ dimensional surface integral}$$

$$f\left(\vec{X}\right) = \frac{1}{\left(x_{12}^2 x_{34}^2\right)^{\Delta_{\mathcal{O}}}} \sum_{\Psi=\text{primary}} \lambda_{\mathcal{O}\mathcal{O}\Psi}^2 \underline{G_\Psi\left(u,v\right)} + \left[-\frac{1}{\left(x_{12}^2 x_{34}^2\right)^{\Delta_{\mathcal{O}}}} - \frac{1}{\left(x_{13}^2 x_{24}^2\right)^{\Delta_{\mathcal{O}}}} - \frac{1}{\left(x_{14}^2 x_{23}^2\right)^{\Delta_{\mathcal{O}}}}\right]$$

→ conformal blocks

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$$= \frac{1}{\left(x_{12}^2 x_{34}^2\right)^{\Delta_{\mathcal{O}}}} g\left(u,v\right)$$

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conformal blocks

$$g(u,v) = \sum_{\Psi=\text{primary}} \lambda_{\mathcal{O}\mathcal{O}\Psi}^2 G_\Psi\left(u,v\right) + \left[-1 - u^{\Delta_{\mathcal{O}}} - \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \right]$$

Euclidean position space

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conformal blocks

$$\sum_{\Psi=\text{primary}} \lambda_{\mathcal{O}\mathcal{O}\Psi}^2 G_\Psi(u,v) + \left[-1 - u^{\Delta_{\mathcal{O}}} - \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \right]$$

Conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Euclidean position space

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conformal blocks

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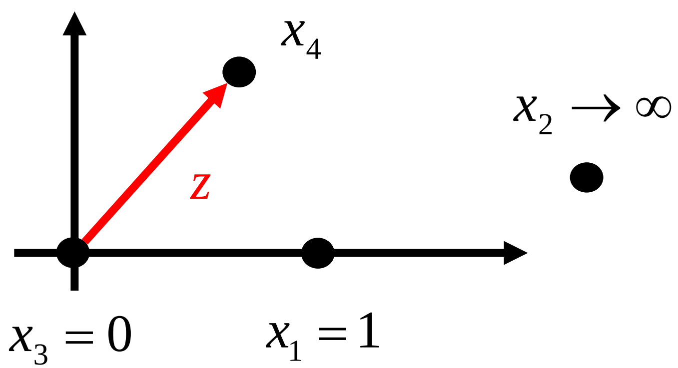

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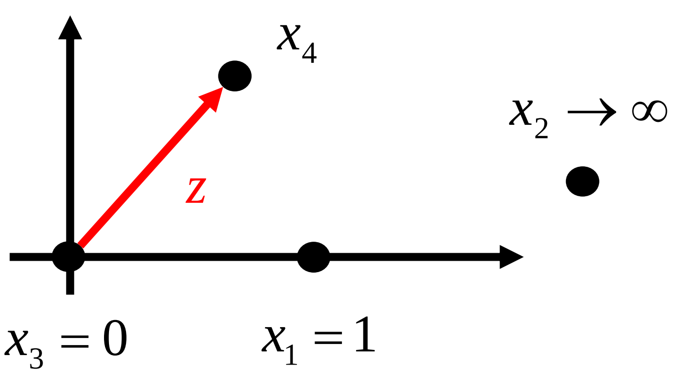
$$= \frac{1}{\left(x_{12}^2 x_{34}^2\right)^{\Delta_{\mathcal{O}}}} \underline{g\left(u,v\right)}$$

$$\sum_{\Psi=\text{primary}} \lambda_{\mathcal{O}\mathcal{O}\Psi}^2 G_\Psi\left(u,v\right) + \left[-1 - u^{\Delta_{\mathcal{O}}} - \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}\right]$$

Conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$= |z|^2 \qquad\qquad = |1-z|^2$$



Euclidean position space

Using Faddeev-Popov technique to "gauge fix" $x_1 = 1,\ x_2 = \infty,\ x_3 = 0$

$$c = \oint d\vec{S} \cdot \vec{X} f\left(\vec{X}\right) = \Omega_{D-1} \int_0^\infty r^{D-1} dr \int_0^\pi \sin^{D-2}\theta d\theta \left(r^2\right)^{-3D/4} A(u,v)\, g(u,v)$$

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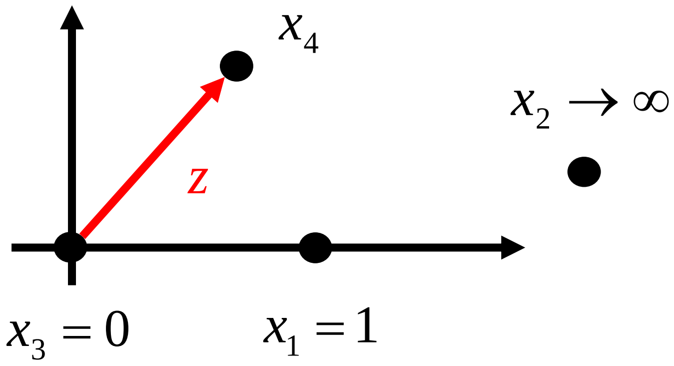

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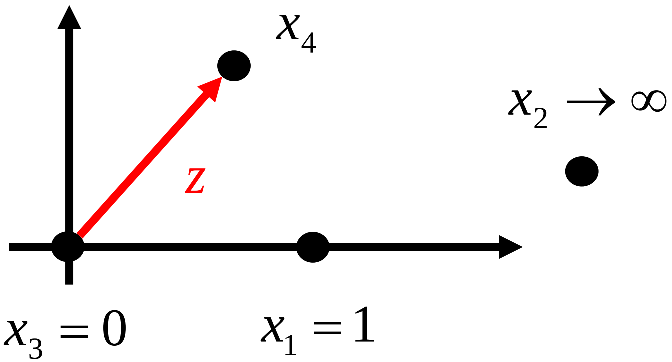

$$A(u,v) = \frac{2\pi^D}{\Gamma(D/2)\left[\Gamma(D/4)\right]^2} \int d\lambda_1 d\lambda_2 d\lambda_3 \delta\left(\lambda_1 + \lambda_2 + \lambda_3 - 1\right) \left(\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 \lambda_2 u + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 v} \right)^{D/4}$$

Euclidean position space

$$c = \Omega_{D-1}\int_0^\infty r^{D-1}dr\int_0^\pi \sin^{D-2}\theta d\theta \left(r^2\right)^{-3D/4} A(u,v)\, g(u,v) \quad , \quad g(u,v) = \sum_{\Psi=\text{primary}} \lambda^2_{\mathcal{O}\mathcal{O}\Psi} G_\Psi(u,v) + \left[-1 - u^{\Delta_\mathcal{O}} - \left(\frac{u}{v}\right)^{\Delta_\mathcal{O}}\right]$$

$$c = \sum_{\Psi=\text{primary}} \lambda^2_{\mathcal{O}\mathcal{O}\Psi} c'_\Psi + c_{\text{disc}}$$

$$c'_\Psi = \Omega_{D-1}\int_0^\infty r^{D-1}dr\int_0^\pi \sin^{D-2}\theta d\theta \left(r^2\right)^{-3D/4} A(u,v)\, G_\Psi(u,v)$$

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Euclidean position space

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- Computable from CFT data

Euclidean position space

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- Numerical Results $\left(D=4,\quad \Delta_\mathcal{O}=3\right)$:

  $c'_\Psi>0$ for $l=0$

  $c'_\Psi<0$ for $l=2,4,6,\cdots$

  $c_{\text{disc}}<0$

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No sign of positivity

Minkowski momentum space

$$\left(\sum_{i=1}^{4} x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}}+3D\right)\left\langle T\left(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right)\right\rangle=-ic\delta^D\left(x_{14}\right)\delta^D\left(x_{24}\right)\delta^D\left(x_{34}\right)$$

Minkowski momentum space

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Couple the CFT to a probe field A : $\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A\right)^2 + \epsilon A\mathcal{O} + \mathcal{L}_{\text{CFT}}$

Minkowski momentum space

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↓ massless scalar

Minkowski momentum space

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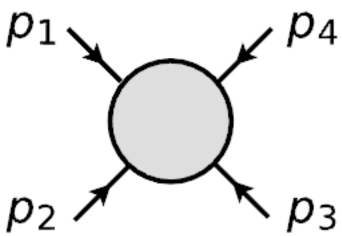

$$i\mathcal{M}_{p_1,p_2\to -p_3,-p_4}(AA\to AA)(2\pi)^D\delta^D\left(\sum_{i=1}^{4}p_i\right) = \epsilon^4\int\left(\prod_{i=1}^{4}d^Dx_i e^{ip_ix_i}\right)\langle T(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4)\rangle_{\text{conn}}$$

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↓

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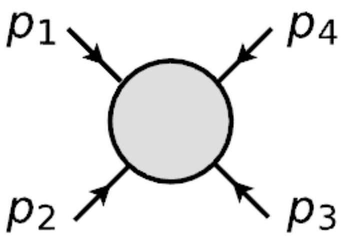

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$p_i^2 = 0$ ↓

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Minkowski momentum space

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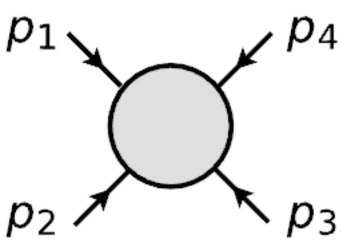

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$$i\mathcal{M}\left(s,t\right) \quad , \quad \begin{cases} s = -\left(p_1+p_2\right)^2 \\ t = -\left(p_1+p_3\right)^2 \end{cases}$$

$$c\epsilon^4 = 2\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)\mathcal{M}\left(s,t\right)$$

Minkowski momentum space

Forward limit: $t \to 0 \quad , \quad \mathcal{M}(s) \equiv \lim_{t \to 0} \mathcal{M}(s,t)$

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Minkowski momentum space

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Need to make this precise

Minkowski momentum space

$$\text{Caveats:} \quad \text{Im}\,\mathcal{M}(s) \to \infty$$

- UV divergence

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A\right)^2 + \epsilon A\mathcal{O} + \mathcal{L}_{\text{CFT}}$$

$$\mathcal{L}_{\text{CFT}} = \mathcal{L}_{\text{free}}(\phi) \quad , \quad \mathcal{O} = \phi^3 \quad , \quad D = 4$$

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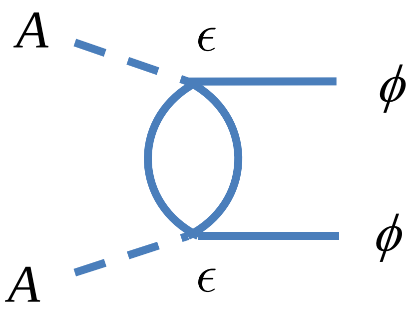

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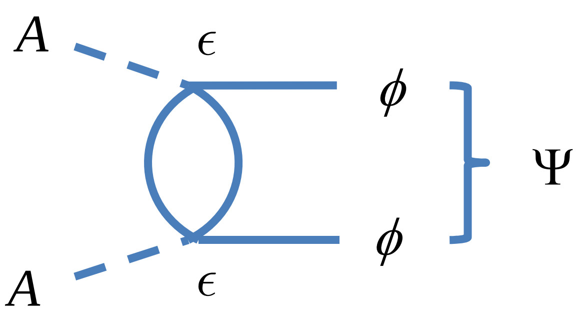

$$\mathcal{L}_{\mathrm{c.t.}} \supset \epsilon^2 A^2 \Psi$$

Minkowski momentum space

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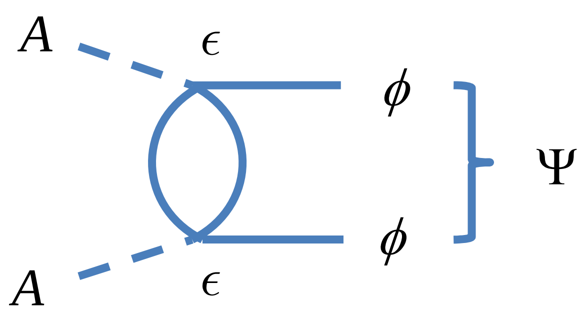

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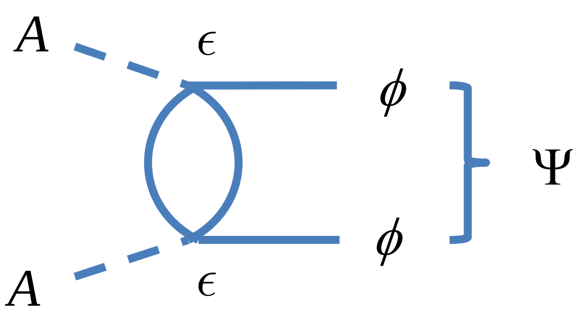

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Minkowski momentum space

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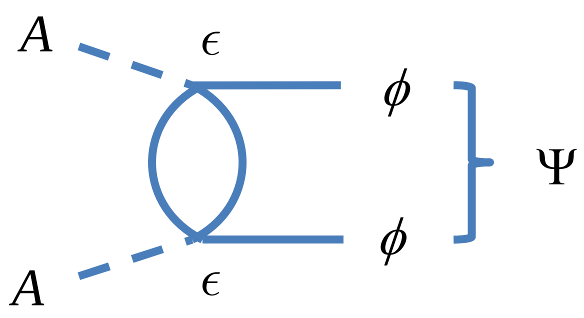

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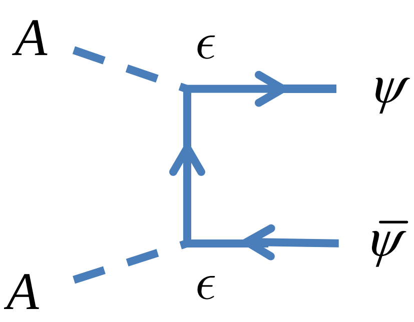

Minkowski momentum space

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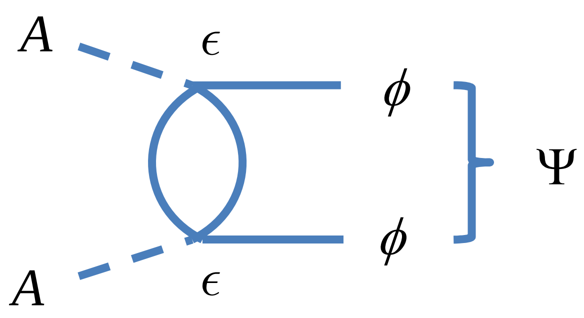

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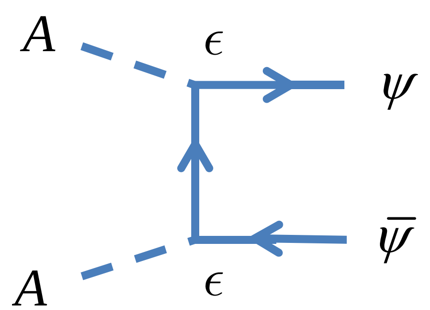

Minkowski momentum space

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$$2\operatorname{Im}\mathcal{M}(s) = -\frac{\epsilon^4}{V}\int\left(\prod_{i=1}^{4} d^D x_i\right) e^{i(p_1x_1+p_2x_2)-i(p_1x_3+p_2x_4)} \left[\begin{array}{l}\left\langle T\left(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right)\right\rangle \\ +\left\langle \bar{T}\left(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right)\right\rangle\end{array}\right]$$

$$= \frac{\epsilon^4}{V}\int\left(\prod_{i=1}^{4} d^D x_i\right) e^{i(p_1x_1+p_2x_2)-i(p_1x_3+p_2x_4)} \left\langle \bar{T}\left(\mathcal{O}_3\mathcal{O}_4\right) T\left(\mathcal{O}_1\mathcal{O}_2\right)\right\rangle$$

Minkowski momentum space

$$c \propto \operatorname{Im}\mathcal{M}(s) \propto \sigma_{\text{total}}(AA \to \text{CFT}) \propto \sum_{\psi} \left|\langle \psi | AA \rangle\right|^2$$

$$i\mathcal{M}_{p_1,p_2\to -p_3,-p_4}(AA \to AA)(2\pi)^D \delta^D\left(\sum_{i=1}^{4} p_i\right) = \epsilon^4 \int \left(\prod_{i=1}^{4} d^D x_i e^{ip_i x_i}\right) \left\langle T\left(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right)\right\rangle_{\text{conn}}$$

$$2\operatorname{Im}\mathcal{M}(s) = -\frac{\epsilon^4}{V} \int \left(\prod_{i=1}^{4} d^D x_i\right) e^{i(p_1x_1+p_2x_2)-i(p_1x_3+p_2x_4)} \left[\begin{matrix}\left\langle T\left(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right)\right\rangle \\ +\left\langle \bar{T}\left(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\right)\right\rangle\end{matrix}\right]$$

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$$1 = \sum_{\psi} |\psi\rangle\langle\psi|$$

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Primary: $\left|\Psi\right\rangle = \Psi_E\left(x_E^\mu = 0\right)\left|0\right\rangle$

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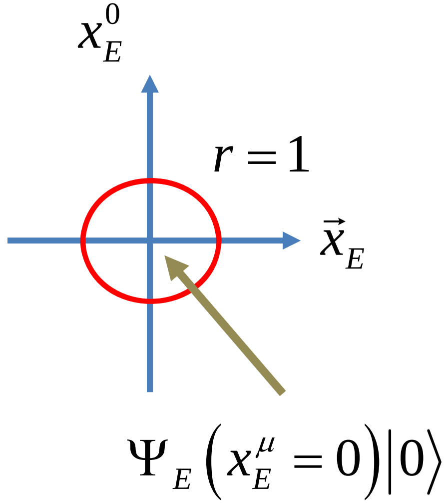
Euclidean $\mathbb{R}^D$

$x_E^\mu \leftrightarrow \left(r,\chi,\Omega^{D-2}\right)$



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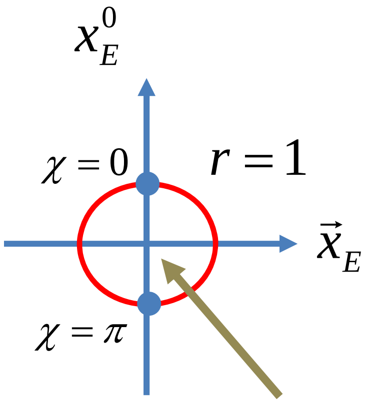

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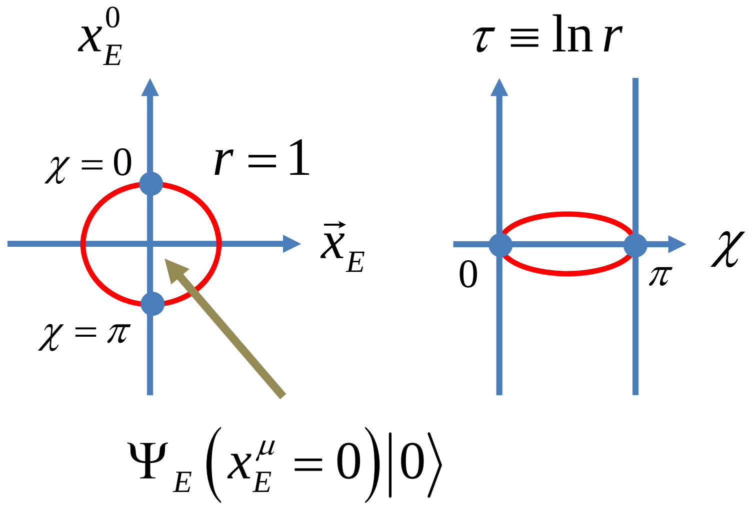
Euclidean $\mathbb{R}^D$ E-cylinder

$x_E^\mu \leftrightarrow \left(r, \chi, \Omega^{D-2}\right) \qquad \left(\tau, \chi, \Omega^{D-2}\right)$



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Euclidean $\mathbb{R}^D$	E-cylinder	M-cylinder
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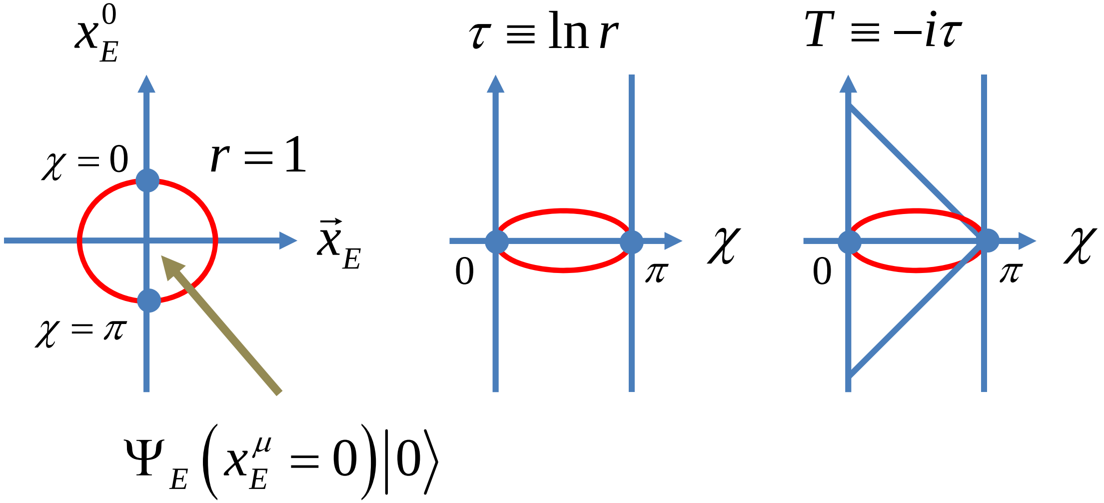

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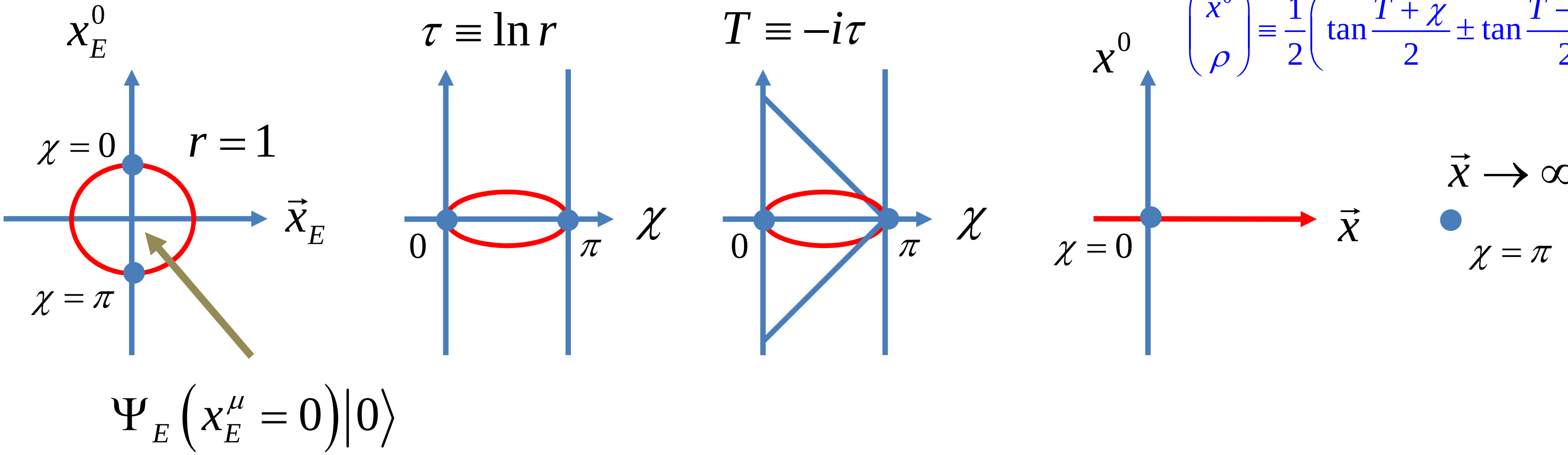

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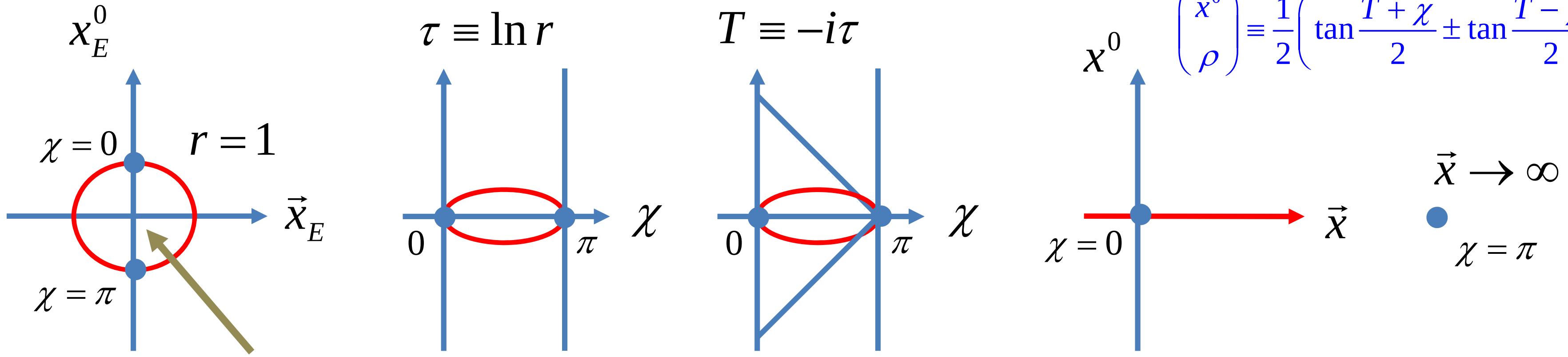

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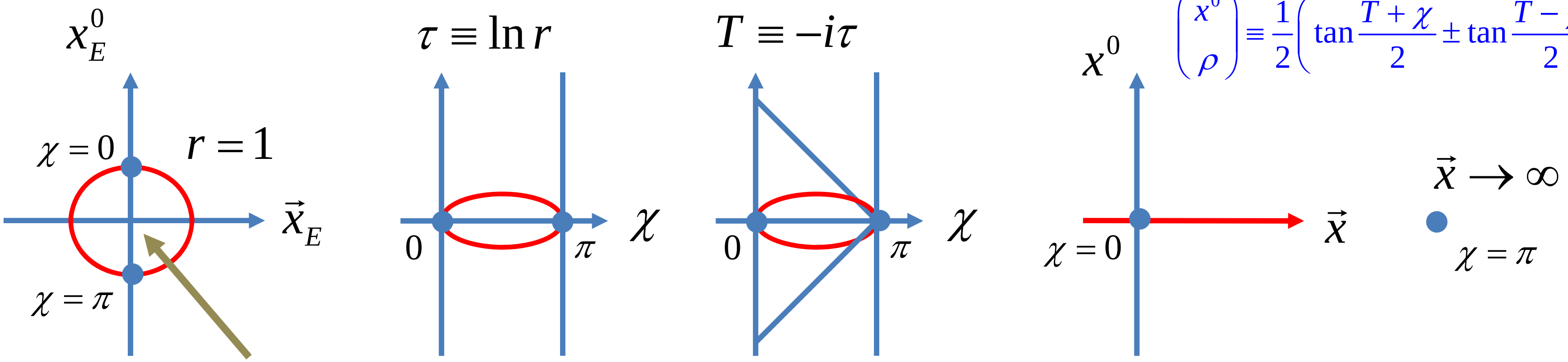

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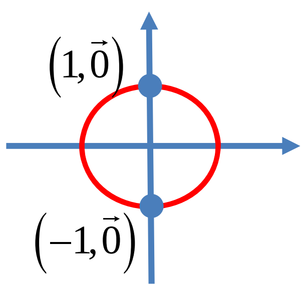

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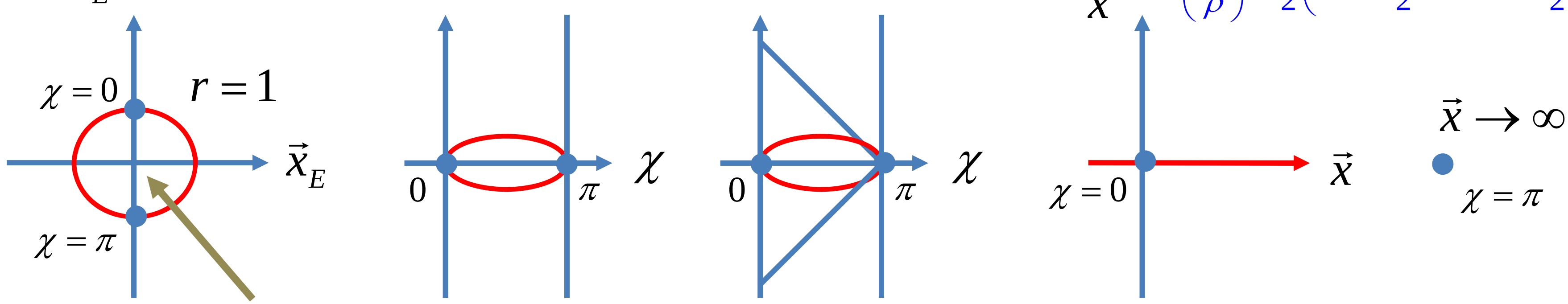

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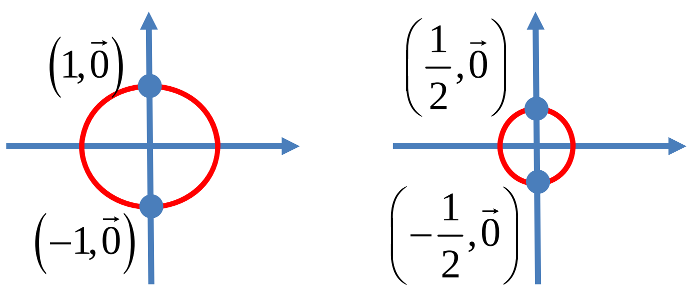

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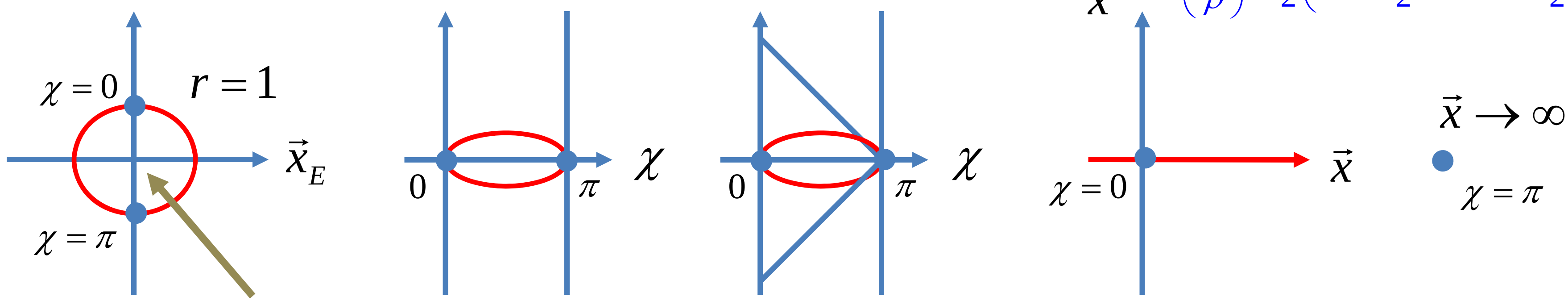

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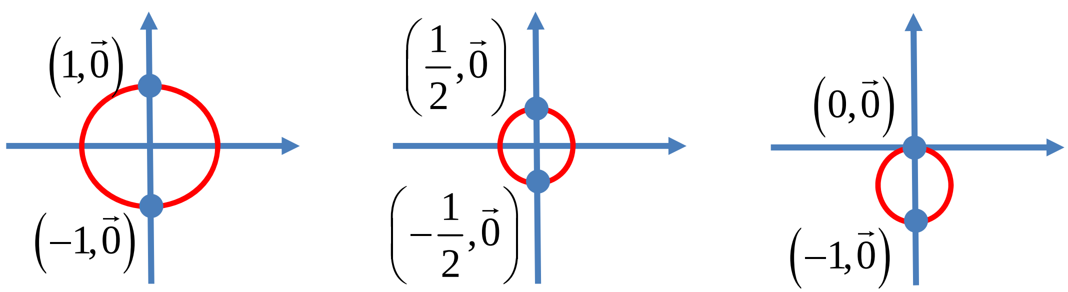

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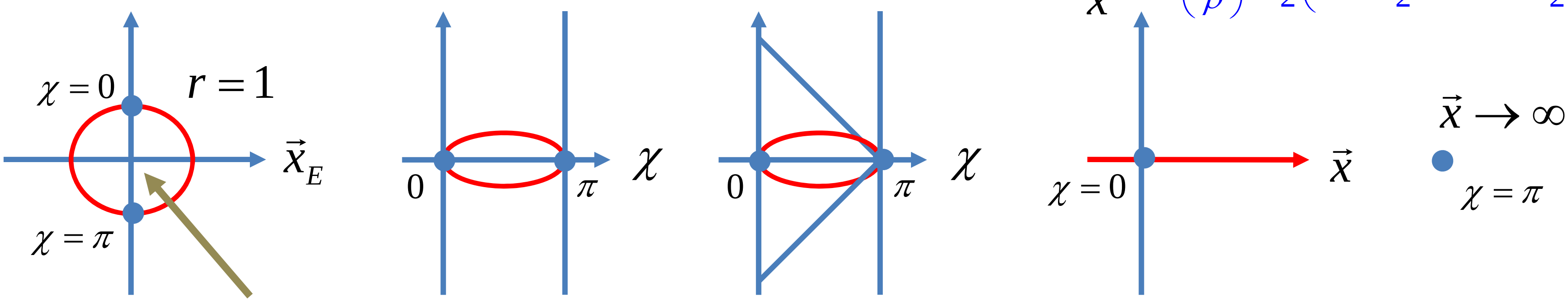

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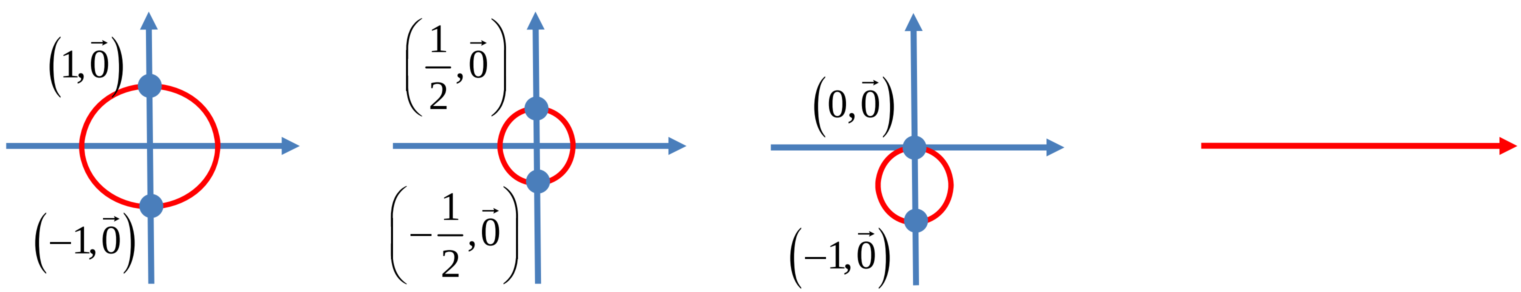

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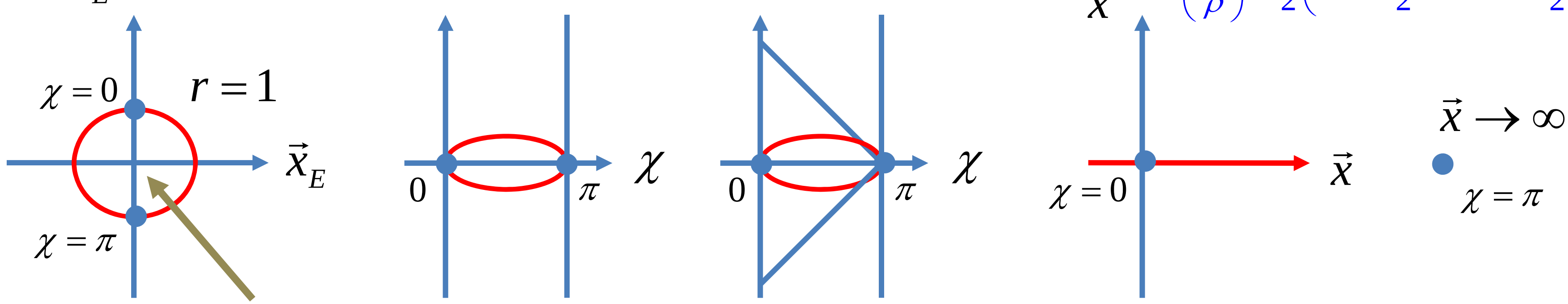

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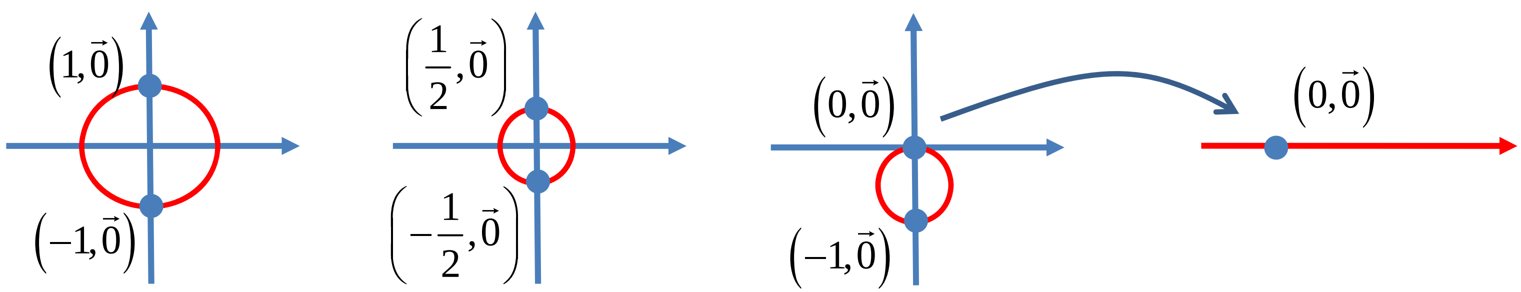

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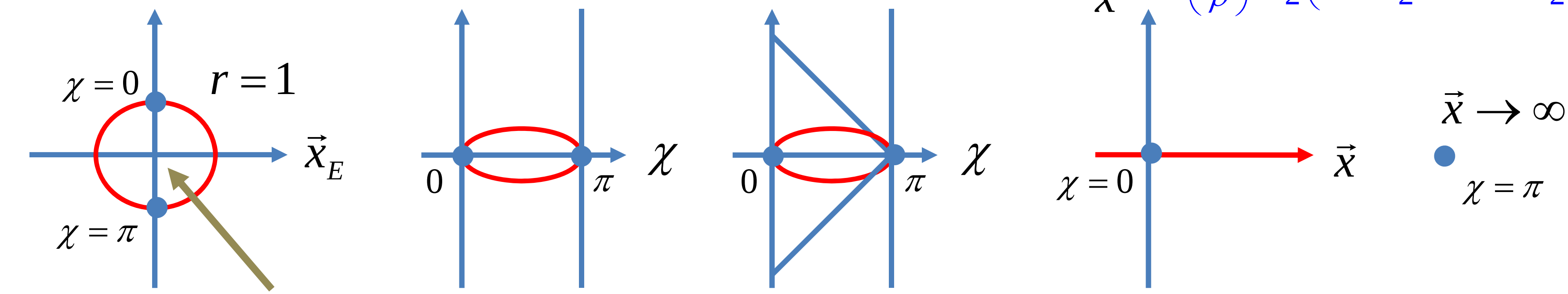

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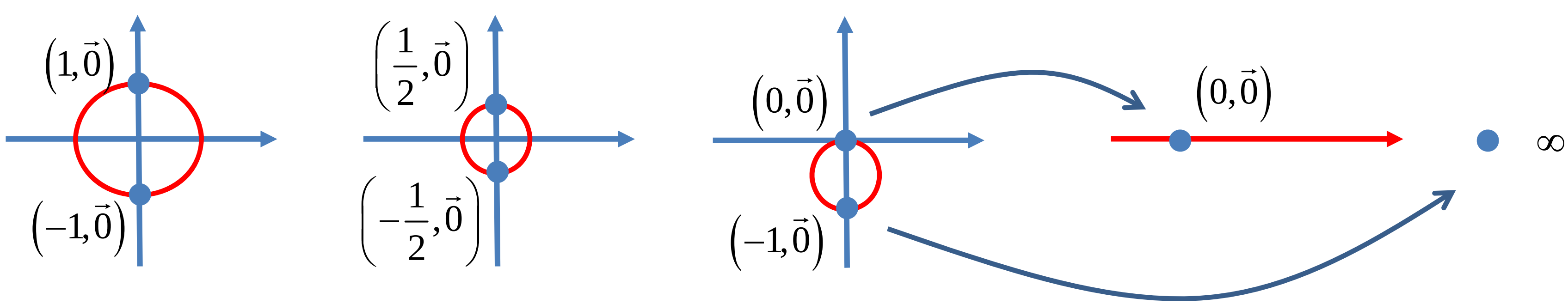

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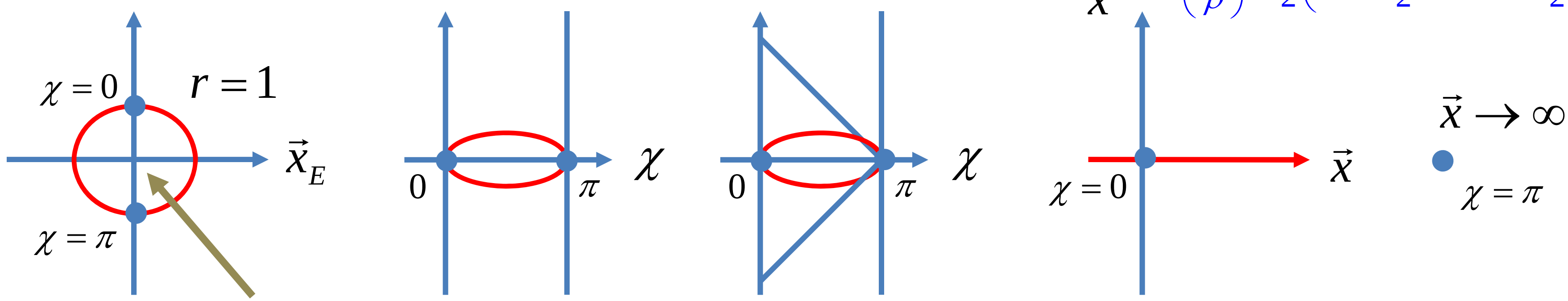

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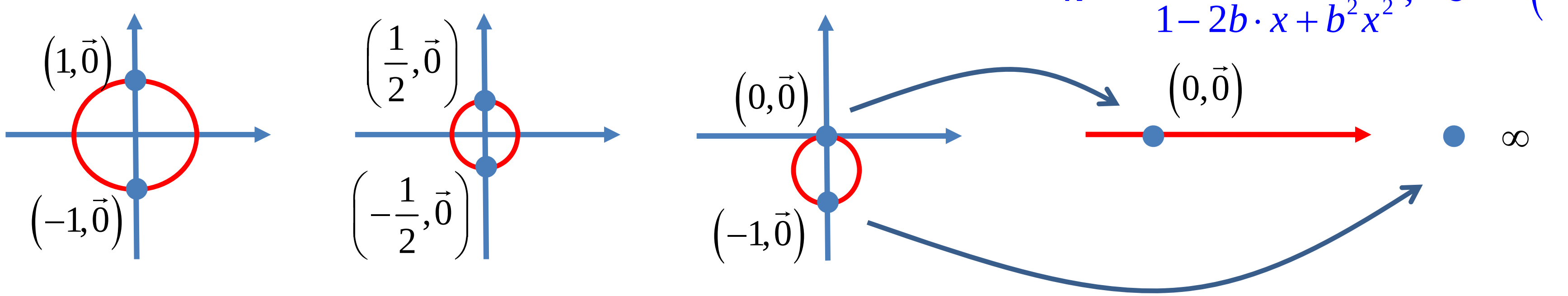

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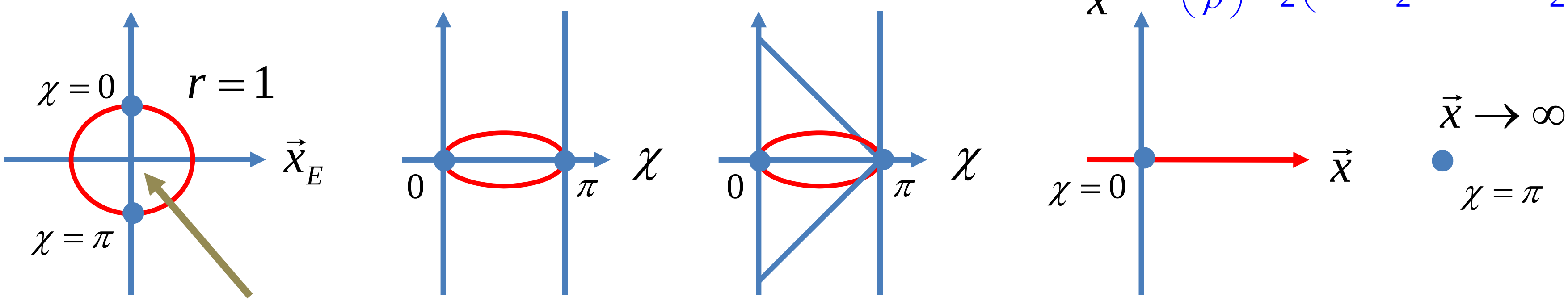

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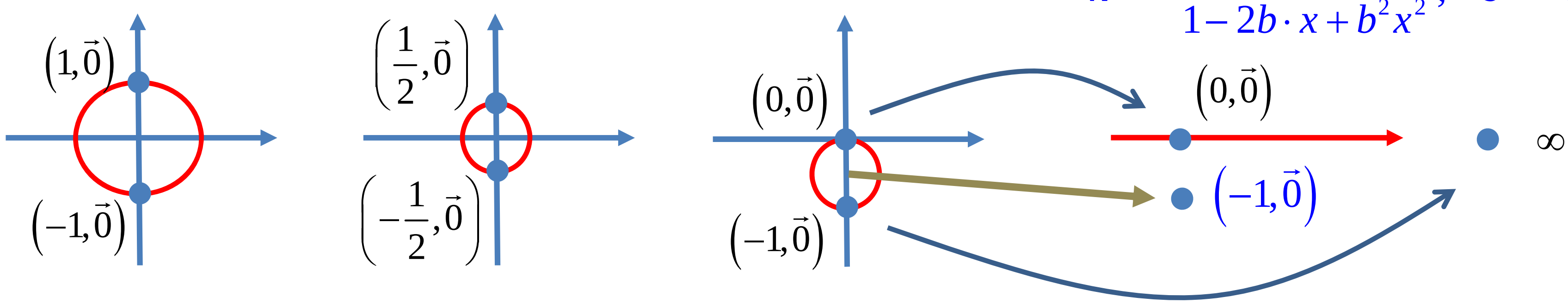

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$$\left|\Psi\left(k\right)\right\rangle \equiv \int d^D\xi e^{-ik\xi} e^{iP\xi}\left|\Psi\right\rangle \quad , \quad P_\mu\left|\Psi\left(k\right)\right\rangle = k_\mu\left|\Psi\left(k\right)\right\rangle$$

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$$\left\langle\Psi_I\left(p\right)\middle|\Psi_J\left(q\right)\right\rangle = \left(2\pi\right)^D\delta^D\left(p-q\right)\theta\left(p^0\right)\theta\left(-p^2\right)e^{-2p^0}2^{2\Delta_\Psi}\Pi_\Psi\left(p\right)\delta_{IJ}$$

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$$\left\langle\Psi_I(p)\middle|\Psi_J(q)\right\rangle = (2\pi)^D \delta^D(p-q)\theta\left(p^0\right)\theta\left(-p^2\right)e^{-2p^0} 2^{2\Delta_\Psi}\Pi_\Psi(p)\delta_{IJ}$$

$$1 = \sum_{\Psi,I}\int \frac{d^Dk}{(2\pi)^D}\frac{\theta\left(k^0\right)\theta\left(-k^2\right)}{e^{-2k^0}2^{2\Delta_\Psi}\Pi_\Psi(k)}\left|\Psi_I(k)\right\rangle\left\langle\Psi_I(k)\right|$$

Minkowski momentum space

- $$2\,\mathrm{Im}\,\mathcal{M}(s) = \frac{\epsilon^4}{V}\int\left(\prod_{i=1}^{4} d^D x_i\right) e^{i(p_1 x_1 + p_2 x_2) - i(p_1 x_3 + p_2 x_4)} \left\langle \bar{T}\left(\mathcal{O}_3 \mathcal{O}_4\right) T\left(\mathcal{O}_1 \mathcal{O}_2\right)\right\rangle$$

- $$1 = \sum_{\Psi, I}\int \frac{d^D k}{(2\pi)^D} \frac{\theta\left(k^0\right)\theta\left(-k^2\right)}{e^{-2k^0}\, 2^{2\Delta_\Psi}\, \Pi_\Psi(k)} \left|\Psi_I(k)\right\rangle\left\langle\Psi_I(k)\right|$$

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$$\frac{\pi}{2}c\epsilon^4 = 2\,\mathrm{Im}\,\mathcal{M}(s) = \epsilon^4\sum_{\Psi,I}\frac{\lambda^2_{\mathcal{O}\mathcal{O}\Psi}}{\Pi_\Psi\left(p_1+p_2\right)}\left|V_{\Psi,I}\left(p_1,p_2\right)\right|^2$$

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$$\lambda_{\mathcal{O}\mathcal{O}\Psi}V_{\Psi,I}(p_1,p_2) \equiv \varepsilon_I^\mu(p_1+p_2)\int d^D x_1 d^D x_2 e^{i(p_1x_1+p_2x_2)}\left\langle\Psi_\mu(0)T\left[\mathcal{O}(x_1)\mathcal{O}(x_2)\right]\right\rangle$$

Minkowski momentum space

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$$c=\frac{2}{\pi}\sum_{\Psi,I}\frac{\lambda^2_{\mathcal{O}\mathcal{O}\Psi}}{\Pi_\Psi(p_1+p_2)}\left|V_{\Psi,I}(p_1,p_2)\right|^2=\sum_{\Psi=\text{primary}}\lambda^2_{\mathcal{O}\mathcal{O}\Psi}c_\Psi$$

Minkowski momentum space

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$$c_\Psi = \frac{2}{\pi}\sum_{I}\frac{1}{\Pi_\Psi(p_1+p_2)}\left|V_{\Psi,I}(p_1,p_2)\right|^2 \geq 0$$

Minkowski momentum space

For scalar Ψ

$$\begin{cases} \left\langle \Psi(\xi)\Psi(0)\right\rangle = \dfrac{1}{\left[-\left(\xi^0 - i\epsilon\right)^2 + \vec{\xi}^{\,2}\right]^{\Delta_\Psi}} \\ \left\langle \Psi(0) T\left[\mathcal{O}(x_1)\mathcal{O}(x_2)\right]\right\rangle = \dfrac{\lambda_{\mathcal{O}\mathcal{O}\Psi}}{\left(x_{12}^2 + i\epsilon\right)^{\Delta_\mathcal{O} - \Delta_\Psi/2}\left[-\left(-x_1^0 - i\epsilon\right)^2 + \vec{x}_1^2\right]^{\Delta_\Psi/2}\left[-\left(-x_2^0 - i\epsilon\right)^2 + \vec{x}_2^2\right]^{\Delta_\Psi/2}} \end{cases}$$

$$\begin{cases} \Pi_\Psi(p_1 + p_2) = \dfrac{2^{D+1-2\Delta_\Psi}\pi^{D/2+1}}{\Gamma(\Delta_\Psi)\Gamma(\Delta_\Psi - D/2+1)} s^{\Delta_\Psi - D/2} \\ V_\Psi(p_1, p_2) = \dfrac{-i\pi^{D+1} 2^{2D+1-2\Delta_\mathcal{O}-\Delta_\Psi}\left[\Gamma(\Delta_\mathcal{O} - D/2)\right]^2 \Gamma(D/2 - \Delta_\mathcal{O} + \Delta_\Psi/2)}{\left[\Gamma(\Delta_\Psi/2)\right]^2 \Gamma(\Delta_\mathcal{O} - \Delta_\Psi/2)\Gamma(\Delta_\mathcal{O} + \Delta_\Psi/2 - D/2)\Gamma(\Delta_\mathcal{O} + \Delta_\Psi/2 - D + 1)} s^{\Delta_\mathcal{O} + \Delta_\Psi/2 - D} \end{cases}$$

$$c_\Psi = \frac{2^{3D+2-4\Delta_\mathcal{O}}\pi^{3D/2}\left[\Gamma(\Delta_\mathcal{O} - D/2)\right]^4\left[\Gamma(D/2 - \Delta_\mathcal{O} + \Delta_\Psi/2)\right]^2\Gamma(\Delta_\Psi)\Gamma(\Delta_\Psi - D/2 + 1)}{\left[\Gamma(\Delta_\Psi/2)\right]^4\left[\Gamma(\Delta_\mathcal{O} - \Delta_\Psi/2)\Gamma(\Delta_\mathcal{O} + \Delta_\Psi/2 - D/2)\Gamma(\Delta_\mathcal{O} + \Delta_\Psi/2 - D + 1)\right]^2}$$

$$V_\Psi(p_1, p_2) = 0, \quad \Delta_\Psi = 2\Delta_\mathcal{O} + 2k, \quad k = 0, 1, 2, \cdots$$

double trace operators
do not contribute

Minkowski momentum space

Example: free scalar theory at $D = 8$

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A\right)^2 + \epsilon A\mathcal{O} + \mathcal{L}_{\text{CFT}}$$

$$\mathcal{L}_{\text{CFT}} = \mathcal{L}_{\text{free}}\left(\phi\right) \quad , \quad \mathcal{O} = \frac{1}{2}\phi^2$$

Free of UV or IR divergence

$$D = 8 \quad , \quad \Delta_{\mathcal{O}} = 6$$

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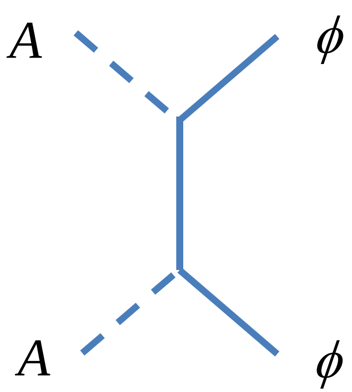

$$2\,\mathrm{Im}\,\mathcal{M}(s) = \int d\Pi_{\phi\phi}\left(p_3, p_4\right)\overline{\left|\mathcal{M}_{p_1 p_2 \to p_3 p_4}\left(AA \to \phi\phi\right)\right|^2}$$

$$= \epsilon^4 s^{D/2-4} 2^{-2D+7} \pi^{-D/2+3/2} \frac{(D-3)(D-5)}{(D-4)(D-6)} \frac{1}{\Gamma(D/2-1/2)}$$

Minkowski momentum space

$$2\,\mathrm{Im}\,\mathcal{M}(s) = \epsilon^4 \sum_{\Psi,I} \frac{\lambda^2_{\mathcal{O}\mathcal{O}\Psi}}{\Pi_\Psi(p_1+p_2)} \left|V_{\Psi,I}(p_1,p_2)\right|^2 \quad , \quad \Psi \sim \partial^{2n}\phi^2$$

$$R_n = 2^{D-5}\pi^{-1/2}\frac{(D-4)(D-6)}{(D-3)(D-5)}\Gamma\left(\frac{D-1}{2}\right)$$

$$\times \frac{\left[\sum_{s=0}^{n} b(n,n-s)\Gamma(2s+1)\sum_{k=0}^{2s}\frac{(-2)^k\,\Gamma(D/2-2+k)}{\Gamma(k+1)\Gamma(2s-k+1)\Gamma(D-3+k)}\right]^2}{\sum_{s=0}^{n}\sum_{t=0}^{n} b(n,n-s)b(n,n-t)\sum_{r=0}^{2s+2t}\frac{(-2)^r\,\Gamma(2s+2t+1)\Gamma(D/2-1+r)}{\Gamma(r+1)\Gamma(2s+2t-r+1)\Gamma(D-2+r)}}$$

$$b(n,s) \equiv \frac{(-1)^s\,\Gamma(D/2-1)\Gamma(2n+1)\Gamma(D-3+4n-2s)}{\Gamma(s+1)\Gamma(D-3+2n)\Gamma(2n-2s+1)\Gamma(D/2-1+2n-s)}$$

$$\sum_{n=0}^{\infty} R_n = 1$$

Minkowski momentum space

$$2\,\mathrm{Im}\,\mathcal{M}(s)=\epsilon^4 s^{D/2-4}2^{-2D+7}\pi^{-D/2+3/2}\frac{(D-3)(D-5)}{(D-4)(D-6)}\frac{1}{\Gamma(D/2-1/2)}$$

$$D=8:\qquad \frac{\pi}{2}c\epsilon^4=2\,\mathrm{Im}\,\mathcal{M}(s)=\frac{\epsilon^4}{512\pi^3}\quad\Rightarrow\quad c=\frac{1}{(4\pi)^4}$$

Agrees with the result from our Euclidean position space formalism

$$c=\Omega_{D-1}\int_0^\infty r^{D-1}dr\int_0^\pi \sin^{D-2}\theta d\theta\left(r^2\right)^{-3D/4}A(u,v)\,g(u,v)$$

$$A(u,v)=\frac{2\pi^D}{\Gamma(D/2)\left[\Gamma(D/4)\right]^2}\int d\lambda_1 d\lambda_2 d\lambda_3\delta(\lambda_1+\lambda_2+\lambda_3-1)\left(\frac{\lambda_1\lambda_2\lambda_3}{\lambda_1\lambda_2 u+\lambda_1\lambda_3+\lambda_2\lambda_3 v}\right)^{D/4}$$

$$\left\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\mathrm{conn}}=\frac{1}{\left(x_{12}^2x_{34}^2\right)^{\Delta_{\mathcal{O}}}}g(u,v)$$

Summary

- Studied the scale anomaly of the scalar operator $\mathcal{O}, \quad n=4, \quad r=0$

$$\left(\sum_{i=1}^{4} x_i^\mu \frac{\partial}{\partial x_i^\mu} + 4\Delta_{\mathcal{O}}\right)\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\right\rangle_{\text{conn}} = c\delta^D(x_{12})\delta^D(x_{13})\delta^D(x_{14})$$

- Showed that c is computable from CFT data

  --- using formalism in Euclidean position space

  --- decomposition has no clear sign of positivity

- Developed formalism for computing c in Minkowski momentum space

  --- formalism hopefully useful elsewhere

  --- positive definite sum: $c = \sum_{\Psi=\text{primary}} \lambda^2_{\mathcal{O}\mathcal{O}\Psi} c_\Psi, \quad c_\Psi \geq 0$

Possible future directions

- Scale anomalies of other type

  $\mathcal{O}$ beyond scalar operator? $r \neq 0$? $\sim c\partial^{2r}\delta^{D}\left(x_{12}\right)\cdots\delta^{D}\left(x_{1n}\right)$

- c anomaly sum rule in $D=4$: $\mathcal{O}=T^{\mu\nu}$

$$c=\lambda_{TTT}^{2}\frac{\left|\left\langle TTT\right\rangle\right|^{2}}{\left\langle TT\right\rangle}+\sum_{\Psi\neq T}\lambda_{TT\Psi}^{2}c_{\Psi} \quad \xrightarrow{\left\langle TT\right\rangle \propto c} \quad c^{2}=\lambda_{TTT}^{2}\left|\left\langle TTT\right\rangle\right|^{2}+c\sum_{\Psi\neq T}\lambda_{TT\Psi}^{2}c_{\Psi}$$

- a anomaly in $D=4$

- Other uses for the momentum space formalism?