Studying the Operator Bases of EFTs

UC Davis Joint Theory Seminar, Nov 30, 2015

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Highlights

- Operator bases of EFTs are organized by the conformal algebra

  The operator basis of an (non-conformal) EFT is spanned by the set of scalar (spin-0) conformal primaries formed from the canonical fields and their derivatives

- A completely automated technique is developed for counting the dimension of an operator basis

B. Henning, XL, T. Melia and H. Murayama:

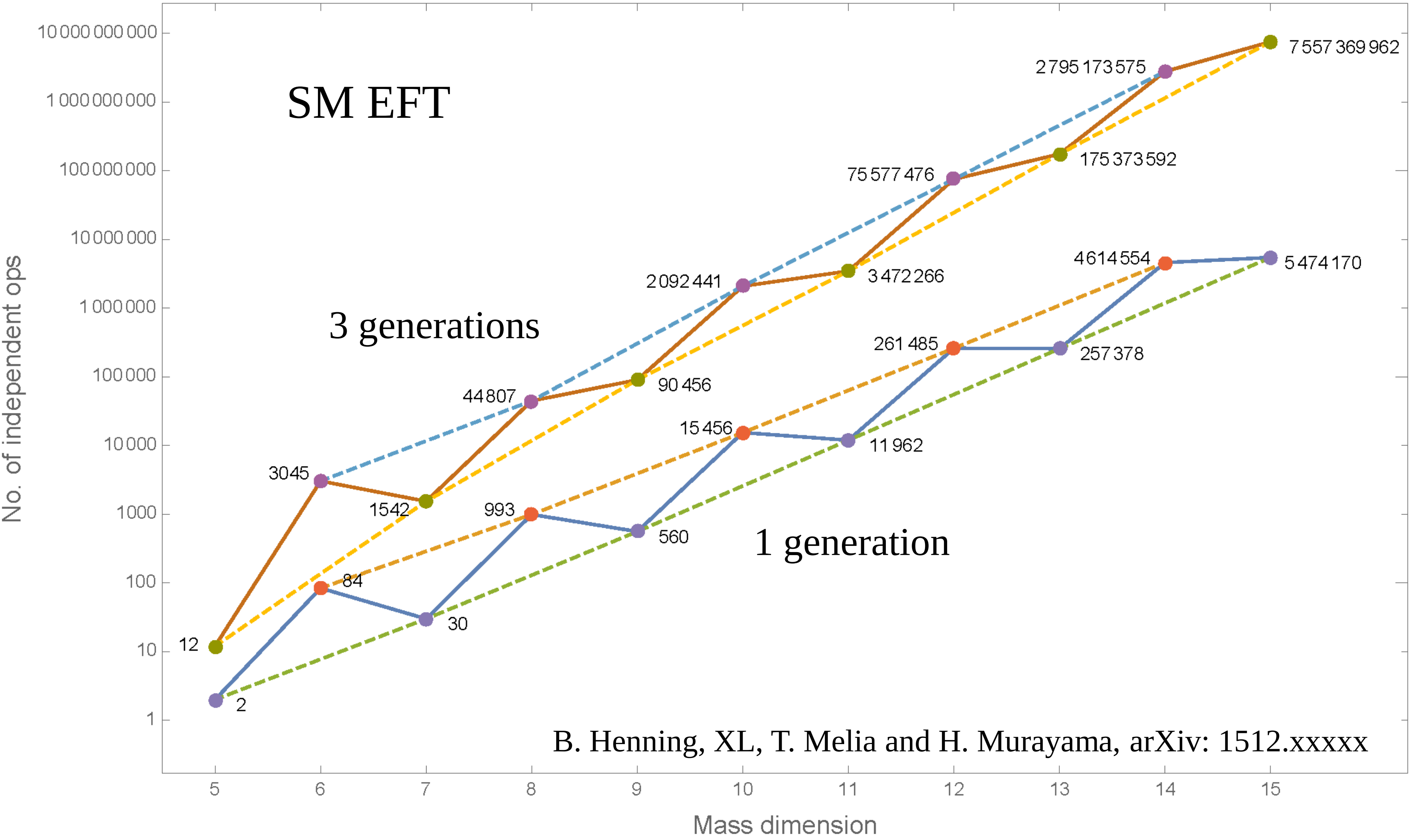
1 dimensional spacetime (arXiv: 1507.07240)

General spacetime dimension (arXiv: 1601.xxxxx)

Application to SM EFT (arXiv: 1512.xxxxx)

……

Highlights



SM EFT
3 generations
1 generation
B. Henning, XL, T. Melia and H. Murayama, arXiv: 1512.xxxxx
No. of independent ops
Mass dimension
10 000 000 000
1 000 000 000
100 000 000
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90 456
2 092 441
3 472 266
75 577 476
175 373 592
2 795 173 575
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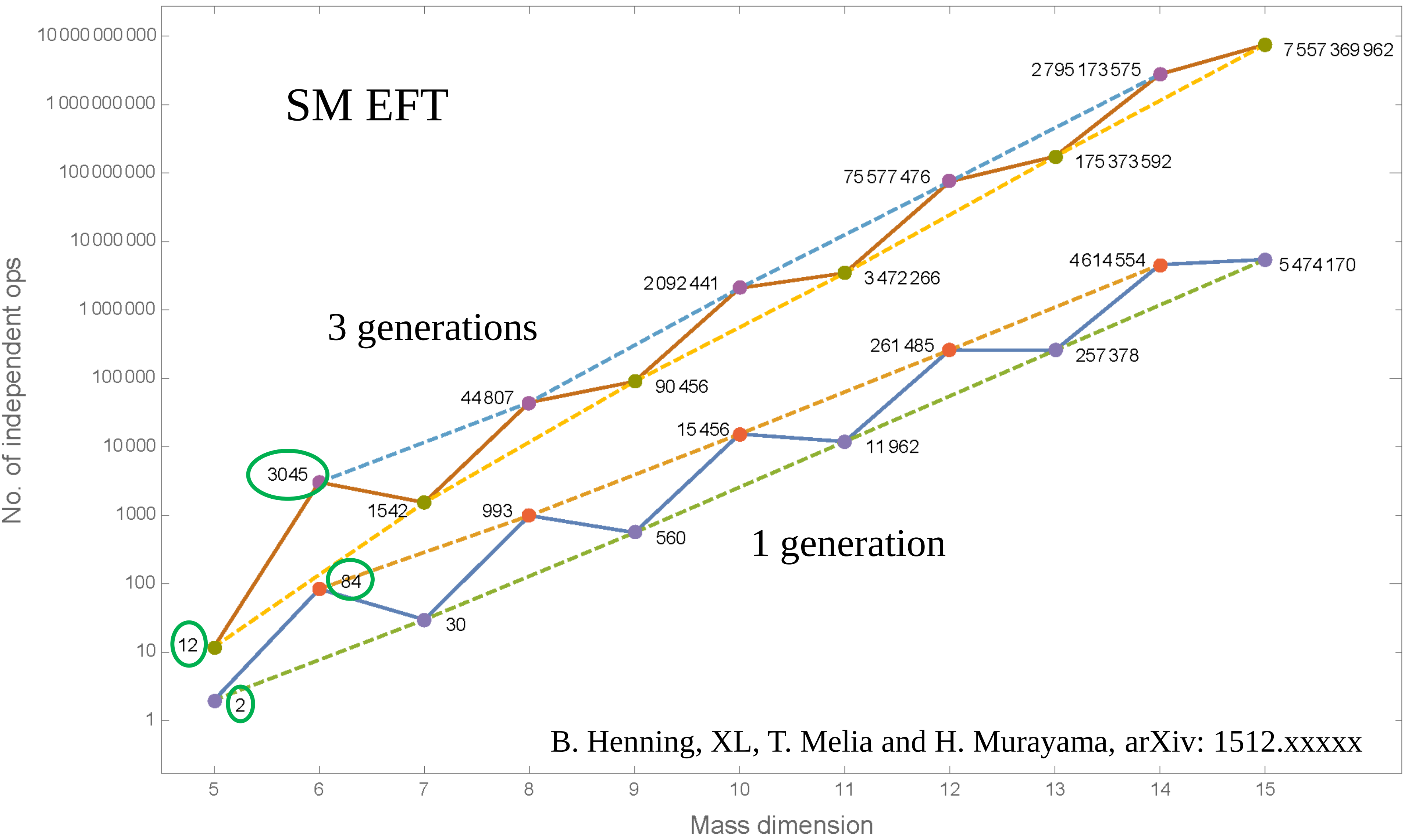
15 456
11 962
261 485
257 378
4 614 554
5 474 170

Highlights



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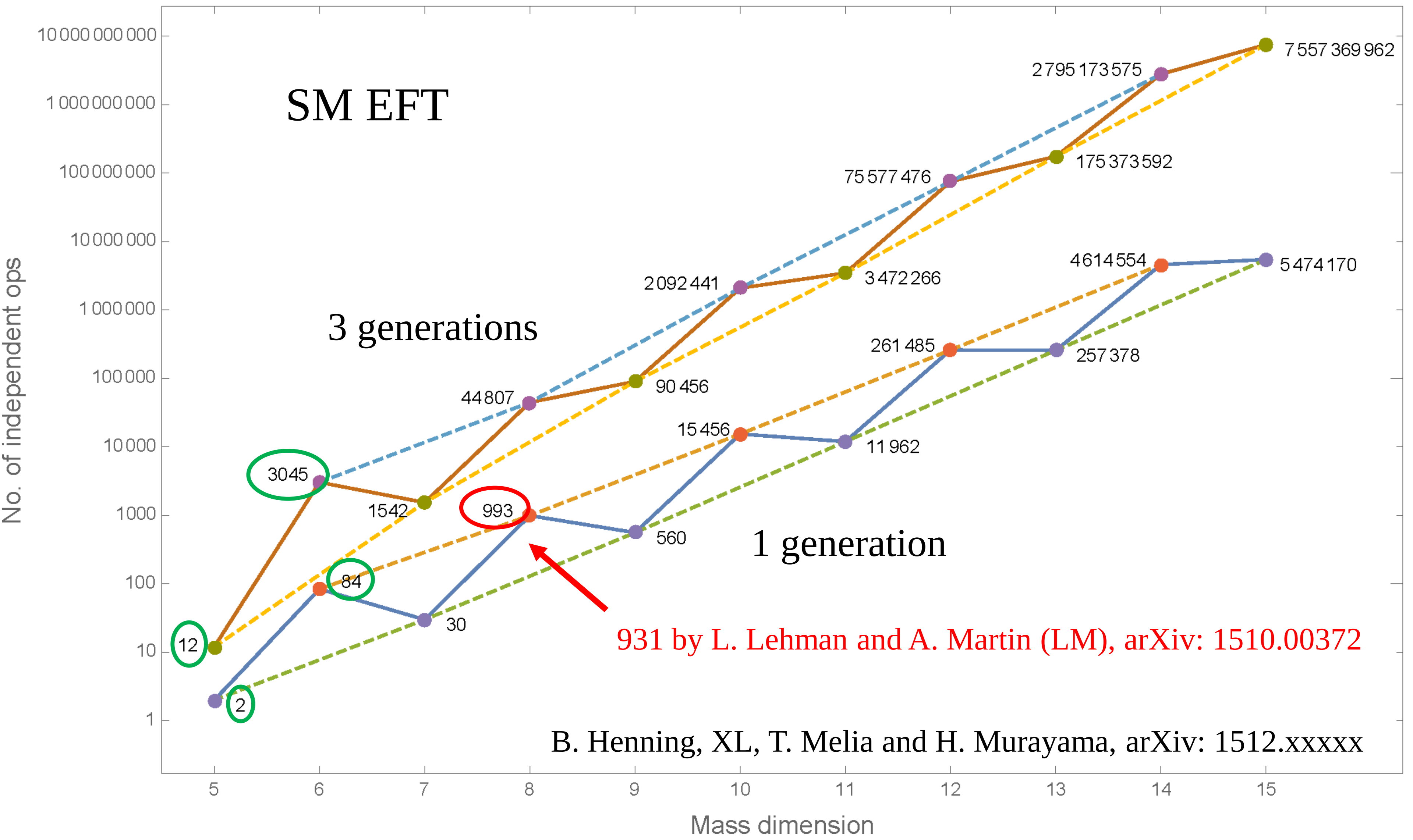

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931 by L. Lehman and A. Martin (LM), arXiv: 1510.00372
B. Henning, XL, T. Melia and H. Murayama, arXiv: 1512.xxxxx

Motivation

Why counting operators?

A theory = field content + symmetries imposed

$$\{\phi, \psi, A_\mu, \cdots\}$$

$$\begin{cases} \text{Lorentz invariance} \\ SU(3)_c \times SU(2)_W \times U(1)_Y \\ \cdots \end{cases}$$

Degree of freedom of the theory

SM	19
MSSM	105

Practical use?

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Practical use? None…

The technique developed may be useful in studying deeper underlying structures, if any

What is hard about counting operators?

The number is large!

What is hard about counting operators?

SM EFT: $\mathcal{L}(\Phi)=\mathcal{L}_{SM}(\Phi)+\frac{1}{\Lambda}\mathcal{O}_5+\frac{1}{\Lambda^2}\mathcal{O}_6+\frac{1}{\Lambda^3}\mathcal{O}_7+\frac{1}{\Lambda^4}\mathcal{O}_8+\cdots$

- Complicated field content $\Phi\in\{H,Q,u,d,L,e,G^a_{\mu\nu},W^a_{\mu\nu},B_{\mu\nu}\}$
- Various symmetries $SU(3)_c\times SU(2)_W\times U(1)_Y$, Lorentz invariance
- Redundancy relations

  - Group Identities $(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$
  - Equations of Motion (EOM) $\mathcal{O}\times D^2\phi=0$
  - Integration by Part (IBP) $D^\mu(\mathcal{O}_\mu)=0$

An operator basis: A set of complete but independent operators

What is hard about counting operators?

- $\text{dim} = 6, \quad N_f = 1$
  - W. Buchmuller and D. Wyler, Nucl. Phys. B 268 (1986) 621

    80 baryon conserving
  - B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, arXiv: 1008.4884

    59 baryon conserving

- $\text{dim} = 8, \quad N_f = 1$
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    535 operators = 931 real coefficients

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    535 operators = ~~931 real coefficients~~ 993 real coefficients

What is hard about counting operators?

counting by hand…

dim = 6 B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, arXiv: 1008.4884

X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar l_p e_r \varphi)$
$Q_{\widetilde G}$	$f^{ABC} \widetilde G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p u_r \widetilde\varphi)$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	$Q_{\varphi D}$	$\left(\varphi^\dagger D^\mu\varphi\right)^\star \left(\varphi^\dagger D_\mu\varphi\right)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p d_r \varphi)$
$Q_{\widetilde W}$	$\varepsilon^{IJK} \widetilde W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$				
$X^2\varphi^2$		$\psi^2 X\varphi$		$\psi^2\varphi^2 D$	
$Q_{\varphi G}$	$\varphi^\dagger\varphi\, G_{\mu\nu}^A G^{A\mu\nu}$	Q_{eW}	$(\bar l_p \sigma^{\mu\nu} e_r)\tau^I \varphi W_{\mu\nu}^I$	$Q_{\varphi l}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar l_p \gamma^\mu l_r)$
$Q_{\varphi\widetilde G}$	$\varphi^\dagger\varphi\, \widetilde G_{\mu\nu}^A G^{A\mu\nu}$	Q_{eB}	$(\bar l_p \sigma^{\mu\nu} e_r)\varphi B_{\mu\nu}$	$Q_{\varphi l}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu^I \varphi)(\bar l_p \tau^I \gamma^\mu l_r)$
$Q_{\varphi W}$	$\varphi^\dagger\varphi\, W_{\mu\nu}^I W^{I\mu\nu}$	Q_{uG}	$(\bar q_p \sigma^{\mu\nu} T^A u_r)\widetilde\varphi\, G_{\mu\nu}^A$	$Q_{\varphi e}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar e_p \gamma^\mu e_r)$
$Q_{\varphi\widetilde W}$	$\varphi^\dagger\varphi\, \widetilde W_{\mu\nu}^I W^{I\mu\nu}$	Q_{uW}	$(\bar q_p \sigma^{\mu\nu} u_r)\tau^I \widetilde\varphi\, W_{\mu\nu}^I$	$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar q_p \gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger\varphi\, B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\bar q_p \sigma^{\mu\nu} u_r)\widetilde\varphi\, B_{\mu\nu}$	$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu^I \varphi)(\bar q_p \tau^I \gamma^\mu q_r)$
$Q_{\varphi\widetilde B}$	$\varphi^\dagger\varphi\, \widetilde B_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\bar q_p \sigma^{\mu\nu} T^A d_r)\varphi\, G_{\mu\nu}^A$	$Q_{\varphi u}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar u_p \gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger\tau^I\varphi\, W_{\mu\nu}^I B^{\mu\nu}$	Q_{dW}	$(\bar q_p \sigma^{\mu\nu} d_r)\tau^I \varphi\, W_{\mu\nu}^I$	$Q_{\varphi d}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar d_p \gamma^\mu d_r)$
$Q_{\varphi\widetilde WB}$	$\varphi^\dagger\tau^I\varphi\, \widetilde W_{\mu\nu}^I B^{\mu\nu}$	Q_{dB}	$(\bar q_p \sigma^{\mu\nu} d_r)\varphi\, B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\widetilde\varphi^\dagger D_\mu \varphi)(\bar u_p \gamma^\mu d_r)$

Table 2: Dimension-six operators other than the four-fermion ones.

$(\bar LL)(\bar LL)$		$(\bar RR)(\bar RR)$		$(\bar LL)(\bar RR)$	
Q_{ll}	$(\bar l_p \gamma_\mu l_r)(\bar l_s \gamma^\mu l_t)$	Q_{ee}	$(\bar e_p \gamma_\mu e_r)(\bar e_s \gamma^\mu e_t)$	Q_{le}	$(\bar l_p \gamma_\mu l_r)(\bar e_s \gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar q_s \gamma^\mu q_t)$	Q_{uu}	$(\bar u_p \gamma_\mu u_r)(\bar u_s \gamma^\mu u_t)$	Q_{lu}	$(\bar l_p \gamma_\mu l_r)(\bar u_s \gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\bar q_p \gamma_\mu \tau^I q_r)(\bar q_s \gamma^\mu \tau^I q_t)$	Q_{dd}	$(\bar d_p \gamma_\mu d_r)(\bar d_s \gamma^\mu d_t)$	Q_{ld}	$(\bar l_p \gamma_\mu l_r)(\bar d_s \gamma^\mu d_t)$
$Q_{lq}^{(1)}$	$(\bar l_p \gamma_\mu l_r)(\bar q_s \gamma^\mu q_t)$	Q_{eu}	$(\bar e_p \gamma_\mu e_r)(\bar u_s \gamma^\mu u_t)$	Q_{qe}	$(\bar q_p \gamma_\mu q_r)(\bar e_s \gamma^\mu e_t)$
$Q_{lq}^{(3)}$	$(\bar l_p \gamma_\mu \tau^I l_r)(\bar q_s \gamma^\mu \tau^I q_t)$	Q_{ed}	$(\bar e_p \gamma_\mu e_r)(\bar d_s \gamma^\mu d_t)$	$Q_{qu}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar u_s \gamma^\mu u_t)$
		$Q_{ud}^{(1)}$	$(\bar u_p \gamma_\mu u_r)(\bar d_s \gamma^\mu d_t)$	$Q_{qu}^{(8)}$	$(\bar q_p \gamma_\mu T^A q_r)(\bar u_s \gamma^\mu T^A u_t)$
		$Q_{ud}^{(8)}$	$(\bar u_p \gamma_\mu T^A u_r)(\bar d_s \gamma^\mu T^A d_t)$	$Q_{qd}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar d_s \gamma^\mu d_t)$
				$Q_{qd}^{(8)}$	$(\bar q_p \gamma_\mu T^A q_r)(\bar d_s \gamma^\mu T^A d_t)$

$(\bar LR)(\bar RL)$ and $(\bar LR)(\bar LR)$		B-violating	
Q_{ledq}	$(\bar l_p^j e_r)(\bar d_s q_t^j)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$
$Q_{quqd}^{(1)}$	$(\bar q_p^j u_r)\varepsilon_{jk}(\bar q_s^k d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{quqd}^{(8)}$	$(\bar q_p^j T^A u_r)\varepsilon_{jk}(\bar q_s^k T^A d_t)$	$Q_{qqq}^{(1)}$	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\varepsilon_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(1)}$	$(\bar l_p^j e_r)\varepsilon_{jk}(\bar q_s^k u_t)$	$Q_{qqq}^{(3)}$	$\varepsilon^{\alpha\beta\gamma}(\tau^I\varepsilon)_{jk}(\tau^I\varepsilon)_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(3)}$	$(\bar l_p^j \sigma_{\mu\nu} e_r)\varepsilon_{jk}(\bar q_s^k \sigma^{\mu\nu} u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$

Table 3: Four-fermion operators.

Motivation

Our goal: find a systematic, automated, and correct way of doing it

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Outline for the rest of this talk

- Review of non-derivative techniques

- derivative warm up: 1d spacetime

- true task: higher d spacetime

non-derivative techniques

Counting technique without derivative (Invariant theory)

- Hilbert series as an organizing tool
- Molien-Weyl formula

Good reviews:

arXiv: 0907.4763, E. E. Jenkins and A. V. Manohar

arXiv: 1010.3161, A. Hanany, E. E. Jenkins, A. V. Manohar, and G. Torri

arXiv: 1503.07537, L. Lehman and A. Martin

Hilbert series as an organizing tool

polynomial ring

$$\mathbb{R}[x_1, x_2, \cdots, x_n] = V\left\{x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, r_i \in \mathbb{N}\right\}$$

$$r = r_1 + r_2 + \cdots + r_n, \quad c_r = ?$$

Organizing all order information:

$$H(u) = \sum_{r=0}^{\infty} c_r u^r, \quad |u| < 1$$

grading

counting information at each order

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$$\mathcal{K} = \mathbb{R}[\phi_1, \phi_2, \cdots, \phi_n]$$

$x_i = \phi_i \to u_i$ multi-graded

$$H = \frac{1}{1-u_1} \frac{1}{1-u_2} \cdots \frac{1}{1-u_n}$$

$u_1 = u_2 = \cdots = u_n = u$

$$H = \frac{1}{(1-u)^n}$$

Hilbert series as an organizing tool

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freely generated

Example for non freely generated ring

$$\mathcal{K} = \mathbb{R}\left[\phi^2, \phi^3\right], \qquad \phi \to u, \qquad H = \left(1 + u + u^2 + u^3 + u^4 + \cdots\right) - u = \frac{1+u^3}{1-u^2}$$

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$$\left.\begin{array}{l} x_1 = \phi^2 \to u^2 \\ x_2 = \phi^3 \to u^3 \end{array}\right\} \Rightarrow H \overset{?}{=} \frac{1}{\left(1-x_1\right)\left(1-x_2\right)} = \frac{1}{\left(1-u^2\right)\left(1-u^3\right)}$$

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$x_1^3 = x_2^2$ redundancy relation $I = \left\langle x_1^3 - x_2^2 \right\rangle$

$\mathcal{K} = R / I = \mathbb{R}\left[x_1, x_2\right] / \left\langle x_1^3 - x_2^2 \right\rangle$ A quotient ring

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the least common multiple

$$H = \frac{1 - x_1^3}{\left(1-x_1\right)\left(1-x_2\right)} = \frac{1-u^6}{\left(1-u^2\right)\left(1-u^3\right)} = \frac{1+u^3}{1-u^2}$$

Imposing symmetries

$$\mathcal{L}(\phi), \qquad \mathcal{K} = \mathbb{R}[\phi]$$

$$Z_2 = \{e, P\} \qquad \begin{cases} e\phi = \phi \\ P\phi = -\phi \end{cases}$$
$$P^2 = e$$

$$\phi \to u$$

$$x = \phi^2 \to u^2$$

$$\mathcal{K}_{inv} = \mathbb{R}[x]$$

$$H = \frac{1}{1-x} = \frac{1}{1-u^2}$$

$$\mathcal{L}(\phi, \phi^*), \qquad \mathcal{K} = \mathbb{R}[\phi, \phi^*]$$

$$U(1), \qquad \begin{cases} \phi \to e^{i\theta}\phi \\ \phi^* \to e^{-i\theta}\phi^* \end{cases}$$

$$\begin{cases} \phi \to u \\ \phi^* \to u^* \end{cases}$$

$$x = \phi^*\phi \to u^* u$$

$$\mathcal{K}_{inv} = \mathbb{R}[x]$$

$$H = \frac{1}{1-x} = \frac{1}{1-u^* u}$$

Nontrivial examples of imposing symmetries

$$\mathcal{L}(\phi_1,\phi_2), \quad Z_2=\{e,P\}$$
$$P\phi_1=-\phi_1, \quad P\phi_2=-\phi_2$$

$$\phi_1\to u_1, \quad \phi_2\to u_2$$

$$x_1=\phi_1^2, \quad x_2=\phi_2^2, \quad x_3=\phi_1\phi_2$$
$$x_1x_2=x_3^2$$

$$\mathcal{K}_{inv}=\mathbb{R}[x_1,x_2,x_3]/\langle x_1x_2-x_3^2\rangle$$

$$H=\frac{1-x_1x_2}{(1-x_1)(1-x_2)(1-x_3)}=\frac{1+u_1u_2}{(1-u_1^2)(1-u_2^2)}$$

$$\mathcal{L}(\phi_1,\phi_1^*,\phi_2,\phi_2^*), \quad U(1)$$
$$\phi_1\to e^{i\theta}\phi_1, \quad \phi_2\to e^{2i\theta}\phi_2$$

$$\phi_1\to u_1, \quad \phi_1^*\to u_1^*, \quad \phi_2\to u_2, \quad \phi_2^*\to u_2^*$$

$$x_1=\phi_1\phi_1^*, \quad x_2=\phi_2\phi_2^*, \quad x_3=\phi_1^2\phi_2^*, \quad x_4=\phi_1^{*2}\phi_2$$
$$x_1^2x_2=x_3x_4$$

$$\mathcal{K}_{inv}=\mathbb{R}[x_1,x_2,x_3,x_4]/\langle x_1^2x_2-x_3x_4\rangle$$

$$H=\frac{1-x_1^2x_2}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}=\frac{1-u_1^2u_1^{*2}u_2u_2^*}{(1-u_1u_1^*)(1-u_2u_2^*)(1-u_1^2u_2^*)(1-u_1^{*2}u_2)}$$

Molien-Weyl formula: representation theory

$$G = \{g\}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \to g = g_\Phi$$

$$H(u) \equiv \sum_{r=0}^{\infty} c_{\text{singlet},r} u^r = ?$$

$$\left\{ \phi_1^{r_1} \phi_2^{r_2} \cdots \phi_n^{r_n}, r_1 + r_2 + \cdots + r_n = r \right\} \to g_r$$

$$\mathbb{R}\left[\phi_1, \phi_2, \cdots, \phi_n \right] \to g_R = \oplus g_r$$

Molien-Weyl formula: representation theory

$$G=\{g\},\quad \Phi=\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix}\rightarrow g=g_\Phi$$

$$H(u)\equiv\sum_{r=0}^{\infty}c_{\text{singlet},r}u^r=?$$

$$\left\{\phi_1^{r_1}\phi_2^{r_2}\cdots\phi_n^{r_n},r_1+r_2+\cdots+r_n=r\right\}\rightarrow g_r$$

$$\mathbb{R}\left[\phi_1,\phi_2,\cdots,\phi_n\right]\rightarrow g_R=\oplus g_r$$

$$\chi_r(g)=\text{tr}(g_r)=\sum_a c_a\chi_a(g)$$

$$\frac{1}{|G|}\sum_g\chi_a^*(g)\chi_b(g)=\delta_{ab}$$

$$\Rightarrow\quad c_{\text{singlet},r}=\frac{1}{|G|}\sum_g 1\cdot\chi_r(g)$$

Molien-Weyl formula: representation theory

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$$\chi_R(u,g)\equiv\sum_{r=0}^{\infty}u^r\chi_r(g)$$ graded Ring (R) character

Molien-Weyl formula: representation theory

$$g_\Phi = U D_\Phi U^\dagger, \quad D_\Phi = \mathrm{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)$$

$$g_r = \mathrm{diag}\left\{\lambda_1^{r_1} \cdots \lambda_n^{r_n}, r_1 + \cdots + r_n = r\right\}$$

$$\chi_r\left(g\right) = \mathrm{tr}\left(g_r\right) = \sum_{r_1 + \cdots + r_n = r} \lambda_1^{r_1} \cdots \lambda_n^{r_n}$$

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$$= \frac{1}{\left(1-u\lambda_1\right)\cdots\left(1-u\lambda_n\right)} = \frac{1}{\det\left(1-uD_\Phi\right)} = \frac{1}{\det\left(1-ug_\Phi\right)}$$

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$$H\left(u\right) = \frac{1}{\left|G\right|} \sum_g \frac{1}{\det\left(1 - u g_\Phi\right)} = \int d\mu_G\left(g\right) \frac{1}{\det\left(1 - u g_\Phi\right)}$$

Weyl integration formula

$$g_1 = h g_2 h^{-1} \rightarrow g_1 \sim g_2$$

class function:

$$g_1 \sim g_2 \rightarrow f(g_1) = f(g_2)$$

$$\int d\mu_G(g)\, f(g) = \int_{\alpha \in T} d\mu_G(\alpha)\, f\left[g(\alpha)\right]$$

maximal torus generated by Cartan subalgebra

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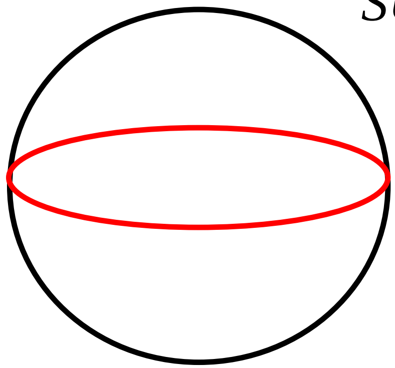
maximal torus generated by Cartan subalgebra



$$SU(2) \supset g = e^{i\theta^a t^a}$$

$$T \supset g(\alpha) = e^{i\theta t^3}, \quad \alpha \equiv e^{i\theta/2}$$

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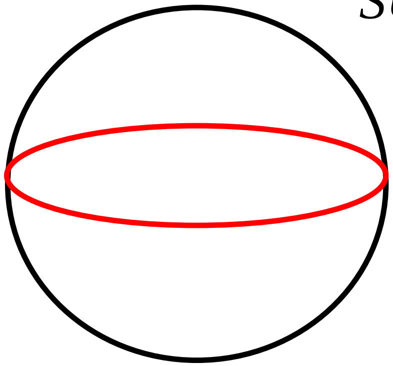

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W. Fulton and J. Harris, *Representation Theory*, Springer, New York, 2004

Weyl integration formula

$$\int d\mu_{U(1)} = \oint_{|x|=1} \frac{dx}{2\pi i}\frac{1}{x}, \qquad \int d\mu_{SU(2)} = \oint_{|y|=1} \frac{dy}{2\pi i}\frac{1}{y}\frac{1}{2}\left(1-y^2\right)\left(1-y^{-2}\right)$$

$$\int d\mu_{SU(3)} = \oint_{|z_1|=|z_2|=1} \frac{dz_1}{2\pi i}\frac{1}{z_1}\frac{dz_2}{2\pi i}\frac{1}{z_2}\frac{1}{6}\left(1-z_1z_2\right)\left(1-\frac{1}{z_1z_2}\right)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2}{z_1^2}\right)\left(1-\frac{z_1}{z_2^2}\right)\left(1-\frac{z_2^2}{z_1}\right)$$

A. Hanany and R. Kalveks, arXiv: 1408.4690

Group	*Haar Measure on Maximal Torus*
$U(r)$	$\frac{1}{(2\pi i)^r r!} \oint_{\lvert x_i\rvert=1} \frac{dx_i}{x_i} \prod_{1\le j<k\le r} \lvert x_j - x_k\rvert^2$
$SU(r+1)$	$\frac{1}{(2\pi i)^r (r+1)!} \prod_{i=1}^{r} \oint_{\lvert x_i\rvert=1} \frac{dx_i}{x_i} \prod_{1\le j<k\le r+1} \lvert x_j - x_k\rvert^2 \qquad x_{r+1} \equiv \prod_{j=1}^{r} \frac{1}{x_j}$
$SO(2r+1)$	$\frac{1}{2^r (2\pi i)^r r!} \prod_{i=1}^{r} \oint_{\lvert x_i\rvert=1} \frac{dx_i}{x_i} \prod_{1\le j<k\le r} \frac{(x_j-x_k)^2(1-x_jx_k)^2}{x^2{}_j x^2{}_k} \prod_{m=1}^{r} \frac{(1-x_m)(x_m-1)}{x_m}$
$USp(2r)$	$\frac{1}{2^r (2\pi i)^r r!} \prod_{i=1}^{r} \oint_{\lvert x_i\rvert=1} \frac{dx_i}{x_i} \prod_{1\le j<k\le r} \frac{(x_j-x_k)^2(1-x_jx_k)^2}{x^2{}_j x^2{}_k} \prod_{m=1}^{r} \frac{(1-x^2{}_m)(x^2{}_m-1)}{x^2{}_m}$
$SO(2r)$	$\frac{1}{2^{r-1} (2\pi i)^r r!} \prod_{i=1}^{r} \oint_{\lvert x_i\rvert=1} \frac{dx_i}{x_i} \prod_{1\le j<k\le r} \frac{(x_j-x_k)^2(1-x_jx_k)^2}{x^2{}_j x^2{}_k}$

Plethystic Exponential

$$\alpha \in T \to \mathrm{tr}\left\{\left[g_\Phi(\alpha)\right]^r\right\} = \chi_\Phi\left(\alpha^r\right)$$

$$\chi_R(u,\alpha) = \frac{1}{\det\left[1-ug_\Phi(\alpha)\right]} = \exp\left\{-\ln\det\left[1-ug_\Phi\right]\right\} = \exp\left\{-\mathrm{tr}\ln\left[1-ug_\Phi\right]\right\}$$
$$= \exp\left\{\sum_{r=1}^{\infty}\frac{1}{r}u^r\mathrm{tr}\left[\left(g_\Phi\right)^r\right]\right\} = \exp\left[\sum_{r=1}^{\infty}\frac{1}{r}u^r\chi_\Phi\left(\alpha^r\right)\right]$$
$$\equiv \mathrm{PE}\left[u\chi_\Phi(\alpha)\right]$$

More than one generate multiplets: $\{\Phi_a\}$

$$\chi_R(u_a,\alpha) = \frac{1}{\prod_a \det\left[1-u_a g_{\Phi_a}(\alpha)\right]} = \mathrm{PE}\left[\sum_a u_a \chi_{\Phi_a}(\alpha)\right]$$

A side note on fermionic fields

$$\sum_{r=0}^{\infty} u^r = \frac{1}{1-r}, \qquad \frac{1}{\det\left(1-ug_\Phi\right)}, \qquad \mathrm{PE}\left[u\chi_\Phi(\alpha)\right] = \exp\left[\sum_{r=1}^{\infty}\frac{1}{r}u^r\chi_\Phi\left(\alpha^r\right)\right]$$

↓ ↓ ↓

$$\sum_{r=0}^{1} u^r = 1+r, \qquad \det\left(1+ug_\Phi\right), \qquad \mathrm{PEF}\left[u\chi_\Phi(\alpha)\right] = \exp\left[\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}u^r\chi_\Phi\left(\alpha^r\right)\right]$$

Summary of Molien-Weyl formula

project out singlet

graded R character

$$H(u) = \frac{1}{|G|}\sum_{g} \chi_R(u,g) = \int_{\alpha\in T} d\mu_G(\alpha)\,\chi_R(u_a,\alpha)$$

$$\chi_R(u_a,\alpha) = \frac{1}{\prod_a \det\left[1 - u_a g_{\Phi_a}(\alpha)\right]} = \mathrm{PE}\left[\sum_a u_a \chi_{\Phi_a}(\alpha)\right]$$

generator multiplet representation matrix

generator multiplet character

Back to examples of imposing symmetry

$$\mathcal{L}(\phi_1,\phi_2), \quad Z_2=\{e,P\}$$
$$P\phi_1=-\phi_1, \quad P\phi_2=-\phi_2$$

$$\Phi_a=\{\phi_1,\phi_2\}\to u_a=\{u_1,u_2\}$$
$$g_{\Phi_a}(e)=\{1,1\}, \quad g_{\Phi_a}(P)=\{-1,-1\}$$

$$\chi_R(e)=\frac{1}{\prod_a \det\left[1-u_a g_{\Phi_a}(e)\right]}=\frac{1}{(1-u_1)(1-u_2)}$$

$$\chi_R(P)=\frac{1}{\prod_a \det\left[1-u_a g_{\Phi_a}(P)\right]}=\frac{1}{(1+u_1)(1+u_2)}$$

$$H=\frac{1}{2}\left[\chi_R(e)+\chi_R(P)\right]=\frac{1+u_1u_2}{\left(1-u_1^2\right)\left(1-u_2^2\right)}$$

Back to examples of imposing symmetry

$$\mathcal{L}\left(\phi_1,\phi_1^*,\phi_2,\phi_2^*\right),\quad U(1)$$

$$\phi_1 \to e^{i\theta}\phi_1,\quad \phi_2 \to e^{2i\theta}\phi_2$$

$$\Phi_a = \left\{\phi_1,\phi_1^*,\phi_2,\phi_2^*\right\} \to u_a = \left\{u_1,u_1^*,u_2,u_2^*\right\}$$

$$g_{\Phi_a}(\alpha) = \left\{e^{i\theta},e^{-i\theta},e^{2i\theta},e^{-2i\theta}\right\} = \left\{\alpha,\alpha^{-1},\alpha^2,\alpha^{-2}\right\}$$

$$\chi_R(u_a,\alpha) = \frac{1}{\prod_a \det\left[1-u_a g_{\Phi_a}(\alpha)\right]} = \frac{1}{(1-u_1\alpha)\left(1-u_1^*\alpha^{-1}\right)\left(1-u_2\alpha^2\right)\left(1-u_2^*\alpha^{-2}\right)}$$

$$H = \int d\mu_{U(1)}(\alpha)\,\chi_R(u_a,\alpha) = \oint_{|\alpha|=1}\frac{d\alpha}{2\pi i}\frac{1}{\alpha}\frac{1}{(1-u_1\alpha)\left(1-u_1^*\alpha^{-1}\right)\left(1-u_2\alpha^2\right)\left(1-u_2^*\alpha^{-2}\right)}$$

$$= \oint_{|\alpha|=1}\frac{d\alpha}{2\pi i}\frac{\alpha^2}{(1-u_1\alpha)\left(1-u_2\alpha^2\right)\left(\alpha-u_1^*\right)\left(\alpha^2-u_2^*\right)}$$

$$= \frac{1-u_1^2u_1^{*2}u_2u_2^*}{\left(1-u_1u_1^*\right)\left(1-u_2u_2^*\right)\left(1-u_1^2u_2^*\right)\left(1-u_1^{*2}u_2\right)}$$

Need to accommodate derivative!

- EOM   - IBP

- warm up: 1d real scalar
- true task: higher d

EOM: removing generators

$$\begin{cases} \phi \to u \\ \partial \to t \end{cases}$$

$$\mathcal{K}_{\text{free}} = \mathbb{R}\left[\phi, \partial\phi, \partial^2\phi, \cdots\right] \quad \Longrightarrow \quad H_{\text{free}}(u,t) = \frac{1}{(1-u)(1-tu)(1-t^2u)\cdots}$$

EOM: $\partial^2\phi = 0$

$$\mathcal{K}_{\text{EOM}} = \mathbb{R}\left[\phi, \partial\phi\right] \quad \Longrightarrow \quad H_{\text{EOM}}(u,t) = \frac{1}{(1-u)(1-tu)}$$

$$\mathcal{K}_{N,\text{EOM}} = \mathbb{R}\left[\phi_1, \partial\phi_1, \phi_2, \partial\phi_2, \cdots, \phi_N, \partial\phi_N\right] \qquad H_{N,\text{EOM}}(u_a,t) = \prod_{a=1}^{N} \frac{1}{(1-u_a)(1-tu_a)}$$

warm up: 1d real scalar

IBP: highest weight states of SO(3,C)

$$\Phi = \begin{pmatrix} \phi \\ \partial\phi \end{pmatrix}$$

A spin ½ representation of SO(3,C)

derivative as a lowering operator

$$\begin{pmatrix} \phi_1 \\ \partial\phi_1 \end{pmatrix} \otimes \begin{pmatrix} \phi_2 \\ \partial\phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1\phi_2 \\ \phi_1\partial\phi_2 + \phi_2\partial\phi_1 \\ \partial\phi_1\partial\phi_2 \\ \hline \phi_1\partial\phi_2 - \phi_2\partial\phi_1 \end{pmatrix}$$

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$$

Highest weights

Vanish by IBP/EOM

Calculation of Hilbert series

$$\Phi_a = \begin{pmatrix} \phi_a \\ \partial\phi_a \end{pmatrix} \to \bar{u}_a, \quad g_{\Phi_a}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

$$\alpha \sim p^{-1/2}$$

$$\begin{cases} u_a = \bar{u}_a \alpha \\ t = \alpha^{-2} \end{cases}$$

$$\chi_R(\bar{u}_a, \alpha) = \frac{1}{\prod_i \det\left[1 - \bar{u}_a g_{\Phi_a}(\alpha)\right]} = \frac{1}{\prod_{a=1}^{N} (1 - \bar{u}_a \alpha)(1 - \bar{u}_a \alpha^{-1})} = H_{N,\mathrm{EOM}}(\bar{u}_a, \alpha)$$

$$\chi_R(\bar{u}_a, \alpha) = \sum_{r=0}^{\infty} c_{r+1}(\bar{u}_a)\chi_{r+1}(\alpha), \quad \chi_{r+1}(\alpha) = \alpha^r + \alpha^{r-2} + \cdots + \alpha^{-r}$$

$$H_N(\bar{u}_0, \bar{u}_a) = \sum_{r=0}^{\infty} c_{r+1}(\bar{u}_a)\bar{u}_0^r = \int_{\alpha \in T} d\mu_{SO(3,\mathrm{C})}(\alpha) \sum_{r=0}^{\infty} \bar{u}_0^r \chi_{r+1}^*(\alpha) \chi_R(\bar{u}_a, \alpha)$$

$$\oint_{|\alpha|=1} \frac{d\alpha}{2\pi i} \frac{1}{2\alpha} (1 - \alpha^2)(1 - \alpha^{-2})$$

$$\frac{1}{(1 - \bar{u}_0 \alpha)(1 - \bar{u}_0 \alpha^{-1})}$$

Closed form results

$$H_N(\bar{u}_0,\bar{u}_a) = \oint_{|\alpha|=1} \frac{d\alpha}{2\pi i}\frac{1}{\alpha}\left(1-\alpha^2\right)\prod_{a=0}^{N}\frac{1}{\left(1-\bar{u}_a\alpha\right)\left(1-\bar{u}_a\alpha^{-1}\right)} = -\frac{1}{2}\sum_{b=0}^{N}\frac{\left(1-\bar{u}_b^2\right)\bar{u}_b^{N-2}}{\prod_{a\neq b}\left[\left(1-\bar{u}_b\bar{u}_a\right)\left(\bar{u}_b-\bar{u}_a\right)\right]}$$

$$H_1 = \frac{1}{1-\bar{u}_0\bar{u}_1}$$

$$H_2 = \frac{1}{\left(1-\bar{u}_0\bar{u}_1\right)\left(1-\bar{u}_0\bar{u}_2\right)\left(1-\bar{u}_1\bar{u}_2\right)}$$

$$H_3 = \frac{1-\bar{u}_0\bar{u}_1\bar{u}_2\bar{u}_3}{\left(1-\bar{u}_0\bar{u}_1\right)\left(1-\bar{u}_0\bar{u}_2\right)\left(1-\bar{u}_0\bar{u}_3\right)\left(1-\bar{u}_1\bar{u}_2\right)\left(1-\bar{u}_1\bar{u}_3\right)\left(1-\bar{u}_2\bar{u}_3\right)}$$

$\ldots\ldots$

warm up: 1d real scalar

Side remark: recursion relation (composition rule)

$$H_N(\bar{u}_0,\bar{u}_a)=\oint_{|\alpha|=1}\frac{d\alpha}{2\pi i}\frac{1}{\alpha}\left(1-\alpha^2\right)H_{N,\mathrm{EOM}}(\bar{u}_a,\alpha)$$

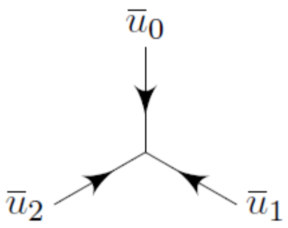
$$H_{N+1,\mathrm{EOM}}(\bar{u}_a,\alpha)=H_{N,\mathrm{EOM}}(\bar{u}_a,\alpha)H_{1,\mathrm{EOM}}(\bar{u}_{N+1},\alpha)$$

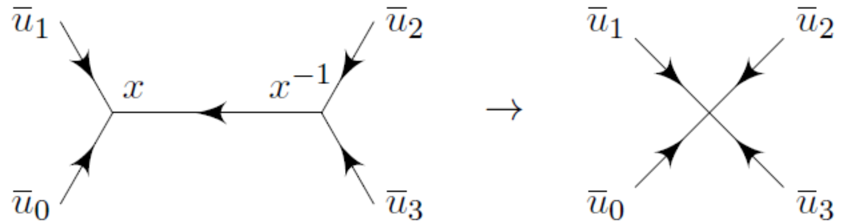
$$H_{N+1}(\bar{u}_0,\bar{u}_1,\cdots,\bar{u}_{N+1})=\oint_{|x|=1}\frac{dx}{2\pi i}\frac{1}{x}H_N(\bar{u}_0,\bar{u}_1,\cdots,\bar{u}_{N-1},x)H_2\left(x^{-1},\bar{u}_N,\bar{u}_{N+1}\right)$$

$$H_{N+M-1}(\bar{u}_0,\bar{u}_1,\cdots,\bar{u}_{N+M-1})=\oint_{|x|=1}\frac{dx}{2\pi i}\frac{1}{x}H_N(\bar{u}_0,\bar{u}_1,\cdots,\bar{u}_{N-1},x)H_M\left(x^{-1},\bar{u}_N,\cdots,\bar{u}_{N+M-1}\right)$$

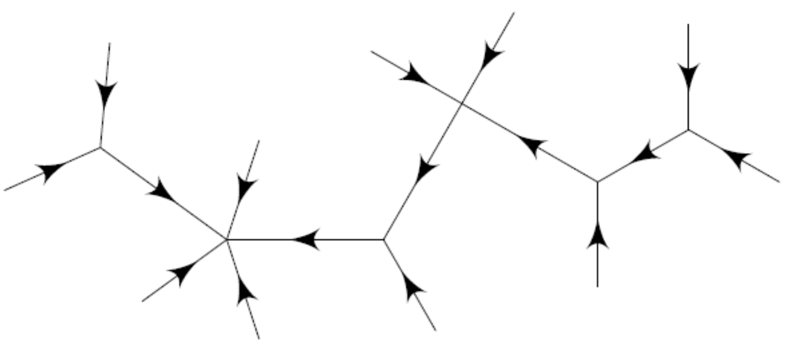
Side remark: composition rule in diagrams

$$H_2\left(\bar{u}_0,\bar{u}_1,\bar{u}_2\right) \qquad \oint_{|x|=1}\frac{dx}{2\pi i}\frac{1}{x}H_2\left(\bar{u}_0,\bar{u}_1,x\right)H_2\left(x^{-1},\bar{u}_2,\bar{u}_3\right)=H_3\left(\bar{u}_0,\bar{u}_1,\bar{u}_2,\bar{u}_3\right)$$





general composition:



1d method recap: two steps

- Find graded R character

$$\chi_R(\bar{u}_a,\alpha)=\frac{1}{\prod_a \det\left[1-\bar{u}_a g_{\Phi_a}(\alpha)\right]} H_{N,\mathrm{EOM}}(\bar{u}_a,\alpha)$$

- Project out highest weight states

$$H_N(\bar{u}_0,\bar{u}_a)=\int d\mu_{SO(3,\mathrm{C})}(\alpha)\sum_{r=0}^{\infty}\bar{u}_0^r\chi_{r+1}^*(\alpha)\chi_R(\bar{u}_a,\alpha)$$

Can we generalize the 1d method to higher d?

- Lorentz invariance

  Euclidean spacetime $SO(d)$ ⟹ Molien-Weyl formula

Can we generalize the 1d method to higher d?

- Lorentz invariance

  Euclidean spacetime $SO(d)$ → Molien-Weyl formula

- EOM does not truncate generators

$$\mathcal{K}_{\text{free}} = \mathbb{R}\left[\phi,\quad \partial_\mu\phi,\quad \partial_\mu\partial_\nu\phi,\quad \partial_\mu\partial_\nu\partial_\rho\phi,\cdots\right] \qquad \partial^2\phi = \partial_\mu\partial_\mu\phi = 0$$

$$\mathcal{K}_{\text{EOM}} = \mathbb{R}\left[\partial^k\phi\right] \qquad \partial^k\phi:\ \ \text{traceless symmetric}\ \ \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_k}\phi$$

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- Lorentz invariance

  Euclidean spacetime $SO(d)$ ➡ Molien-Weyl formula

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- IBP

  A group where each ∂_μ acts as a lowering operator?

Can we generalize the 1d method to higher d?

- Lorentz invariance

  Euclidean spacetime $SO(d)$ ➡ Molien-Weyl formula

- EOM does not truncate generators

  $\mathcal{K}_{\text{free}} = \mathbb{R}\left[\phi, \quad \partial_\mu \phi, \quad \partial_\mu \partial_\nu \phi, \quad \partial_\mu \partial_\nu \partial_\rho \phi, \cdots\right]$ $\qquad \partial^2 \phi = \partial_\mu \partial_\mu \phi = 0$

  $\mathcal{K}_{\text{EOM}} = \mathbb{R}\left[\partial^k \phi\right]$ $\qquad \partial^k \phi:$ traceless symmetric $\partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_k} \phi$

- IBP

  A group where each ∂_μ acts as a lowering operator?

$SO(3,\mathbb{C})$ ➡ $SO(d+2,\mathbb{C})$ complexified conformal group

Can we generalize the 1d method to higher d?

conformal group

$$SO(d,2) \supset \{L_{\mu\nu}, P_\mu, K_\mu, D\} \quad \Rightarrow \quad SO(d+2,\mathbb{C}) \supset \begin{cases} [D, P_\mu] = -P_\mu \\ [D, K_\mu] = +K_\mu \end{cases}$$

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real scalar

$$\mathcal{K}_{\mathrm{EOM}} = \mathbb{R}[\Phi] \qquad \Phi = \begin{pmatrix} \phi \\ \partial\phi \\ \partial^2\phi \\ \vdots \\ \partial^k\phi \\ \vdots \end{pmatrix} \quad \Big\downarrow P_\mu$$

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Lorentz invariance, EOM, and IBP → scalar conformal primaries

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Lorentz invariance, EOM, and IBP → scalar conformal primaries

$$p_i^2 = 0, \quad \sum_i p_i = 0 \longrightarrow \#(p_i p_j) = \binom{n}{2} - n = \frac{n(n-3)}{2} = \#(\text{cross ratios})$$

4d real scalar example

- Find graded R character

$$\chi_R(\phi,q,\alpha,\beta) = \mathrm{PE}\left[\phi\chi_\Phi(q,\alpha,\beta)\right] \equiv \exp\left[\sum_{r=1}^{\infty}\frac{1}{r}\phi^r\chi_\Phi(q^r,\alpha^r,\beta^r)\right]$$

$$\Phi \supset \partial^k\phi \sim \left(\frac{k}{2},\frac{k}{2}\right)$$

Lorentz group

$$SO(4) \simeq SU(2)_L \times SU(2)_R$$

$$g_\Phi(q,\alpha,\beta) = \bigoplus_k\left[q^{k+1}\mathrm{diag}\left(\alpha^k,\alpha^{k-2},\cdots,\alpha^{-k}\right)\otimes\mathrm{diag}\left(\beta^k,\beta^{k-2},\cdots,\beta^{-k}\right)\right]$$

$$\chi_\Phi(q,\alpha,\beta) = \mathrm{tr}\left[g_\Phi(q,\alpha,\beta)\right] = \sum_{k=0}^{\infty}\sum_{m,n=0}^{k} q^{k+1}\alpha^{2m-k}\beta^{2n-k} = q\left(1-q^2\right)P(q,\alpha,\beta)$$

$$P(q,\alpha,\beta) \equiv \frac{1}{(1-q\alpha\beta)(1-q\alpha\beta^{-1})(1-q\alpha^{-1}\beta)(1-q\alpha^{-1}\beta^{-1})}$$

4d real scalar example

- Project out 0-spin primaries

$$\chi_\Delta(q,\alpha,\beta)=q^\Delta P(q,\alpha,\beta)$$

$$\chi_{[\Delta;\bar{l}]}(q,\alpha,\beta)=q^\Delta \chi_{\bar{l}}^{Lorentz}(\alpha,\beta)P(q,\alpha,\beta)$$

F. A. Dolan, arXiv: hep-th/0508031

Orthonormality: $\int d\mu_{SO(4,2)}(q,\alpha,\beta)\chi_\Delta^*(q,\alpha,\beta)\chi_{\Delta'}(q,\alpha,\beta)=\delta_{\Delta\Delta'}$

$$\int d\mu_{SO(4,2)}(q,\alpha,\beta)=\int d\mu_{SO(4)}(\alpha,\beta)\oint_{|q|=1}\frac{dq}{2\pi i}\frac{1}{q}\frac{1}{P(q,\alpha,\beta)P(q^{-1},\alpha^{-1},\beta^{-1})}$$

$$\int d\mu_{SO(4)}(\alpha,\beta)=\oint_{|\alpha|=|\beta|=1}\frac{d\alpha}{2\pi i}\frac{d\beta}{2\pi i}\frac{1}{4\alpha\beta}(1-\alpha^2)(1-\alpha^{-2})(1-\beta^2)(1-\beta^{-2})$$

$$H(\phi,p)=\int d\mu_{SO(4,2)}\sum_{\Delta=0}^{\infty}p^\Delta\chi_\Delta^*\chi_R=\int d\mu_{SO(4)}\frac{1}{P}\chi_R$$

$$\sum_{\Delta=0}^{\infty}p^\Delta\chi_\Delta^*=\sum_{\Delta=0}^{\infty}p^\Delta q^{-\Delta}P(q^{-1},\alpha^{-1},\beta^{-1})=\frac{1}{1-pq^{-1}}P(q^{-1},\alpha^{-1},\beta^{-1})$$

One subtlety: unitarity bound

$$H(\phi, p) = \int d\mu_{SO(4)} \frac{1}{P} \chi_R = \left(1 + p^4\right) + \left(p - p^3\right)\phi + O\left(\phi^2\right)$$

$$\chi_\Delta(q,\alpha,\beta) = q^\Delta P(q,\alpha,\beta)$$

$\tilde{\chi}_\Delta(q,\alpha,\beta) \neq \chi_\Delta(q,\alpha,\beta)$ unitarity bound is saturated

unitarity bound $\quad \tilde{\chi}_\Delta(q,\alpha,\beta) = \chi_\Delta(q,\alpha,\beta), \quad \Delta > 1$

$\Delta \geq 1, \quad l = 0 \quad \tilde{\chi}_1(q,\alpha,\beta) = \chi_1(q,\alpha,\beta) - \chi_3(q,\alpha,\beta)$

F. A. Dolan, arXiv: hep-th/0508031

$$\chi_\Phi(q,\alpha,\beta) = q\left(1 - q^2\right)P(q,\alpha,\beta) = \tilde{\chi}_1(q,\alpha,\beta)$$

One subtlety: unitarity bound

$$\chi_R(\phi,q,\alpha,\beta)=1+c_1(\phi)\tilde{\chi}_1(q,\alpha,\beta)+c_2(\phi)\tilde{\chi}_2(q,\alpha,\beta)+c_3(\phi)\tilde{\chi}_3(q,\alpha,\beta)+\cdots$$

$$H(\phi,p)=1+c_1(\phi)p+c_2(\phi)p^2+c_3(\phi)p^3+\cdots$$

$$\chi_R-1=c_1\chi_1+c_2\chi_2+(c_3-c_1)\chi_3+\cdots$$

$$c_\Delta=\int d\mu_{SO(4,2)}\chi_\Delta^*(\chi_R-1),\qquad \Delta\neq 3$$

$$c_3=\int d\mu_{SO(4,2)}\left(\chi_1^*+\chi_3^*\right)(\chi_R-1)$$

$$H(\phi,p)=1+\int d\mu_{SO(4,2)}\left(\sum_{\Delta=0}^{\infty}p^\Delta\chi_\Delta^*+p^3\chi_1^*-\chi_0^*\right)(\chi_R-1)$$

$$=\int d\mu_{SO(4)}\frac{1}{P}\chi_R+p^3\phi-p^4$$

4d SM EFT

- Find graded R character

$$\chi_R\left(\{\phi_a\},q,\alpha,\beta,x,y,z_1,z_2\right)=\mathrm{PE}\left[\sum_a \phi_a\chi_{\Phi_a}\left(q,\alpha,\beta,x,y,z_1,z_2\right)\right]$$

$$\chi_{\Phi_a}\left(q,\alpha,\beta,x,y,z_1,z_2\right)=\chi_{\Phi_a}^{conformal}\left(q,\alpha,\beta\right)\chi_{\Phi_a}^{gauge}\left(x,y,z_1,z_2\right)$$

- Project out gauge invariant 0-spin primaries

$$H\left(\{\phi_a\},p\right)=\int d\mu_{gauge}\left(x,y,z_1,z_2\right)\int d\mu_{SO(4)}\left(\alpha,\beta\right)\frac{1}{P\left(p,\alpha,\beta\right)}\chi_R\left(\{\phi_a\},p,\alpha,\beta,x,y,z_1,z_2\right)$$

$$=\int d\mu_{gauge}\int d\mu_{SO(4)}\frac{1}{P}\mathrm{PE}\left[\sum_a \phi_a\chi_{\Phi_a}^{conformal}\chi_{\Phi_a}^{gauge}\right]$$

4d SM EFT

- $H\left(\{\phi_a\},p\right)=\int d\mu_{gauge}\int d\mu_{SO(4)}\frac{1}{P}\mathrm{PE}\left[\sum_a \phi_a \chi_{\Phi_a}^{conformal}\chi_{\Phi_a}^{gauge}\right]$

Field content

$$\Phi_a \in \{H,Q,u_c,d_c,L,e_c,G_{\mu\nu}^L,W_{\mu\nu}^L,B_{\mu\nu}^L,c.c.\}$$

Lorentz group representations

$$\phi \sim (0,0)$$
$$\psi_L \sim (1/2,0),\quad \psi_R \sim (0,1/2)$$
$$X_{\mu\nu}^L \sim (1,0),\quad X_{\mu\nu}^R \sim (0,1)$$

$$\begin{cases} X_{\mu\nu}^L \equiv X_{\mu\nu} + i\tilde{X}_{\mu\nu} \\ X_{\mu\nu}^R \equiv X_{\mu\nu} - i\tilde{X}_{\mu\nu} \end{cases}, \qquad \tilde{X}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}X^{\rho\sigma}$$

□	$SU(2)_L$	$SU(2)_R$	$SU(3)_c$	$SU(2)_W$	$U(1)_Y$
H	1	1	1	2	3
Q	2	1	3	2	1
u_c	2	1	$\overline{3}$	1	−4
d_c	2	1	$\overline{3}$	1	2
L	2	1	1	2	−3
e_c	2	1	1	1	6
B_L	3	1	1	1	0
W_L	3	1	1	3	0
G_L	3	1	8	1	0

4d SM EFT

- $H(\{\phi_a\},p)=\int d\mu_{gauge}\int d\mu_{SO(4)}\frac{1}{P}\mathrm{PE}\left[\sum_a \phi_a \chi_{\Phi_a}^{conformal}\chi_{\Phi_a}^{gauge}\right]$

$$\text{EOM}\begin{cases} D^2\phi=0 \\ \not{D}\psi_L=0 \\ D^\mu X_{\mu\nu}^L=0\end{cases} \Rightarrow \begin{cases} D^k\phi\sim\left(\frac{k}{2},\frac{k}{2}\right) \\ D^k\psi_L\sim\left(\frac{k+1}{2},\frac{k}{2}\right) \\ D^k X_L\sim\left(\frac{k+2}{2},\frac{k}{2}\right)\end{cases}$$

$$\chi_{(m/2,n/2)}^{Lorentz}(\alpha,\beta)=\frac{\alpha^{m+1}-\alpha^{-m-1}}{\alpha-\alpha^{-1}}\frac{\beta^{n+1}-\beta^{-n-1}}{\beta-\beta^{-1}}$$

$$\chi_\phi^{conformal}(q,\alpha,\beta)=\sum_{k=0}^{\infty}q^{k+1}\frac{\alpha^{k+1}-\alpha^{-k-1}}{\alpha-\alpha^{-1}}\frac{\beta^{k+1}-\beta^{-k-1}}{\beta-\beta^{-1}}=q(1-q^2)P(q,\alpha,\beta)$$

$$\chi_{\psi_L}^{conformal}(q,\alpha,\beta)=\sum_{k=0}^{\infty}q^{k+3/2}\frac{\alpha^{k+2}-\alpha^{-k-2}}{\alpha-\alpha^{-1}}\frac{\beta^{k+1}-\beta^{-k-1}}{\beta-\beta^{-1}}=q^{3/2}\left[(\alpha+\alpha^{-1})-q(\beta+\beta^{-1})\right]P(q,\alpha,\beta)$$

$$\chi_{X_L}^{conformal}(q,\alpha,\beta)=\sum_{k=0}^{\infty}q^{k+2}\frac{\alpha^{k+3}-\alpha^{-k-3}}{\alpha-\alpha^{-1}}\frac{\beta^{k+1}-\beta^{-k-1}}{\beta-\beta^{-1}}=q^2\left[(\alpha^2+1+\alpha^{-2})-q(\alpha+\alpha^{-1})(\beta+\beta^{-1})+q^2\right]P(q,\alpha,\beta)$$

4d SM EFT

- $$H\left(\{\phi_a\},p\right)=\int d\mu_{gauge}\int d\mu_{SO(4)}\frac{1}{P}\mathrm{PE}\left[\sum_a \phi_a \chi_{\Phi_a}^{conformal}\chi_{\Phi_a}^{gauge}\right]$$

Gauge invariance $SU(3)_c \times SU(2)_W \times U(1)_Y$

$$\chi_{\Phi_a}^{gauge}(x,y,z_1,z_2)=\chi_{\Phi_a}^{U(1)}(x)\,\chi_{\Phi_a}^{SU(2)}(y)\,\chi_{\Phi_a}^{SU(3)}(z_1,z_2)$$

$$\chi_Q^{U(1)}(x)=x^Q,\qquad \chi_2^{SU(2)}(y)=\left[\chi_{\bar{2}}^{SU(2)}(y)\right]^*=y+y^{-1},\qquad \chi_{\mathrm{adj}}^{SU(2)}(y)=y^2+1+y^{-2}$$

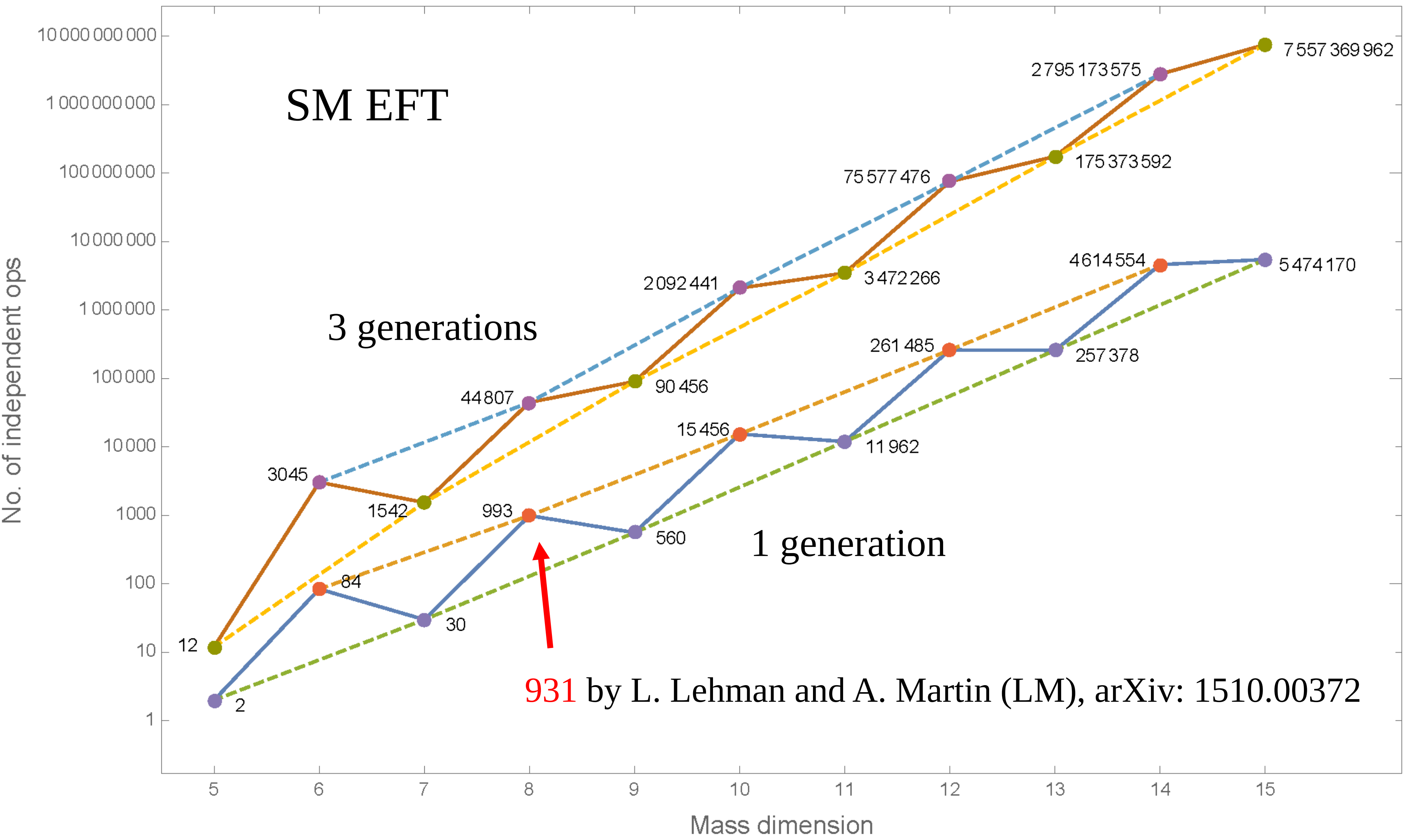
$$\chi_3^{SU(3)}(z_1,z_2)=\left[\chi_{\bar{3}}^{SU(3)}(z_1,z_2)\right]^*=z_1+\frac{z_2}{z_1}+\frac{1}{z_2},\qquad \chi_{\mathrm{adj}}^{SU(3)}(z_1,z_2)=z_1z_2+\frac{z_1^2}{z_2}+\frac{z_2^2}{z_1}+2+\frac{z_2}{z_1^2}+\frac{z_1}{z_2^2}+\frac{1}{z_1z_2}$$

$$\int d\mu_{gauge}(x,y,z_1,z_2)=\int d\mu_{U(1)}(x)\int d\mu_{SU(2)}(y)\int d\mu_{SU(3)}(z_1,z_2)$$

$$\int d\mu_{U(1)}=\oint_{|x|=1}\frac{dx}{2\pi i}\frac{1}{x},\qquad \int d\mu_{SU(2)}=\oint_{|y|=1}\frac{dy}{2\pi i}\frac{1}{y}\frac{1}{2}\left(1-y^2\right)\left(1-y^{-2}\right)$$

$$\int d\mu_{SU(3)}=\oint_{|z_1|=|z_2|=1}\frac{dz_1}{2\pi i}\frac{1}{z_1}\frac{dz_2}{2\pi i}\frac{1}{z_2}\frac{1}{6}\left(1-z_1z_2\right)\left(1-\frac{1}{z_1z_2}\right)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2}{z_1^2}\right)\left(1-\frac{z_1}{z_2^2}\right)\left(1-\frac{z_2^2}{z_1}\right)$$

true task: higher d



What mistake did LM make?

LM method: $H = \max\left\{H_{\text{EOM}} - pH_{D,\text{EOM}}, 0\right\}$

$$H_{\text{EOM}}\left(\{\phi_a\}, p\right) = \int d\mu_{SO(4)}(\alpha,\beta)\,\chi_R\left(\{\phi_a\}, p, \alpha, \beta\right)$$

$$H_{D,\text{EOM}}\left(\{\phi_a\}, p\right) = \int d\mu_{SO(4)}(\alpha,\beta)\,\chi_{(1/2,1/2)}^{Lorentz}(\alpha,\beta)\,\chi_R\left(\{\phi_a\}, p, \alpha, \beta\right)$$

$$A_\mu \in \mathcal{K}_{\text{EOM}} \to D_\mu A_\mu = 0$$

$$A_\mu = D_\nu A_{\mu\nu} \to D_\mu A_\mu = D_\mu D_\nu A_{\mu\nu} = 0$$

L. Lehman and A. Martin, arXiv: 1510.00372

Class $\mathcal{O}(D^2\, H^2\, H^{\dagger 2}\, X)$: there is just one operator in this class (plus h.c.)

$$D^2((H^\dagger H)^2 W^L), \tag{3.26}$$

B. Henning, XL, T. Melia and H. Murayama, arXiv: 1512.xxxxx

Class $H^4 X \mathcal{D}^2$:

$$H^2 H^{\dagger\,2} B_L \mathcal{D}^2,\ 2H^2 H^{\dagger\,2} W_L \mathcal{D}^2 \quad \text{all} + \text{h.c.} \tag{3.30}$$

$$\mathcal{K}_{\text{EOM}} \supset \left\{ \left(D_\mu H^\dagger\right)\left(D_\nu H\right)\left(H^\dagger H\right) B^L_{\mu\nu} \quad , \quad \left(D_\mu H^\dagger H\right)\left(H^\dagger D_\nu H\right) B^L_{\mu\nu} \right\}$$

$$\mathcal{K}_{\text{EOM}} \supset \left\{ \left(H^\dagger D_\nu H\right)\left(H^\dagger H\right) B^L_{\mu\nu} \quad , \quad \left(D_\nu H^\dagger H\right)\left(H^\dagger H\right) B^L_{\mu\nu} \right\}$$

$$\left(H^\dagger D_\nu H\right)\left(H^\dagger H\right) B^L_{\mu\nu} + \left(D_\nu H^\dagger H\right)\left(H^\dagger H\right) B^L_{\mu\nu} = D_\nu \left[\frac{1}{2} \left(H^\dagger H\right)^2 B^L_{\mu\nu} \right]$$

true task: higher d

$$\text{The \# of IBP} = \#\left(D_\mu A_\mu = 0\right)$$

$$= \quad A_\mu = D_\nu A_{\mu\nu}$$

$$\#\left(A_\mu\right) - \#\left(D_\nu A_{\mu\nu}\right)$$

true task: higher d

$$\text{The \# of IBP} \;=\; \#\left(D_\mu A_\mu = 0\right)$$

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$$\#\left(A_\mu\right) - \#\left(D_\nu A_{\mu\nu}\right)$$

$$\|$$

$$\#\left(A_{\mu\nu}\right) - \#\left(D_\rho A_{\mu\nu\rho}\right)$$

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$$=$$

$$\#\left(A_{\mu\nu\rho}\right) - \#\left(D_\sigma A_{\mu\nu\rho\sigma}\right)$$

true task: higher d

The # of IBP $= \#\left(D_\mu A_\mu = 0\right)$

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$$\|$$

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$$\|$$

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$$\|$$

$$\#\left(A_{\mu\nu\rho\sigma}\right)$$

true task: higher d

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$$= \quad A_\mu = D_\nu A_{\mu\nu}$$

$$\#\left(A_\mu\right) - \#\left(D_\nu A_{\mu\nu}\right)$$

$$=$$

$$\#\left(A_{\mu\nu}\right) - \#\left(D_\rho A_{\mu\nu\rho}\right)$$

$$=$$

$$\#\left(A_{\mu\nu\rho}\right) - \#\left(D_\sigma A_{\mu\nu\rho\sigma}\right)$$

$$=$$

$$\#\left(A_{\mu\nu\rho\sigma}\right)$$

The # of IBP $= \#\left(A_\mu\right) - \#\left(A_{\mu\nu}\right) + \#\left(A_{\mu\nu\rho}\right) - \#\left(A_{\mu\nu\rho\sigma}\right)$

What mistake did LM make?

$$H = H_{\text{EOM}} - pH_{\text{1-form,EOM}} + p^2 H_{\text{2-form,EOM}} - p^3 H_{\text{3-form,EOM}} + p^4 H_{\text{4-form,EOM}}$$

$$= \int d\mu_{SO(4)} \left\{1 - p\chi^{Lorentz}_{(1/2,1/2)} + p^2\left[\chi^{Lorentz}_{(1,0)} + \chi^{Lorentz}_{(0,1)}\right] - p^3 \chi^{Lorentz}_{(1/2,1/2)} + p^4\right\} \chi_R$$

$$= \int d\mu_{SO(4)} \frac{1}{P(p,\alpha,\beta)} \chi_R$$

Then it is almost correct

$$\left(1 - p\alpha\beta\right)\left(1 - p\alpha\beta^{-1}\right)\left(1 - p\alpha^{-1}\beta\right)\left(1 - p\alpha^{-1}\beta^{-1}\right)$$

General d case

Lorentz invariance, EOM, and IBP → scalar conformal primaries

$$H(\phi, p) = \int d\mu_{SO(d,2)}(q, x_i) \sum_{\Delta=0}^{\infty} p^{\Delta} \chi_{\Delta}^{*}(q, x_i) \chi_R(\phi, q, x_i)$$

$$\chi_{\Delta}(q, x_i) = q^{\Delta} P(q, x_i)$$

$$P(q, x_i) = \begin{cases} \dfrac{1}{(1-qx_i)(1-qx_i^{-1})} \\ \dfrac{1}{(1-qx_i)(1-qx_i^{-1})} \dfrac{1}{1-q} \end{cases}$$

$$\tilde{\chi}_{\Delta} \neq \chi_{\Delta}$$

when unitarity bound is saturated

$$\int d\mu_{SO(d,2)}(q, x_i) = \int d\mu_{SO(d)}(x_i) \oint_{|q|=1} \frac{dq}{2\pi i} \frac{1}{q} \frac{1}{P(q, x_i) P(q^{-1}, x_i)}$$

F. A. Dolan, arXiv: hep-th/0508031

Conformal representation of each field content: $\chi_R(\phi, q, x_i) = \mathrm{PE}\left[\phi \chi_{\Phi}(q, x_i)\right]$

Possible future direction

- Application: counting issue in other theories, such as ChPT?
- Deeper understanding: connection between Hilbert series and other observables?
- technical: generalizing the recursion relation at 1d?

Thank you!