A three-dimensional bridge between physics and mathematics

Tudor Dimofte
Institute for Advanced Study

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I'd like to discuss an example of this symbiosis,
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"3d-3d correspondence"

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M , $\mathfrak{g}$ $\rightsquigarrow$ $T_{\mathfrak{g}}[M]$

first hints:
[Dimofte-Gukov-Hollands ’10]

3-manifold A,D,E 3d (N=2) SUSY field theory

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depending only on topology of M!

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- its observables (quantities one can compute) all correspond to classical, quantum, or categorical topological invariants, some old, but many new.

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e.g.: hyperbolic volume $\mathrm{Vol}(M)$ [Mostow ’73]

space of flat $G_{\mathbb{C}}$ conn’s $\mathcal{M}_{\text{flat}}(M, G_{\mathbb{C}})$

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cf. Jones poly, WRT [Witten '89, Reshetikhin-Turaev '91]

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"Most" M, $\mathfrak{g} = sl_2$: explicit construction of $T_{\mathfrak{g}}[M]$

[Dimofte-Gaiotto-Gukov '11]
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Main tool: (topological) ideal triangulations
+ a generalization of Thurston-Neumann-Zagier gluing methods
from hyperbolic geometry (’80’s) [Dimofte ’11]

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Math: New “quantum” topological invariants,

a comb’r definition of the $G_{\mathbb{C}}$ Chern-Simons part’n function

$$\mathcal{Z}_{CS}^{G_{\mathbb{C}}}(M)$$

for all CS levels $k \in \mathbb{Z}$

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- analyzing asymptotics of $\mathcal{Z}_{CS}^{G_{\mathbb{C}}}(M)$ (easy!)

$\rightsquigarrow$ simple, conjectured (tested) formula for $G_{\mathbb{C}}$-twisted Reidemeister-Ray-Singer torsion of M

[Dimofte-Garoufalidis ’12]

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[Dimofte-Garoufalidis ’12]

$\rightsquigarrow$ predictions for asymptotics of colored Jones poly’s (hard!; play a role in Volume Conjecture)

[Kashaev ’97, Murakami-Murakami ’99, Gukov ‘03]

[Dimofte-Gukov-Lenells-Zagier ’08]
[Dimofte-Garoufalidis ’15]

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Math: a comb’r definition of the $G_{\mathbb{C}}$ Chern-Simons part’n function

$\mathcal{Z}^{G_{\mathbb{C}}}_{CS}(M)$

Hopefully: a combinatorial definition for $G_{\mathbb{C}}$ 3-manifold homology!

in progress w/ Gaiotto-Moore

(Analogous to Khovanov homology for G)

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Properties of 3d N=2 theories are governed by the geometry of 3-manifolds!

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$T_{\mathfrak{g}}[M]$

[Dimofte-Gaiotto-v.d.Veen ’13]

4d N=2 T | 4d N=2 T’

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	4d N=2 T	4d N=2 T’
S-duality:	g_{YM}	$g'_{YM} \sim 1/g_{YM}$

Remainder of the talk:

$T_{\mathfrak{g}}[M]$

[Dimofte-Gaiotto-v.d.Veen '13]

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- a few more details on the correspondence, and observables of $T_{\mathfrak{g}}[M]$
- tetrahedra, formulas, and examples
- first look at homological/categorical invariants

The correspondence

super-conformal-field-theory

Starting point: 6d (2,0) SCFT [Strominger, Witten ’90’s]

“theory $\mathcal{X}$”

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$\mathcal{X}_{\mathfrak{g}}$ on $M \times \mathbb{R}^3$ (topological twist on M)

$\rightsquigarrow$ effective theory $T_{\mathfrak{g}}[M]$ on $\mathbb{R}^3$

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- nevertheless, can infer many properties of $T_{\mathfrak{g}}[M]$ + its compact'ns

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Basic property:

{vacua of $T_{\mathfrak{g}}[M]$ on $\mathbb{R}^2 \times S^1$} = {flat $G_{\mathbb{C}}$ connections on M }

$\mathcal{M}_{\text{flat}}(M, G_{\mathbb{C}})$

To see this:

$$\begin{array}{lll} \text{6d} & \mathcal{X}_{\mathfrak{g}} & M \times \mathbb{R}^2 \times S^1 \\ & & \downarrow \\ \text{3d} & T_{\mathfrak{g}}[M] & \mathbb{R}^2 \times S^1 \\ & & \downarrow \\ \text{2d} & & \mathbb{R}^2 \end{array}$$

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flat

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[Dimofte-Gukov-Hollands '10]

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$G_{\mathbb{C}}$ Chern-Simons theory on M

$\mathcal{A}$: $\mathfrak{g}_{\mathbb{C}}$-valued 1-form

$$\mathcal{Z}_{CS}[M] = \int \mathcal{D}\mathcal{A}\,\mathcal{D}\overline{\mathcal{A}}\, e^{\frac{k+i\sigma}{8\pi i} I_{\mathrm{CS}}(\mathcal{A}) + \frac{k-i\sigma}{8\pi i} I_{\mathrm{CS}}(\overline{\mathcal{A}})}$$

[Witten '91]

$$I_{CS}(\mathcal{A}) := \int_M \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

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[Witten '89]

- cf. *compact* G CS thy: on knot complements, get Jones polys (combinatorial definition) [Reshetikhin-Turaev '90, etc.]

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$k \in \mathbb{Z} \qquad \sigma \in \mathbb{R}\ \ (\text{or } \mathbb{C}) \qquad I_{CS}(\mathcal{A}) := \int_M \mathrm{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)$

- classical sol'ns are flat $G_{\mathbb{C}}$ connections

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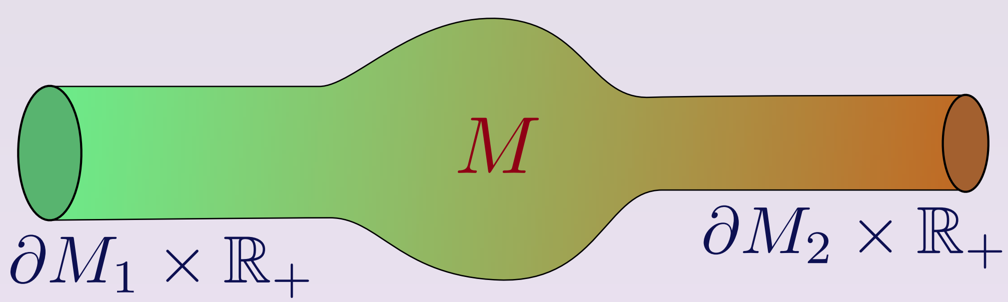

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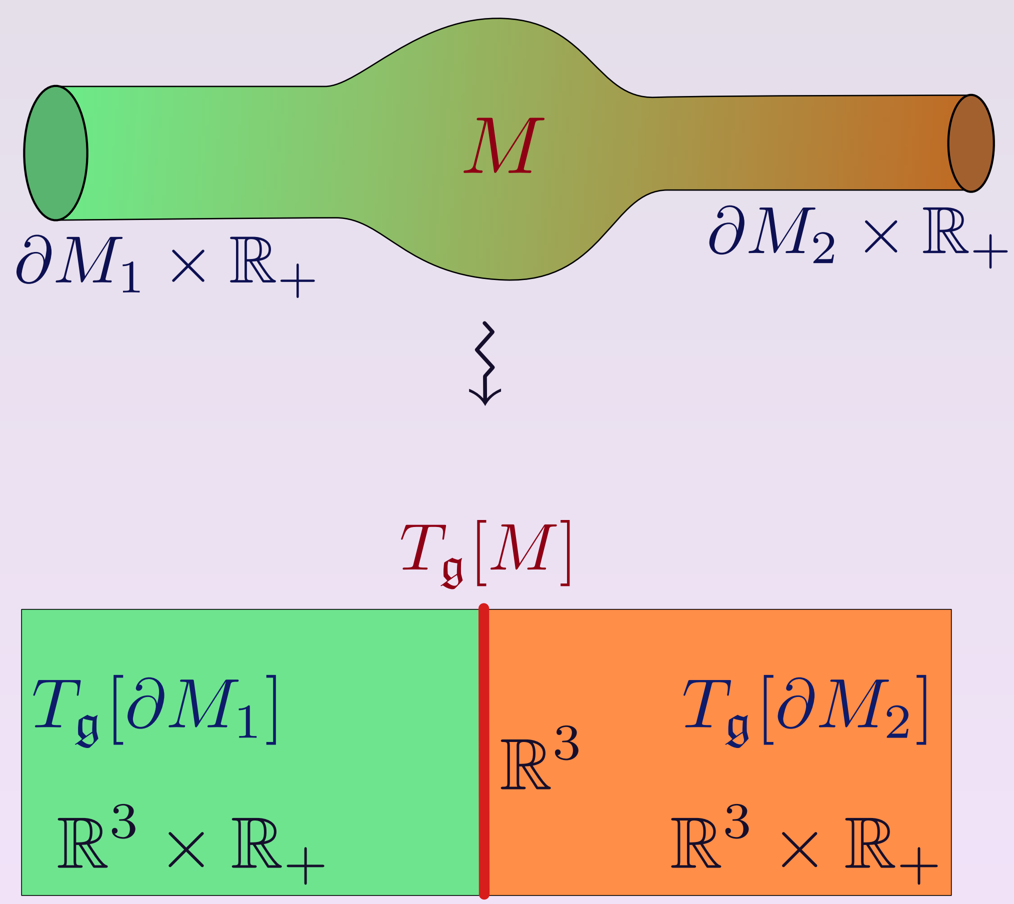

3d interface
between 4d (N=2)
SUSY theories

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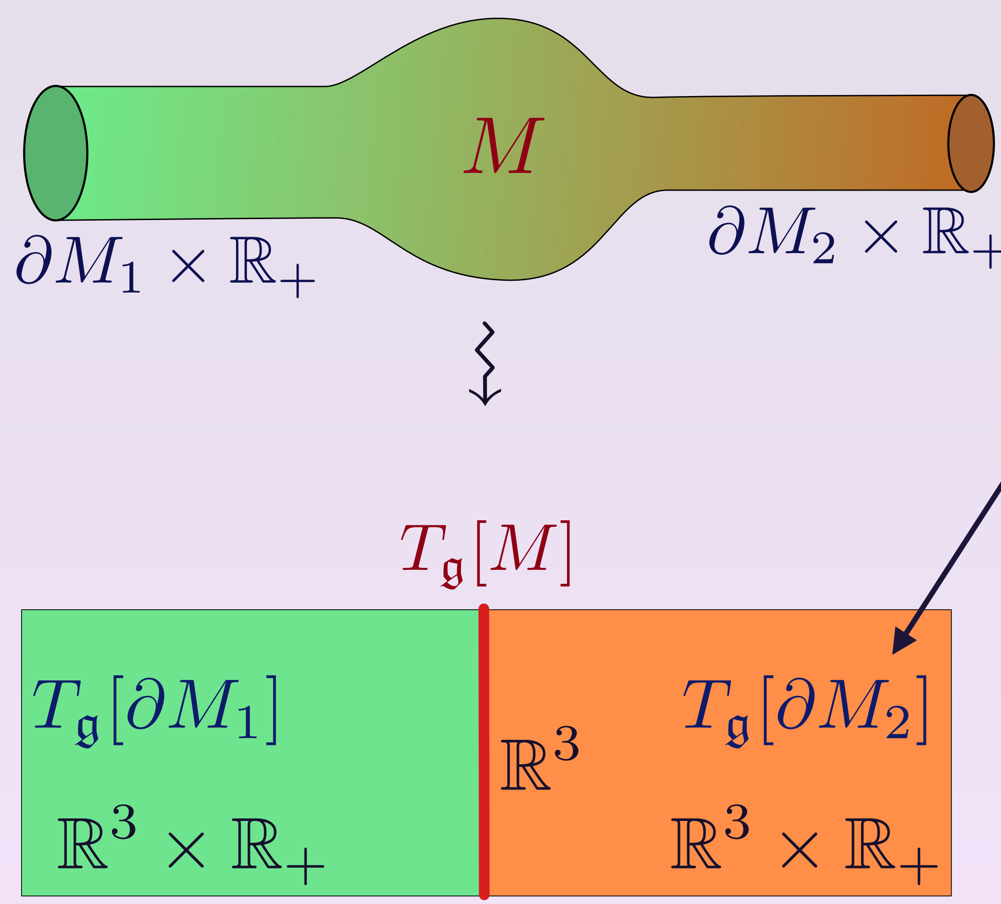

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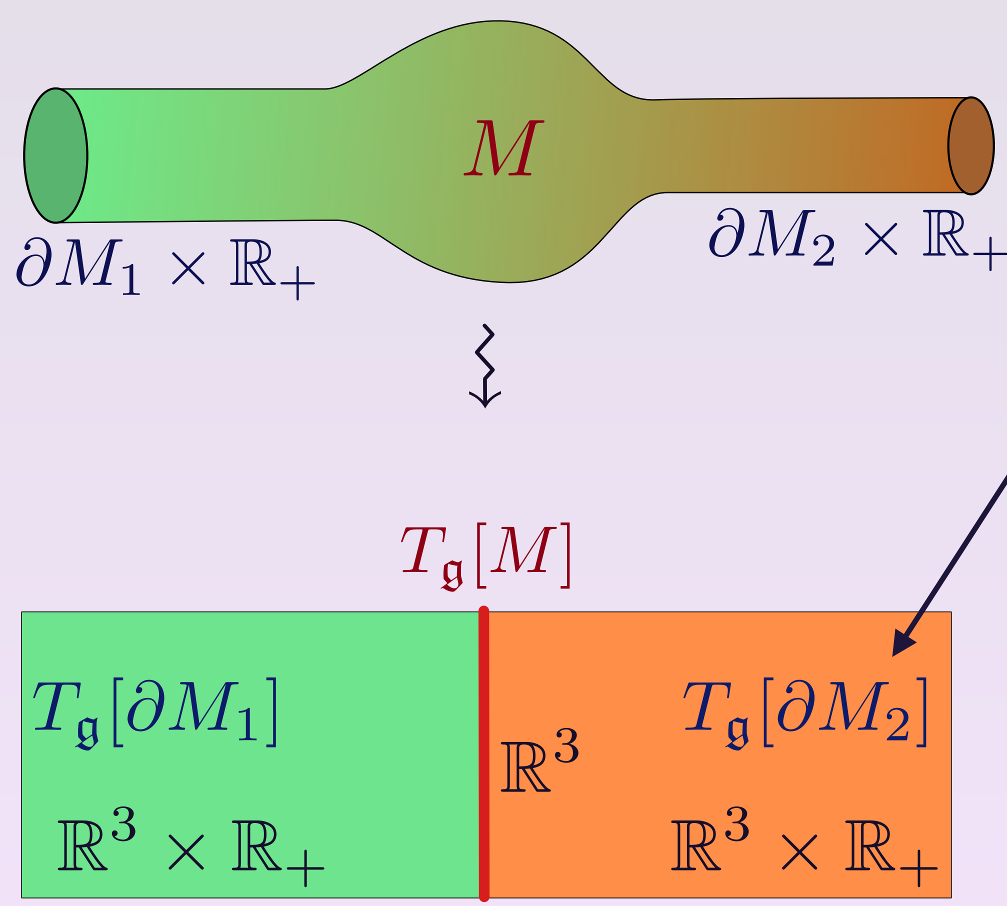

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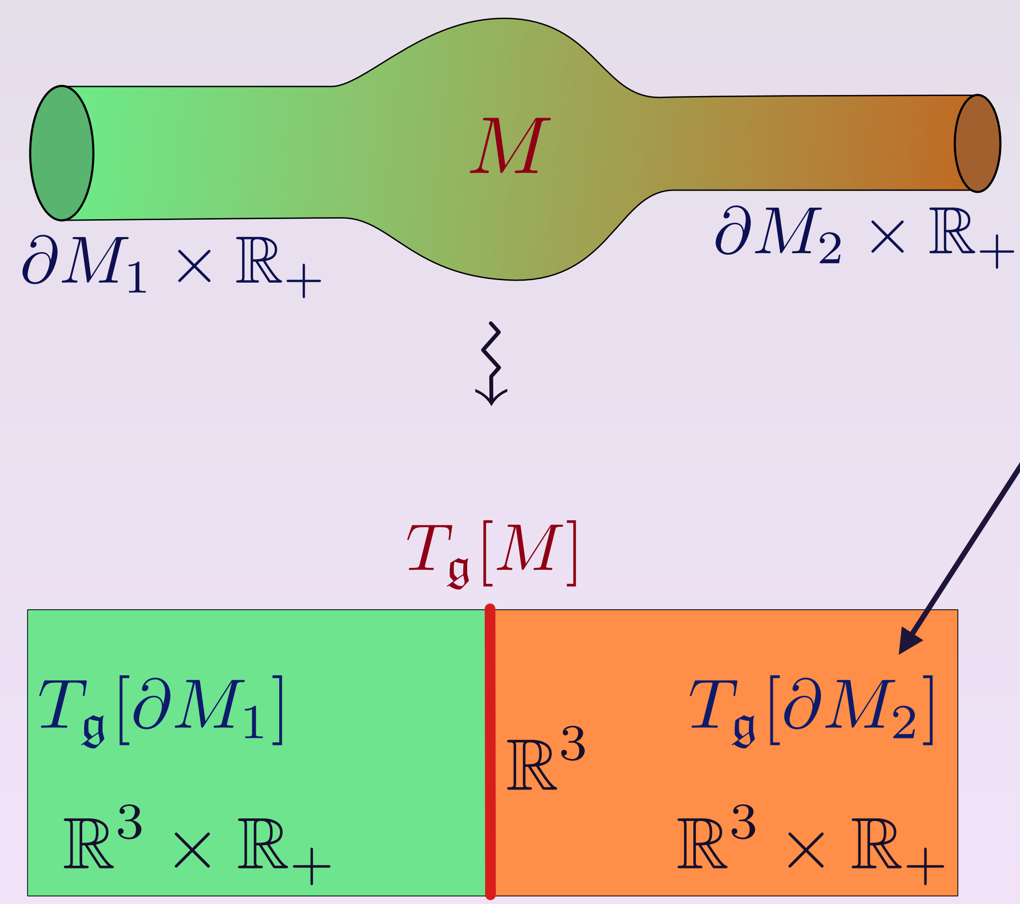

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cf. [Gadde-Gukov-Putrov '13]

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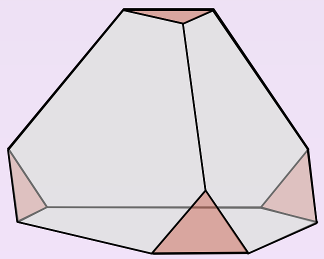

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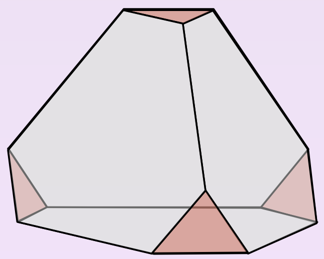

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Remainder of the talk: $\mathfrak{g} = sl_2$ $G_{\mathbb{C}} = SL(2,\mathbb{C})$ (or $PSL(2,\mathbb{C})$
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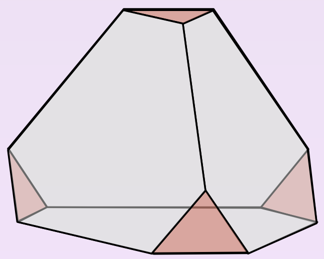

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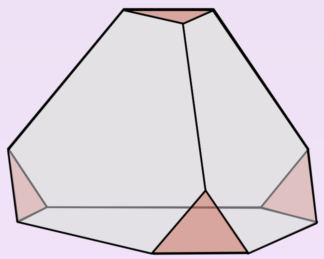

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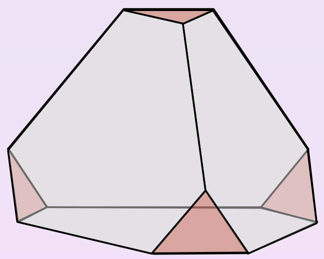
- $PSL(2,\mathbb{C})$ flat connections are (roughly) hyperbolic metrics

So: $T_{\mathfrak{g}}[M]$ quantizes, categorifies, etc. classical hyperbolic geometry!

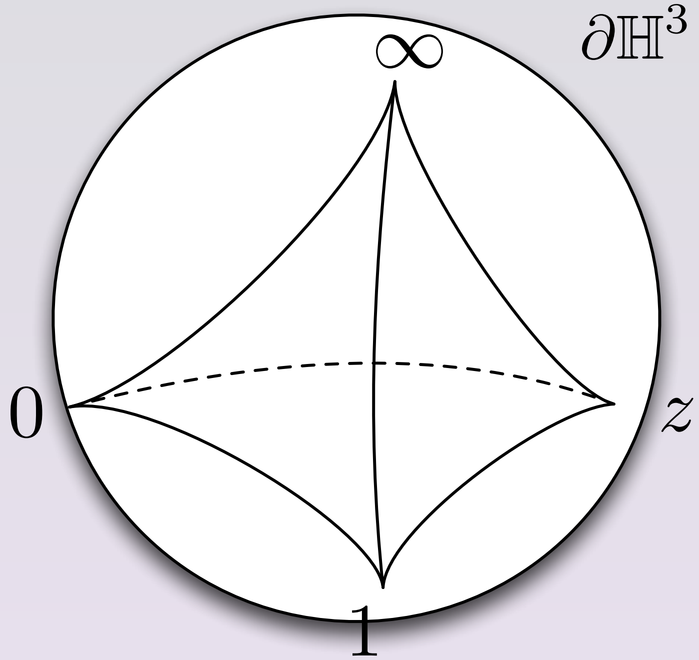
$$M = \bigcup_{i=1}^{N} \Delta_i \qquad T_{\mathfrak{g}}[M] = \Big(\bigotimes_{i=1}^{N} T_{\mathfrak{g}}[\Delta_i] \Big) \Big/ \sim$$



The correspondence is effective

Single (ideal, hyperbolic) tetrahedron:

$\partial\mathbb{H}^3 \simeq \mathbb{C} \cup \{\infty\}$



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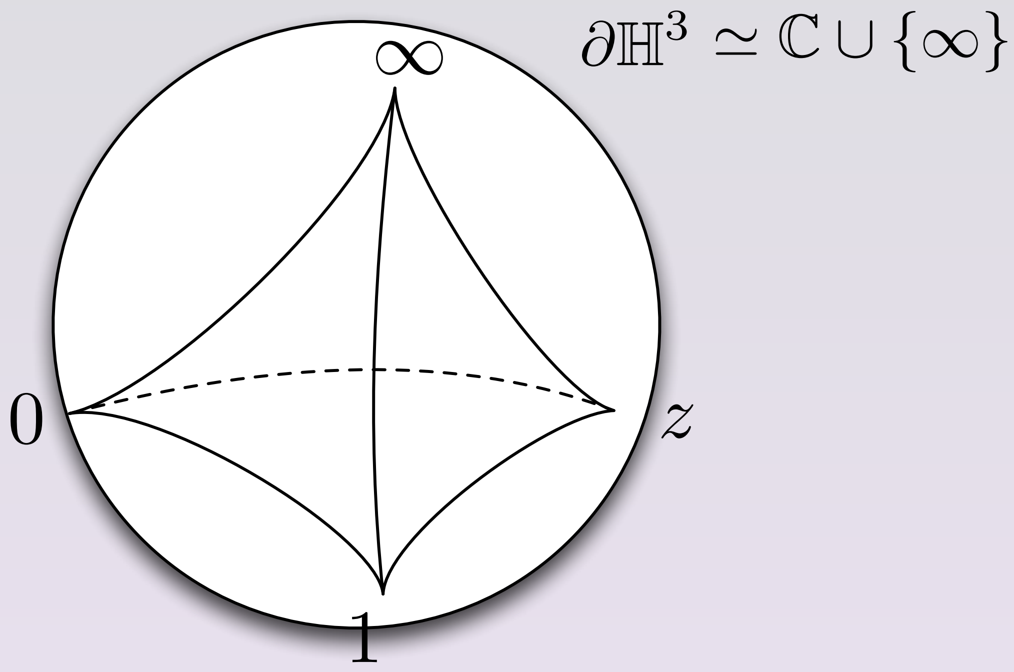

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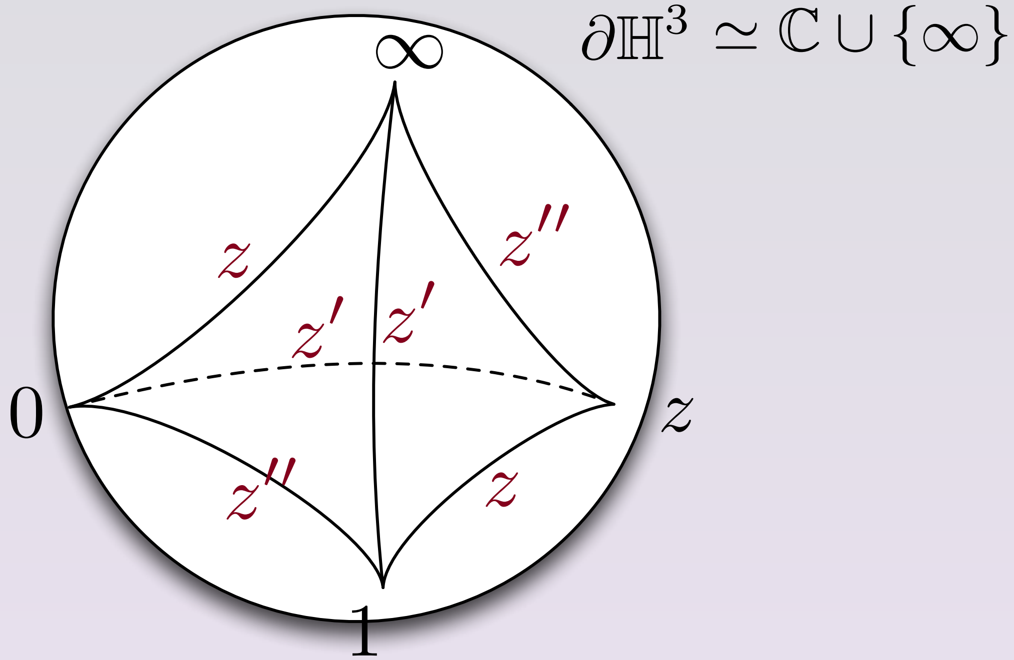

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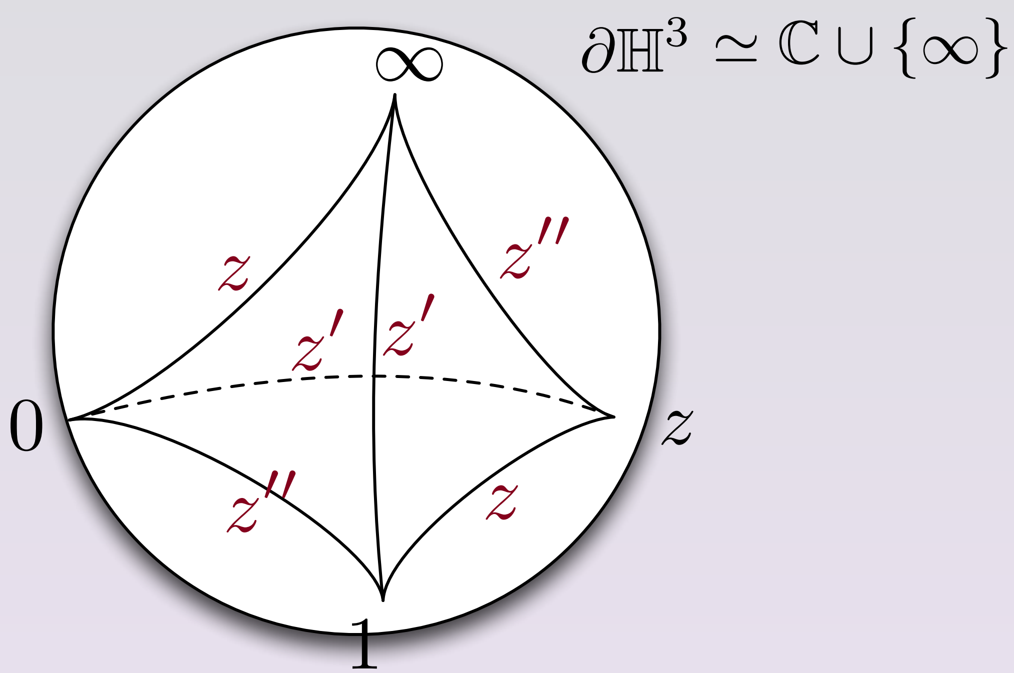

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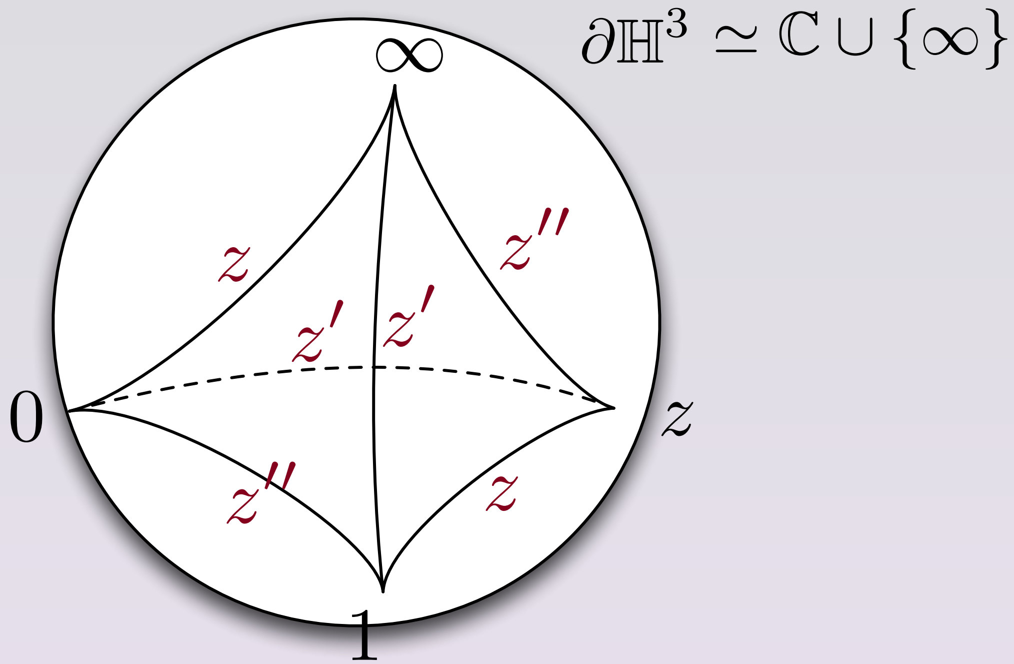

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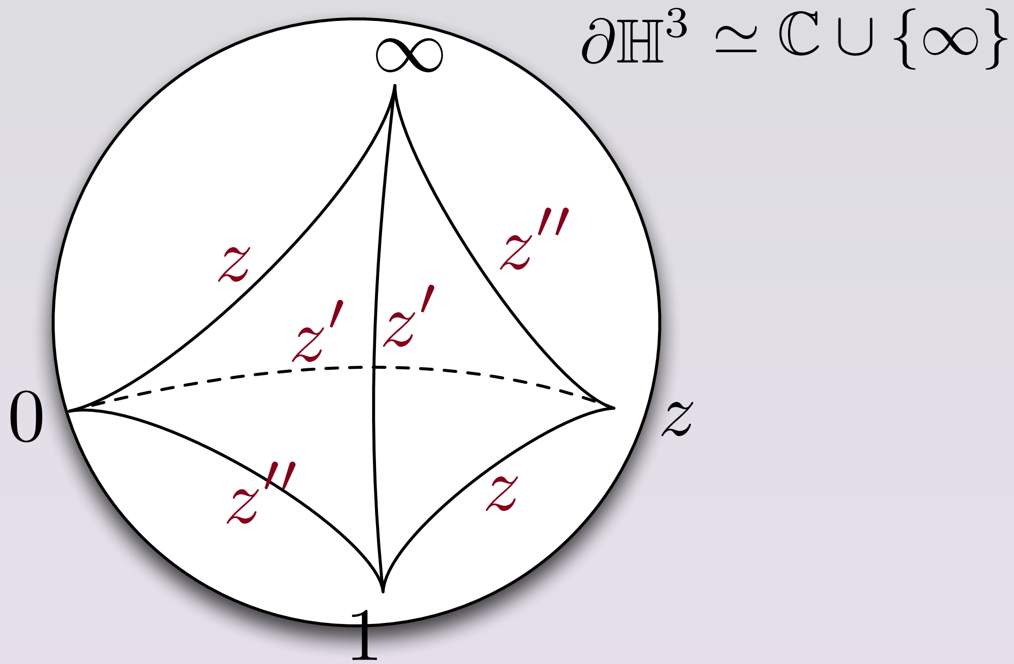

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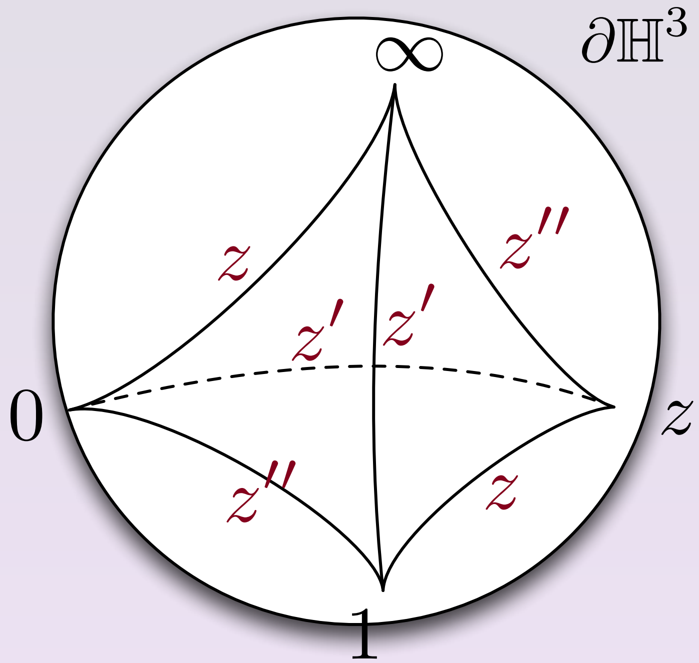

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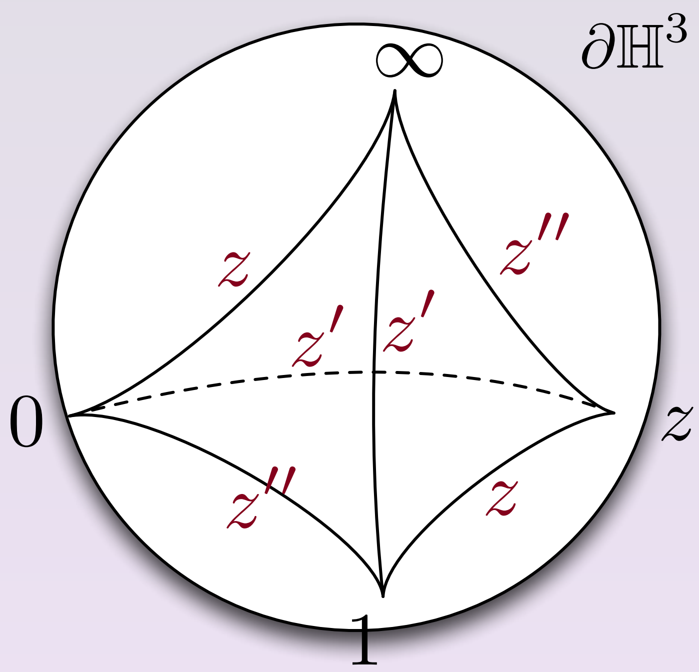

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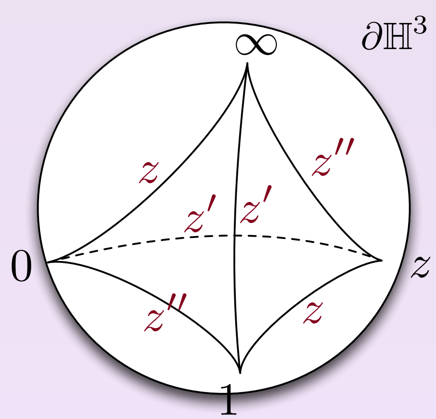

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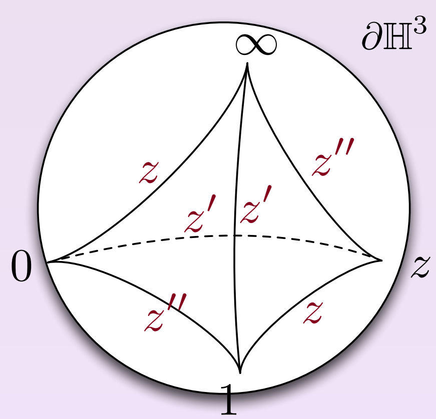

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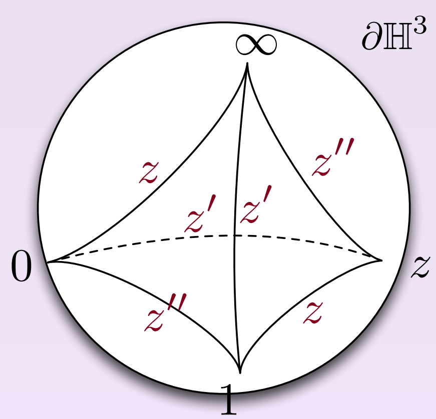

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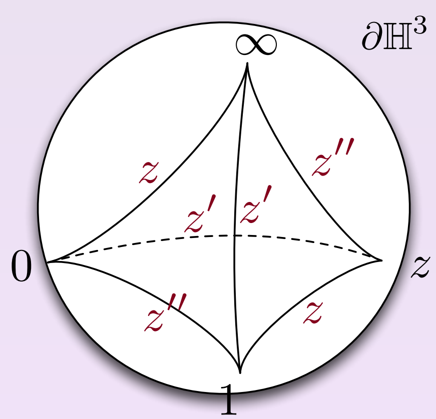

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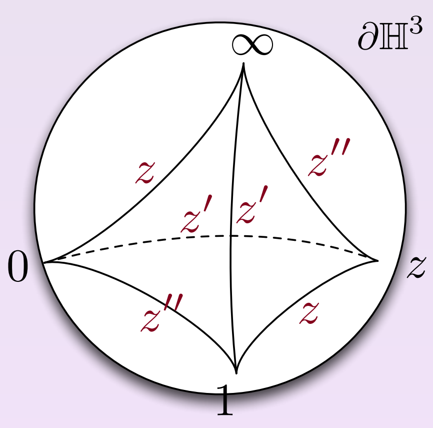

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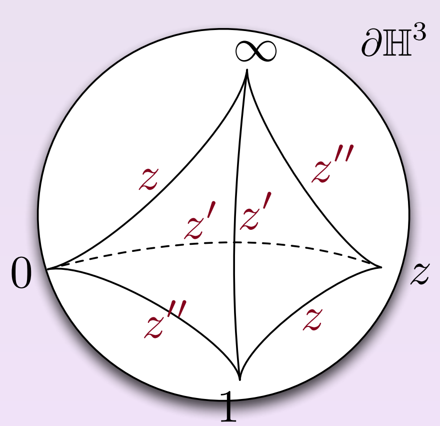

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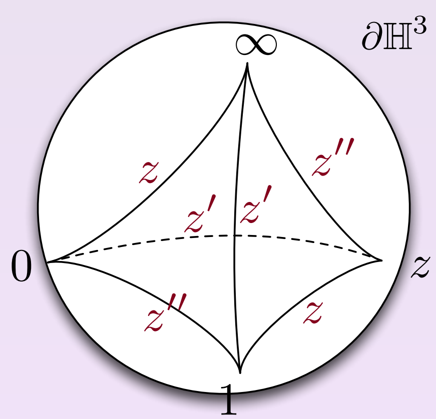

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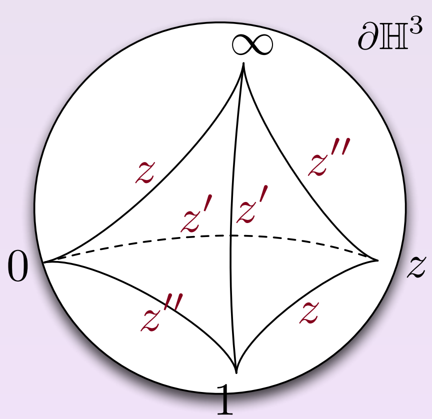

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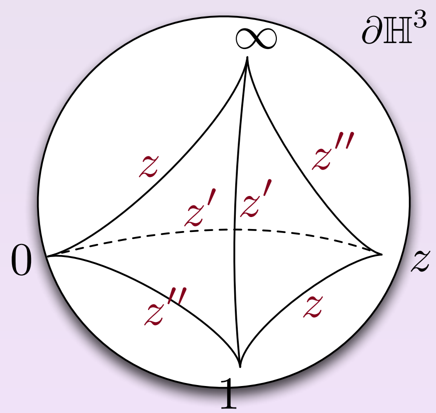

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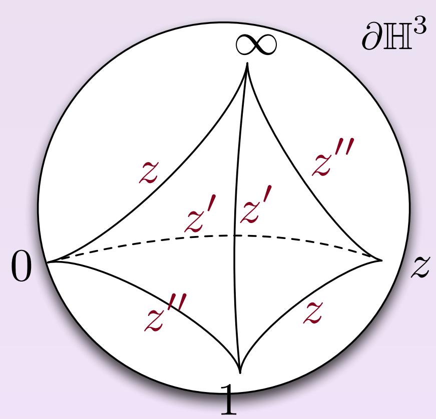

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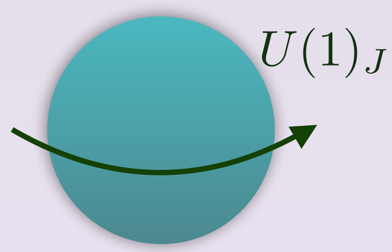

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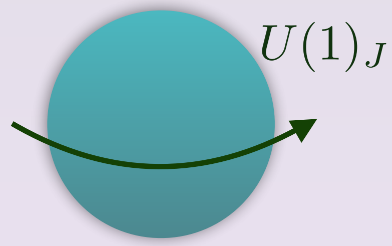

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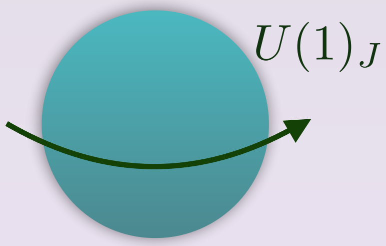

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E.g. $$\mathcal{Z}_{T[\Delta]}[S^2 \times S^1] = \prod_{r=0}^{\infty} \frac{1 - q^{1-\frac{m}{2}}\zeta^{-1}}{1 - q^{-\frac{m}{2}}\zeta}$$ $$q = e^{\frac{2\pi}{\sigma}}$$

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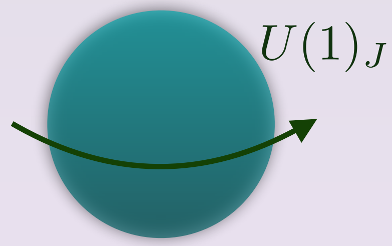

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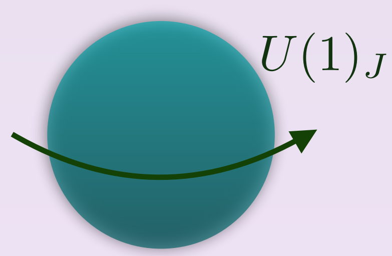

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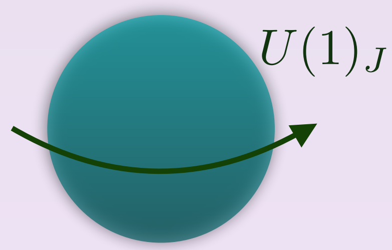

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That categorifies the volume of a hyperbolic tetrahedron.

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In general, *glue*

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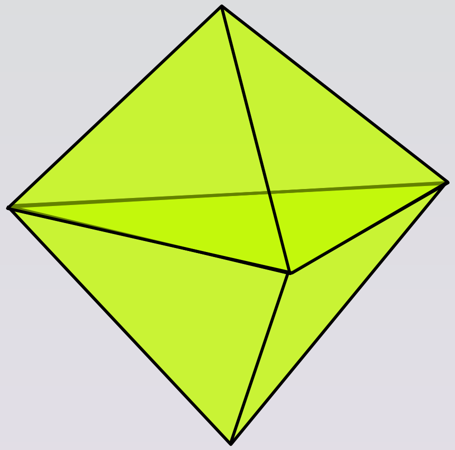

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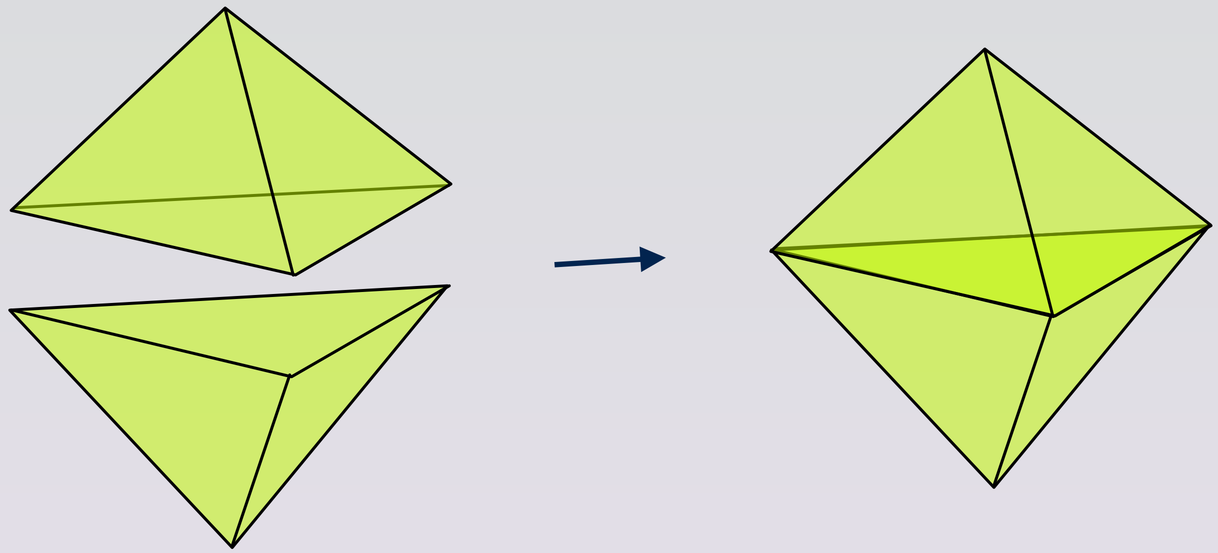

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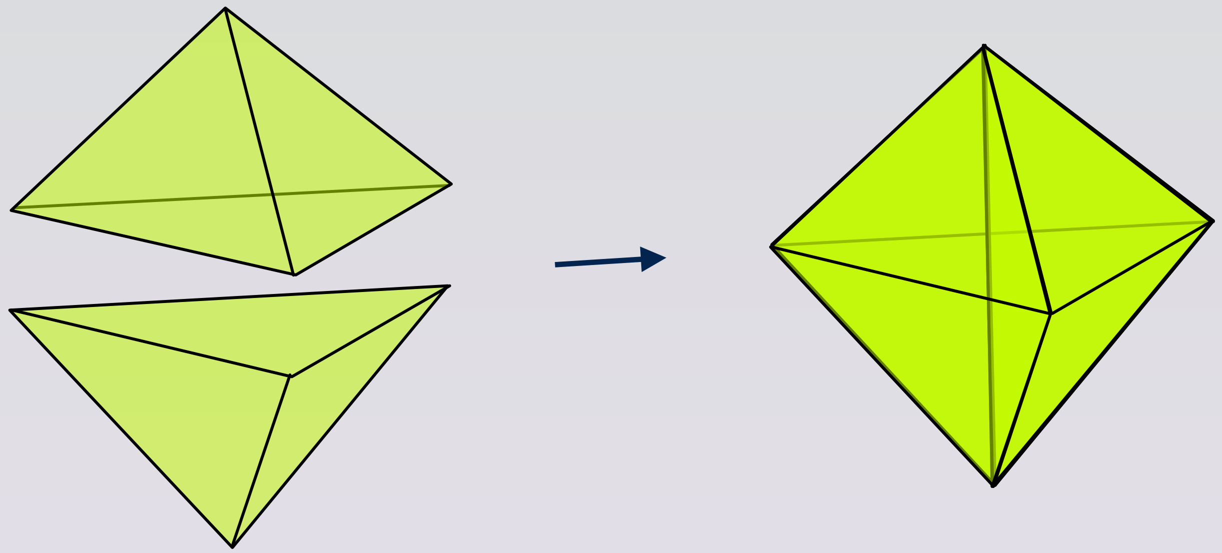

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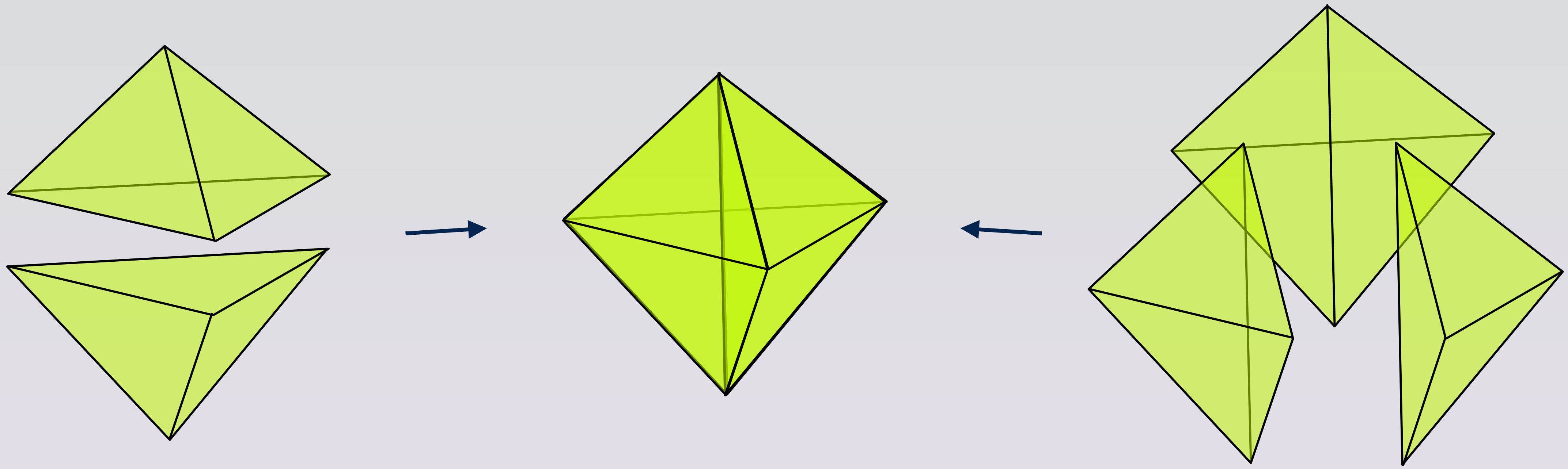

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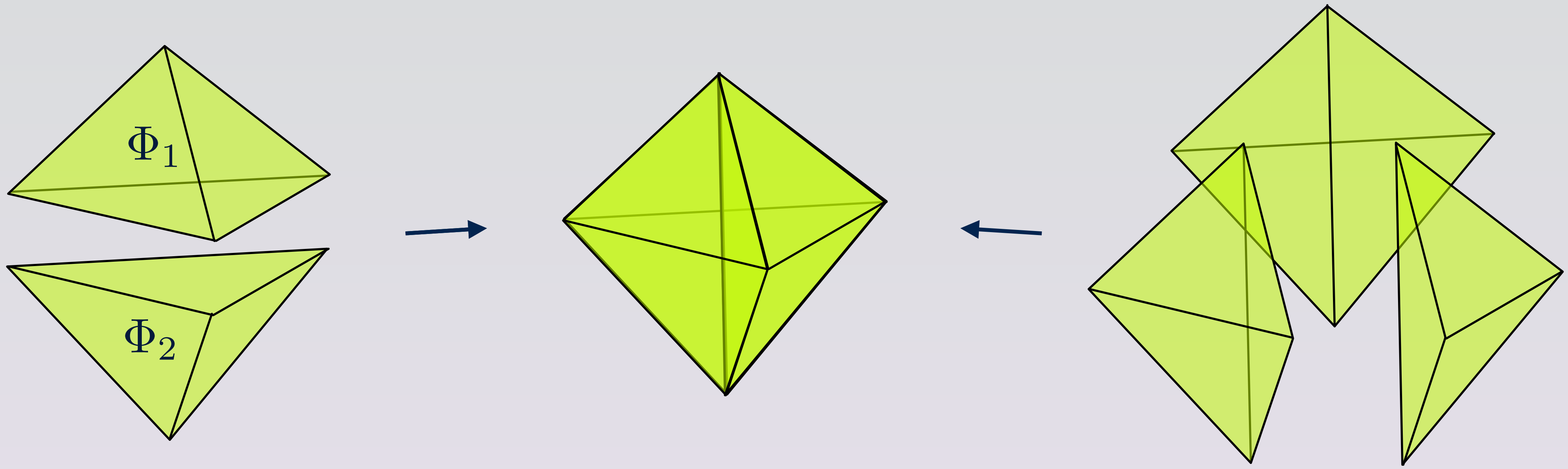

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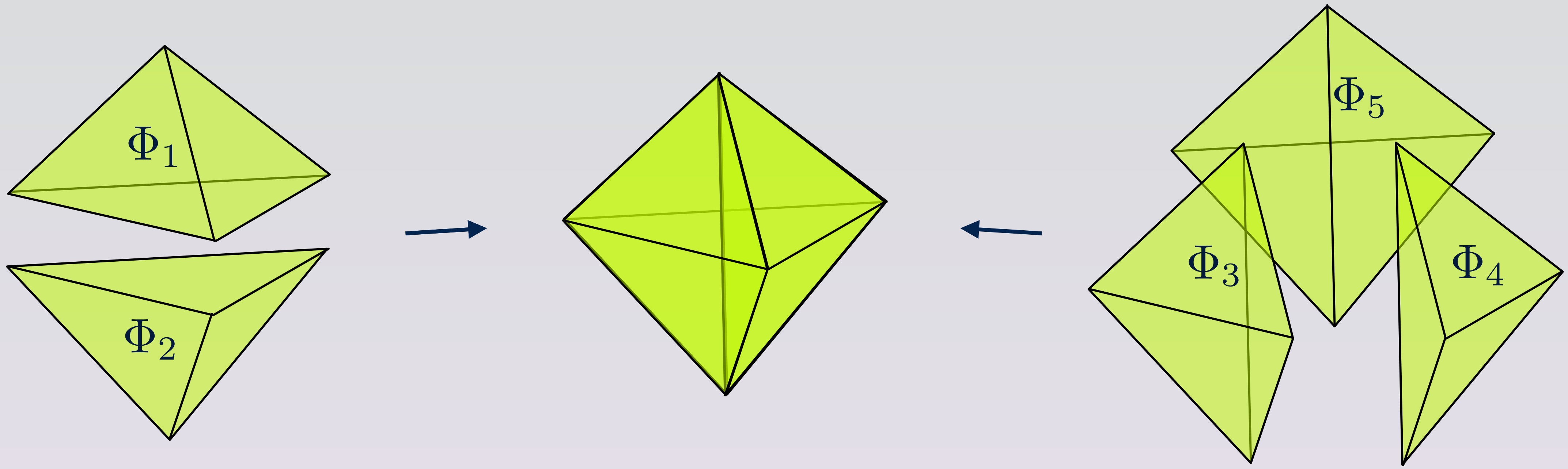

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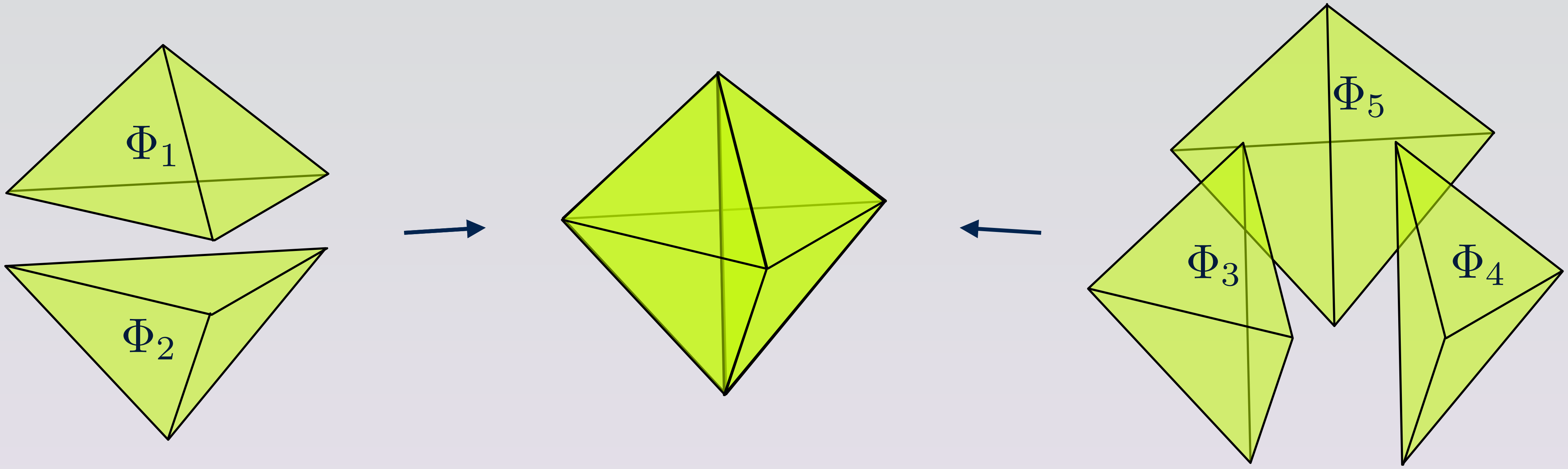

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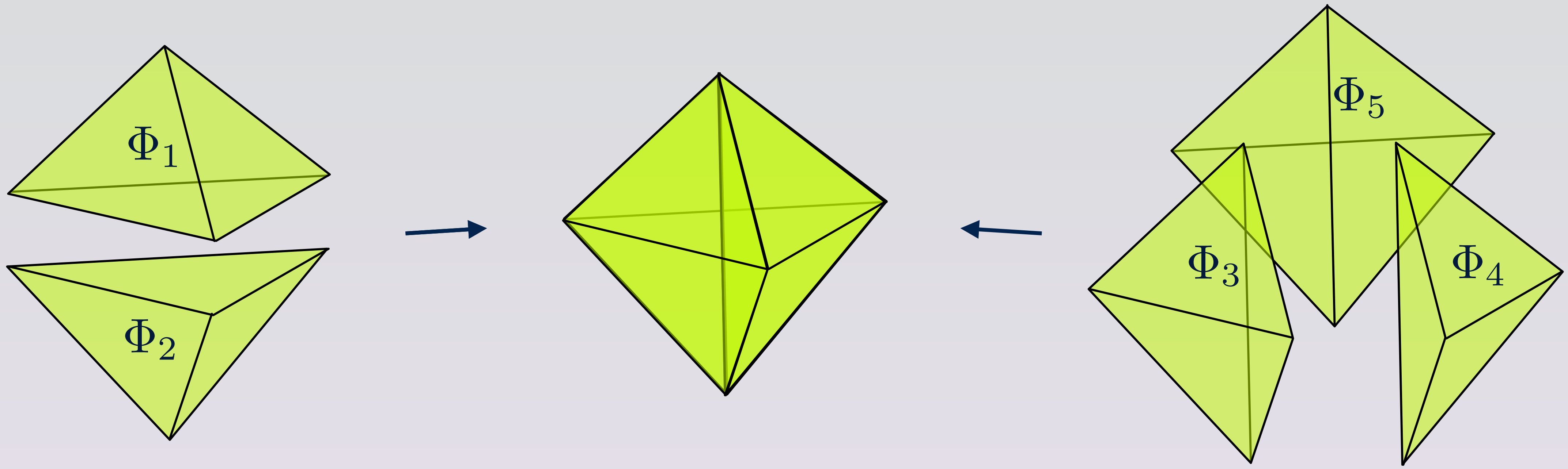

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cf. classical $\mathrm{Li}_2(x) + \mathrm{Li}_2(y) = \mathrm{Li}_2\Big(\frac{x}{1-y}\Big) + \mathrm{Li}_2\Big(\frac{y}{1-x}\Big) + \mathrm{Li}_2\Big(\frac{(1-x)(1-y)}{xy}\Big) + \mathrm{logs}$

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2. 3d N=2 theories on interfaces in 4d
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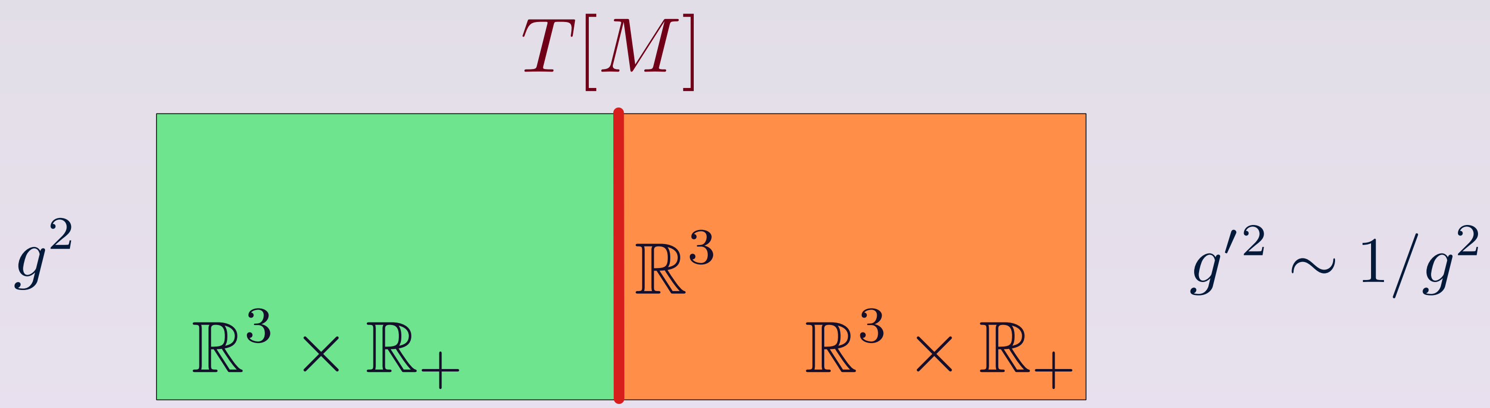

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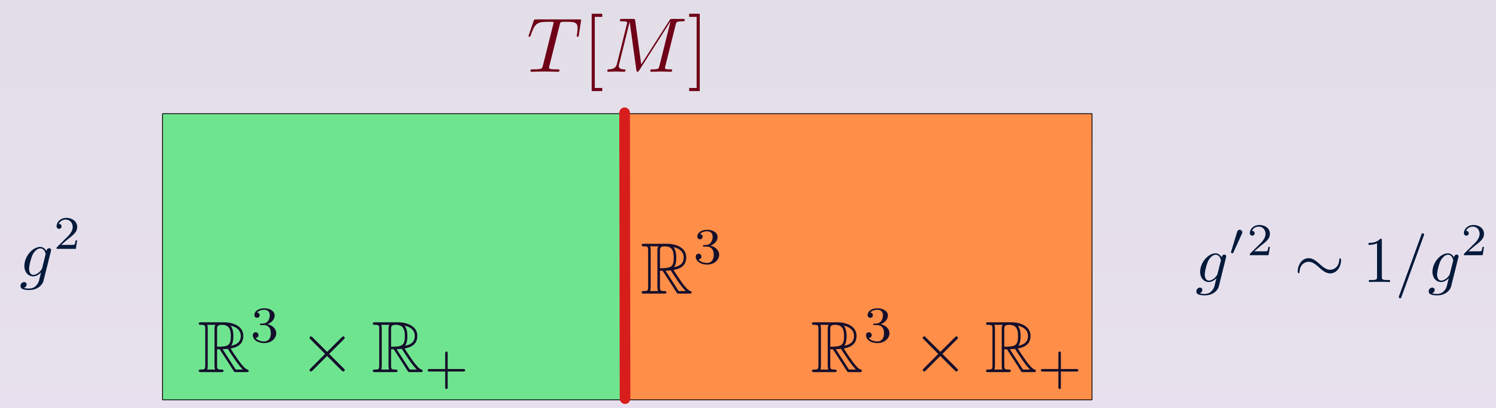

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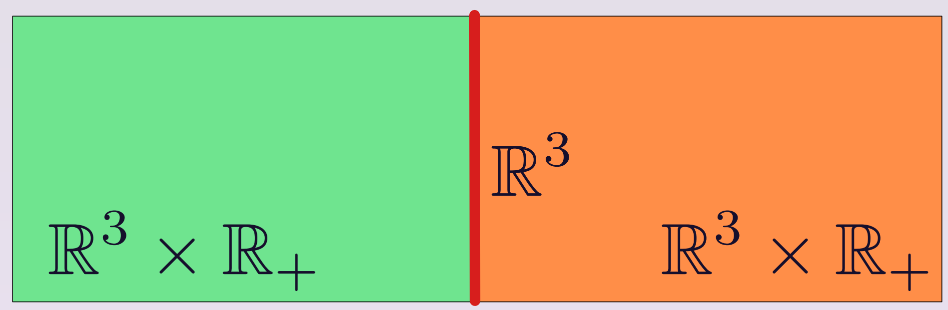
$T[M]$

g^2



$g'^2 \sim 1/g^2$

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$\longrightarrow$ 4 or 5 Δ's

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Moral

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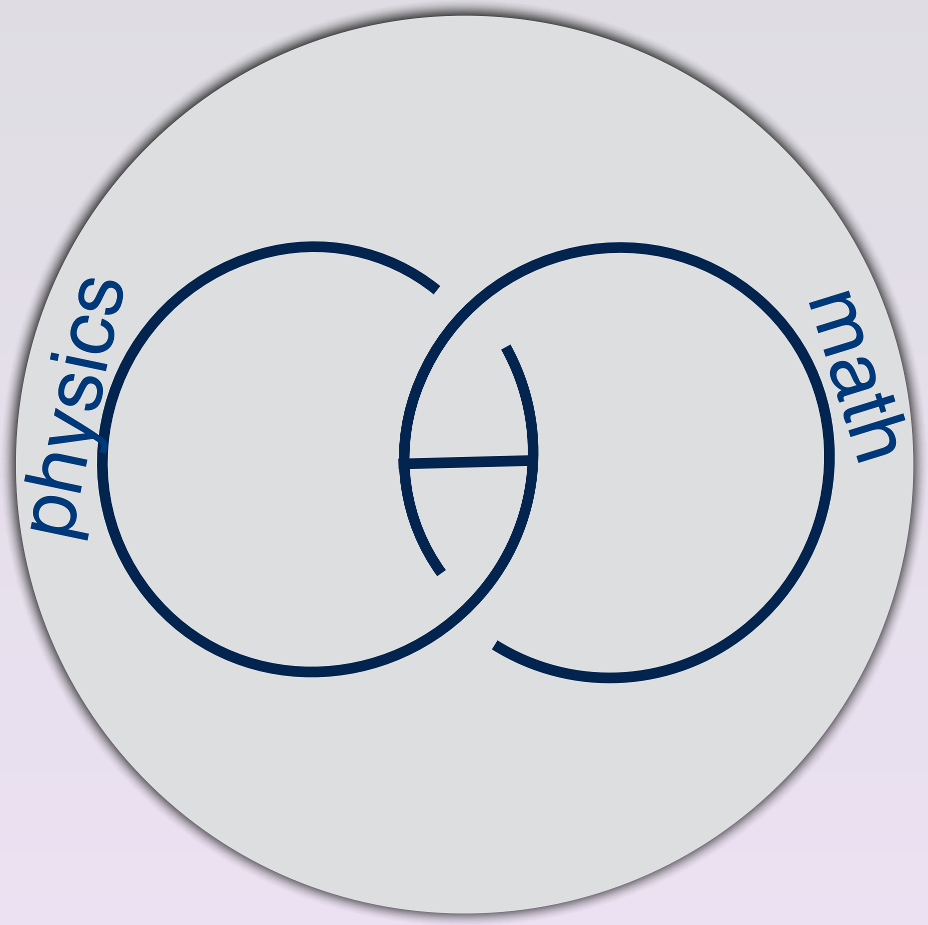
- moduli spaces
- partition functions
- (SUSY) Hilbert spaces



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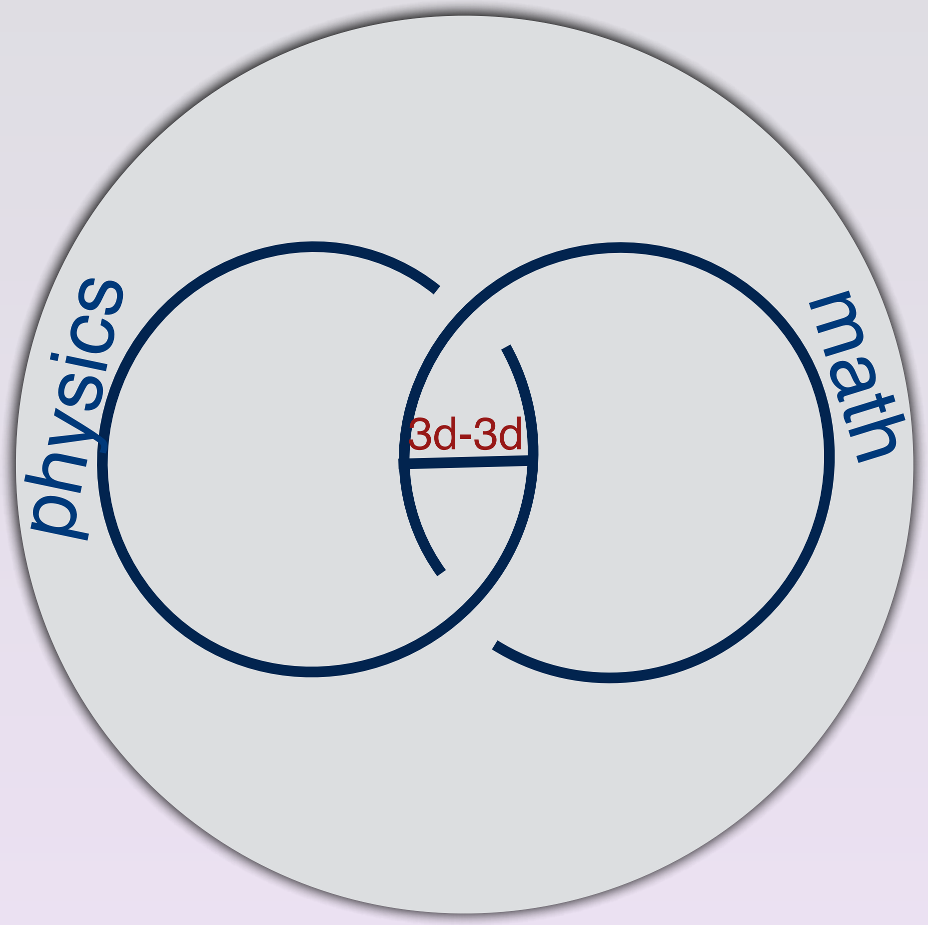

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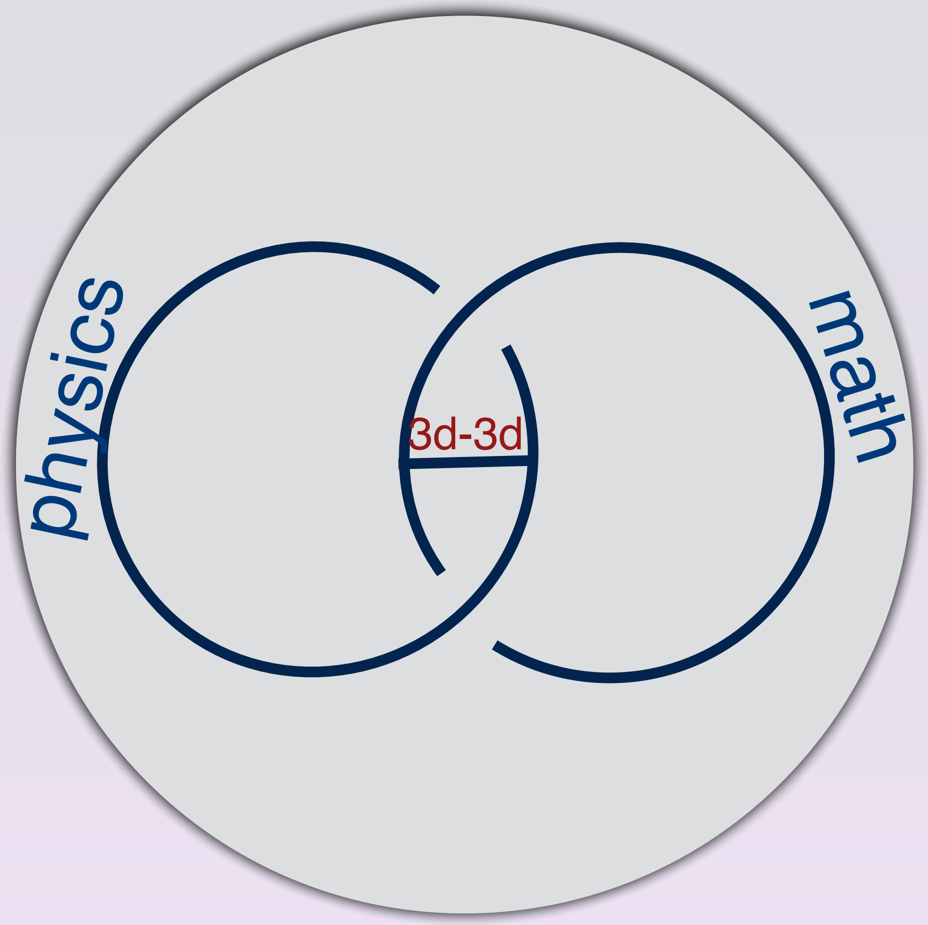

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I hope this type of work will find a place here at Davis.