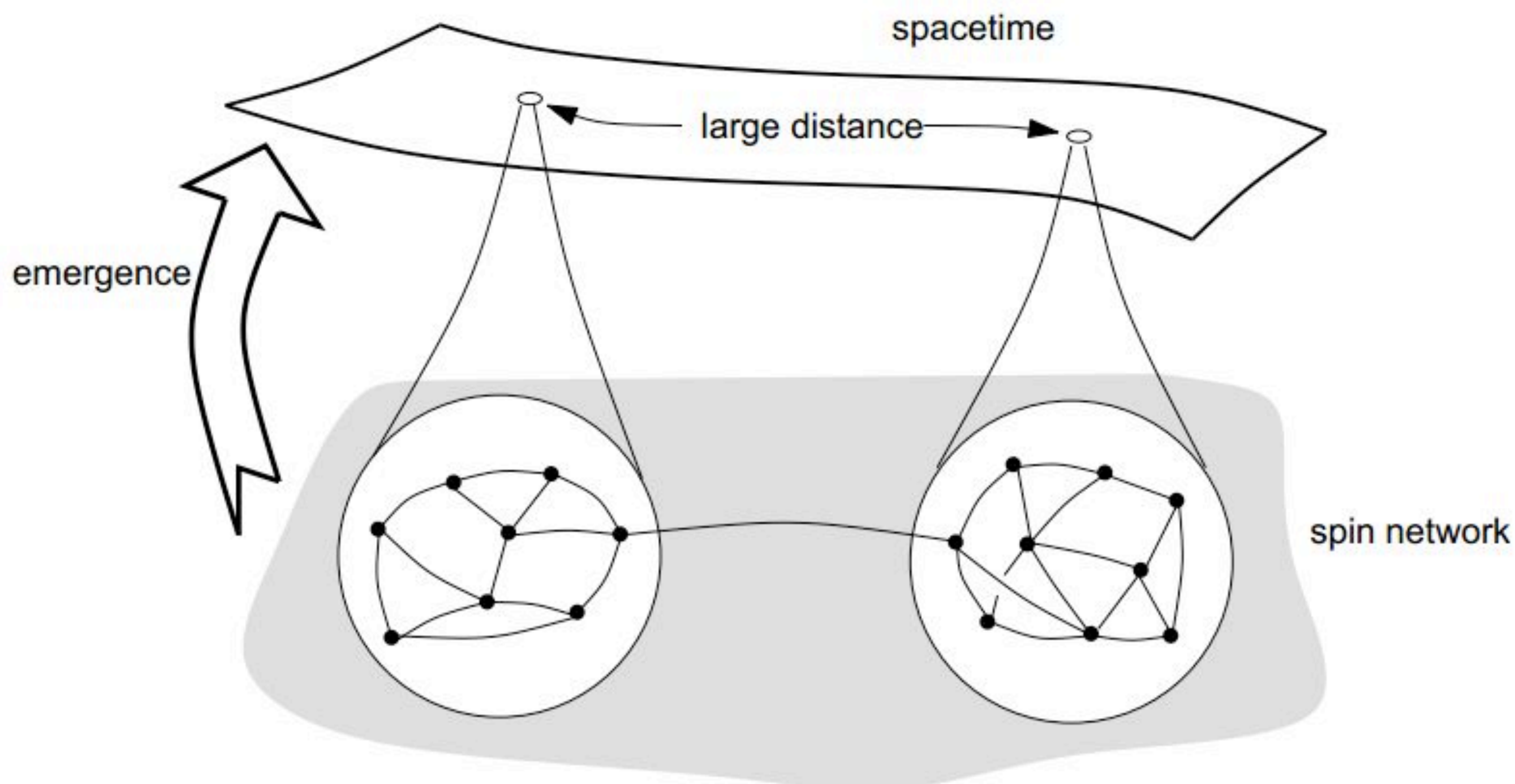
Which quantum field theories have emergent gravity?



Shamit Kachru (Stanford)

Based in part on:

arXiv : 1503.04800 (with Benjamin, Cheng, Moore, Paquette)
arXiv : 1512.00010 (with Benjamin, Keller, Paquette)
Work in progress with Benjamin, Dyer, Zimet

Outline of the talk

I. Introduction

...where the big questions which won't be answered,
are introduced...

II. Why are holographic theories gauge theories?

...where we say that the spectrum of the
gravity theory ought to be reasonable...

III. Density of states bounds

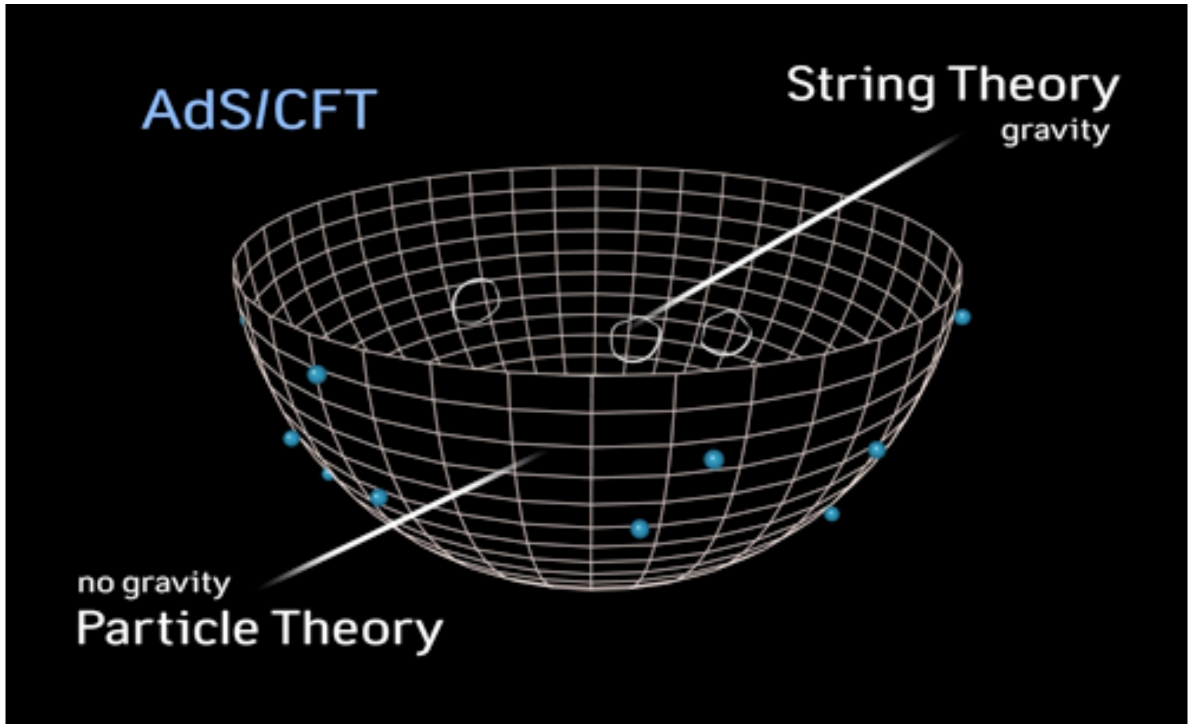
...where we use properties of black holes and
phase transitions...

IV. Other bounds and equivalences

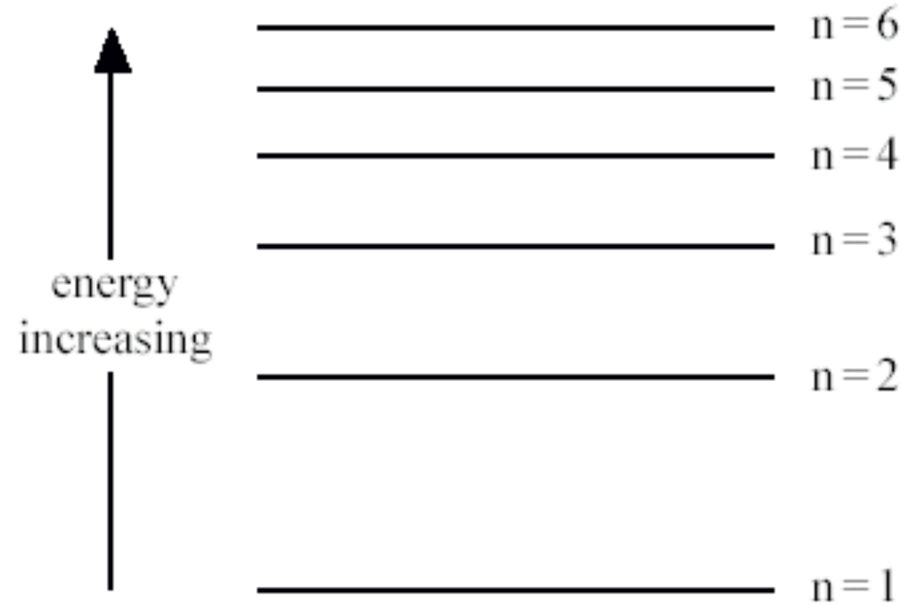
...entanglement, bulk locality, and more may
enter the story...

1. Introduction

Two (related) big questions:



Which quantum field theories give rise to macroscopic theories of gravity (via holography)? Can we find precise criteria?



Which spectra of elementary particles can arise in a UV complete theory including quantum gravity?

This question underlies theoretical particle physics.

Complete answers to these questions far outrun present understanding.

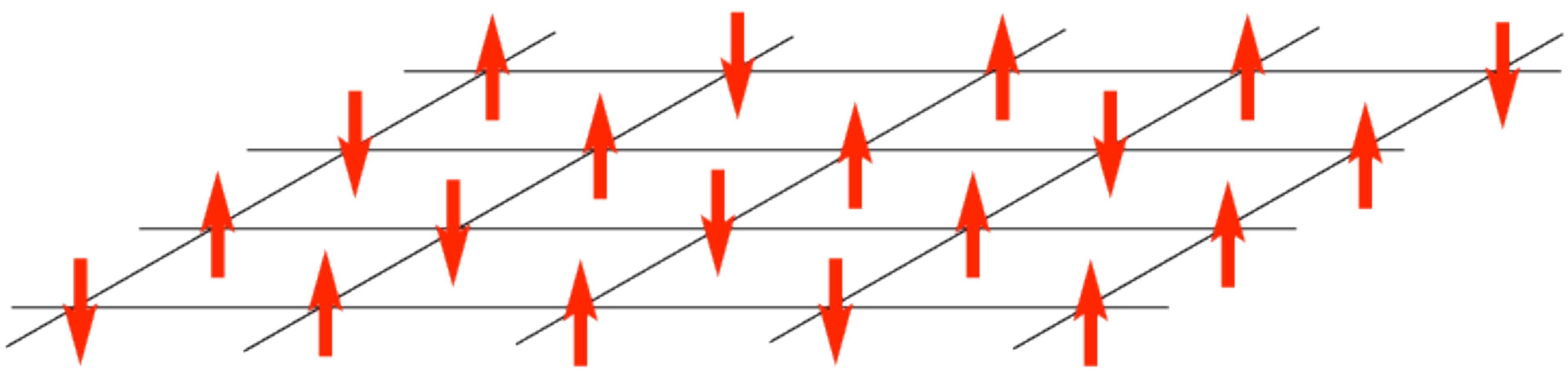
One logical place to search for answers is in studying 2d CFTs and 3d gravity; the extra constraints due to Virasoro symmetry may simplify life.

The first obvious fact is that one needs to focus on CFTs of large central charge to get a macroscopic space-time:

$$c = \frac{3}{2}\frac{L_{\mathrm{AdS}}}{G}$$

Brown, Henneaux

But this is far from sufficient. For instance, no one expects a tensor product of ten million copies of the 2d Ising model to have a weakly curved gravity dual.



2–D Ising Model

So, what criteria should we impose, that go beyond mere large central charge?

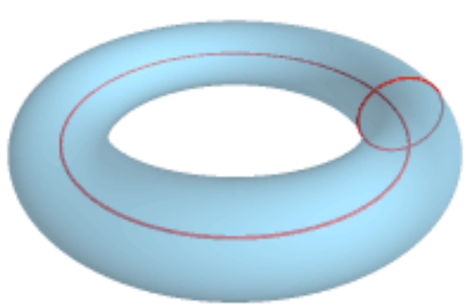
Going beyond that basic element of the AdS/CFT map, we find:

$$conformal\ symmetry \leftrightarrow AdS\ isometries$$

$$primary\ field\ of\ dimension\ \Delta \leftrightarrow bulk\ quantum\ field\ of\ mass\ m(\Delta)$$

$$\cdots\cdots$$

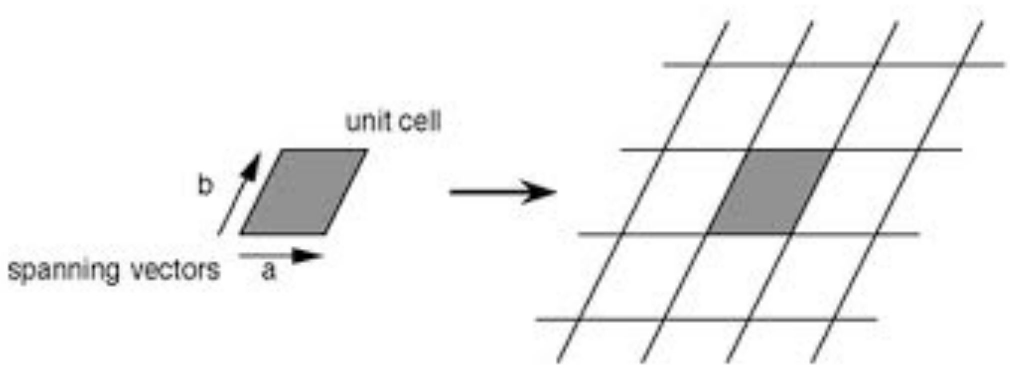
The spectrum of bulk fields is summarized nicely in the torus partition function of the 2d theory:



$$Z = \sum_n c_n q^n \qquad q = e^{2\pi i\tau}$$

$$c_n = \#\ \text{of states at mass level n}$$

Each torus is related to a 2d lattice:



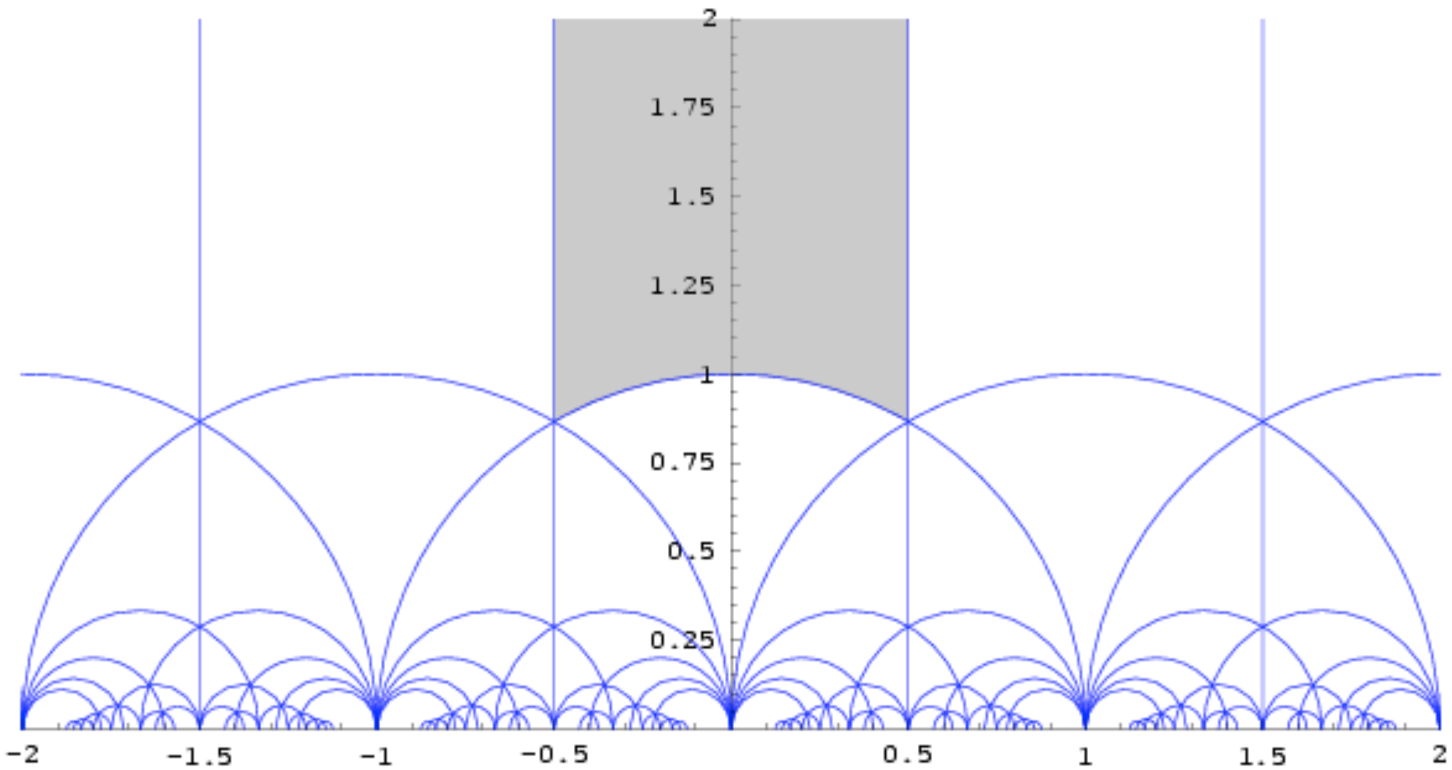
the lattice is generated
by complex numbers
$1, \tau$

But one can choose a different basis for the lattice, and still get the same torus:

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad ad - bc = 1$$

Because of this, we expect that the partition function will be modular:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$



Modular functions map the "fundamental domain" to the complex #s

We will heavily use modularity -- in particular, the existence of the S-transformation

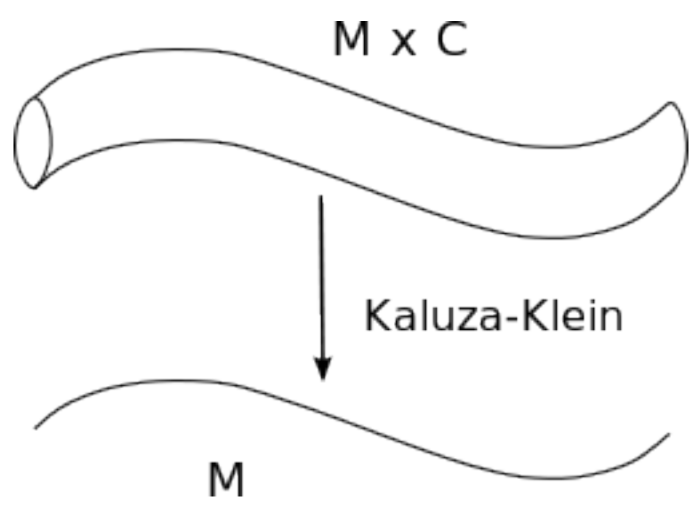
$$S \ : \ \tau \to -\frac{1}{\tau}$$

in finding constraints on partition functions of 2d quantum field theories which are to give rise to emergent AdS3 gravity.

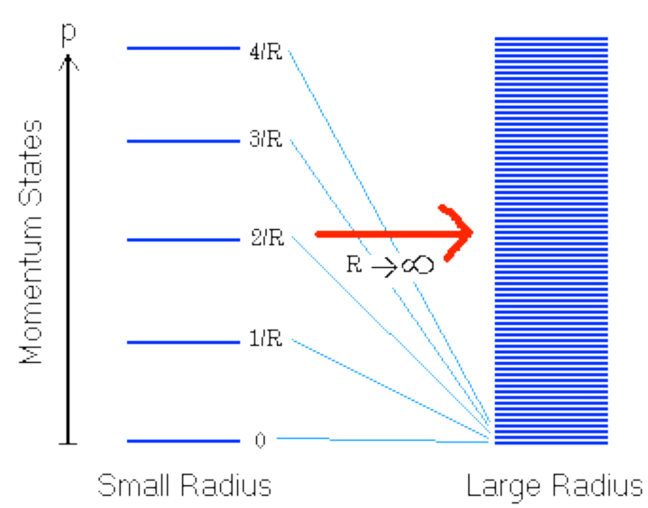
II. Why are holographic theories gauge theories?

Before doing this, however, let us discuss a more basic bound.

Suppose we want a CFT dual to a conventional gravity theory, compactified to AdS3:



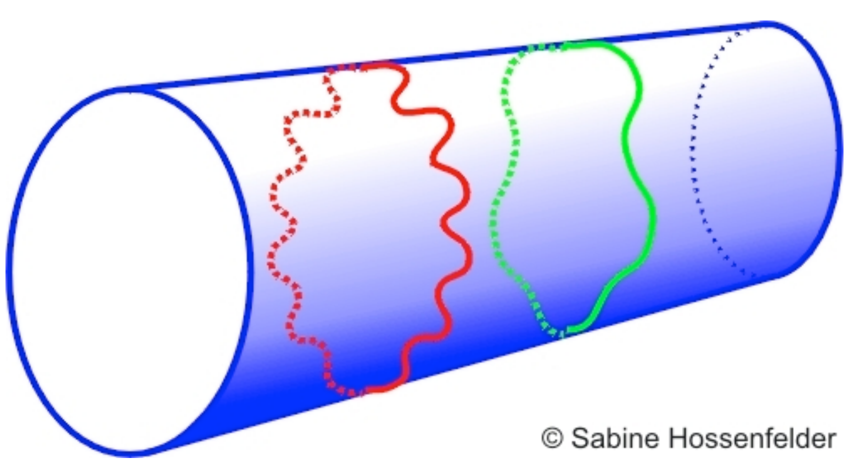
Then we expect a discrete set of states in AdS, with energies split by the compactification radius:



This means that we expect a finite number of CFT operators with dimensions below dimension Δ_*:

$$\sum_{\Delta<\Delta_*} N(\Delta) < \infty$$

This is the dual to the statement that there are finitely many KK modes beneath a given mass, in conventional gravity theories.



Note that even in a low tension string theory, we expect a finite (Hagedorn) degeneracy

$$N(\Delta) \sim e^{\Delta}$$

Now consider a theory at large central charge, or its analogue in higher dimensions, large N.

$$Fields \ : \ \phi_i, \ i = 1, \cdots, N$$

$$Operators : \ \mathcal{O}_k^{i_1 \cdots i_k} \sim \phi_{i_1} \ldots \phi_{i_k}$$

In a theory that isn't a gauge theory but with N fields, we'd expect $\mathcal{O}(N^k)$ operators at dimension ~k.

This would be a disaster -- the spectrum doesn't even stabilize in the limit of large AdS radius, $N \to \infty$.

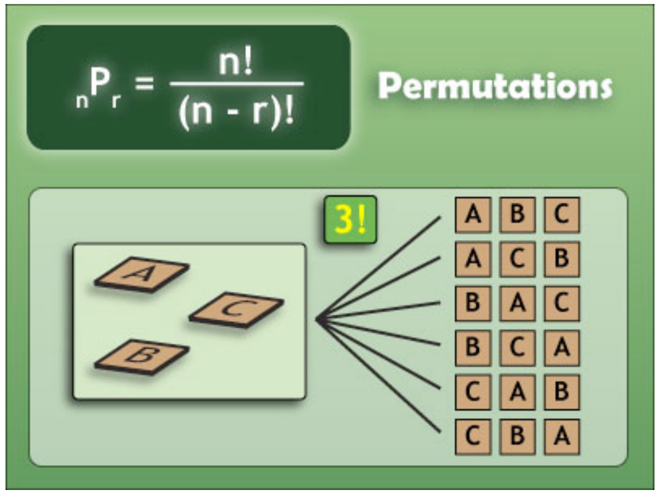
The way out is clear.

* d>2 field theories avoid this plethora of states by imposing a gauge constraint; gauge invariance restricts the set of operators at dimenson ~k to a number that doesn't grow with N.

* Typical constructions that have been studied in AdS3/CFT2 work by a close relative of this:

$$(\mathcal{C})^N/\mathcal{G}_\mathcal{N}$$

$$\mathcal{G}_\mathcal{N} \subset \mathcal{S}_N$$



Symmetric orbifolds of "seed theory" $\mathcal{C}$.

The precise condition one wants for stabilization of the spectrum is that:

$$\rho_\infty(\Delta) = \mathcal{O}(c^0) \text{ as } c \to \infty$$

Subgroups of the permutation group that accomplish this are called "oligomorphic groups," and have been recently studied in both the mathematics and the physics literature.

Cameron;
Haehl, Rangamani

Of course, this is still a relatively trivial kind of constraint. Lets try to get more physical...

III. Density of states bounds

Beyond a zeroth order reasonable spectrum of light states, there is more that we expect of a gravity theory in AdS3.

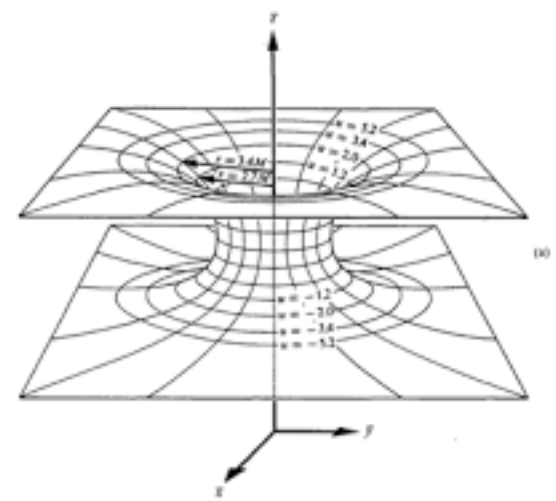
The basic object we'll discuss will be a partition function:

$$Z(q) = \sum_n c_n q^n, \quad q = \exp(2\pi i \tau)$$

Two qualitative facts that we can turn into quantitative bounds:

c.f. Hartman, Keller, Stoica

1. 3d gravity has black holes.



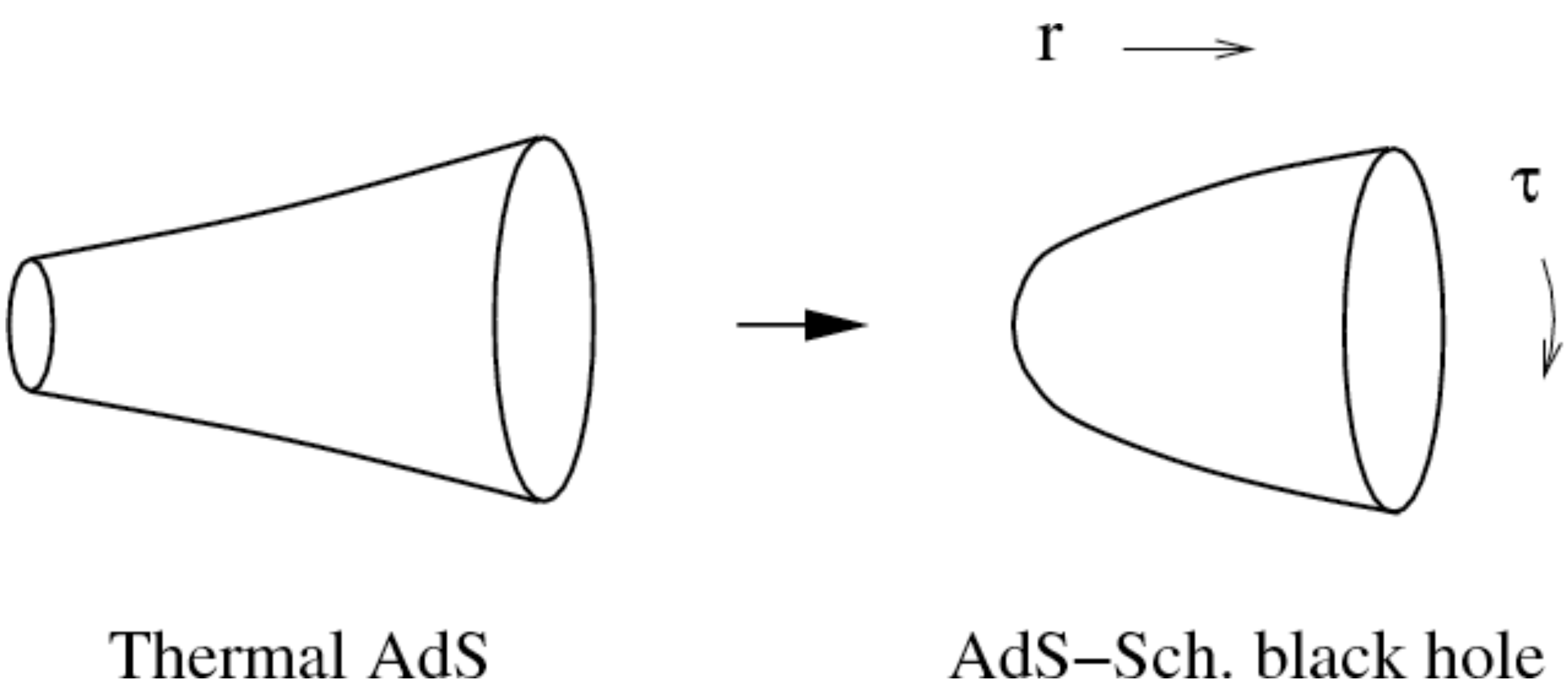
The Bekenstein-Hawking entropy should come out "right" for the black hole states. Cardy only guarantees this for CFT states with

$$\Delta \gg c$$

But we expect in AdS3 gravity that this should work also when

$$\Delta \sim c \ .$$

2. Known large radius models have a phase structure governed by a Hawking-Page transition:



Low temperatures dominated by “gas of particles,” high temperatures by black brane geometry.

While one can try to impose these constraints on a CFT partition function, we chose instead to study an index, the elliptic genus, of 2d SCFTs.

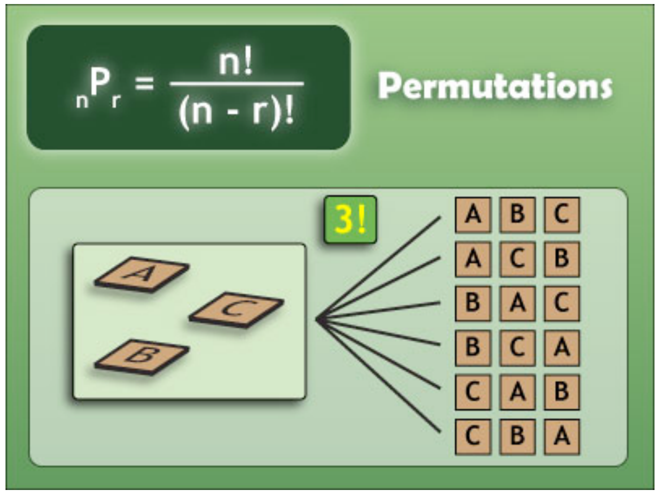
Benjamin, Cheng, SK, Moore, Paquette

Why would one study an index of SCFTs when one could instead study the partition function?

Partition functions are hard to calculate. They are typically calculated (in this literature) for CFTs of the form

$$(\mathcal{C})^N/\mathcal{G}_N$$

$$\mathcal{G}_N \subset \mathcal{S}_N$$



But at such a symmetric orbifold point, any emergent space-time is expected to be "stringy" at best, with

$$L_{\mathrm{AdS}} \sim l_{\mathrm{string}}$$

This is borne out in calculations. One can see that a permutation orbifold has, basically, a Hagedorn density of states beneath the mass where one forms black holes:

$$\rho_N(\Delta) \sim e^{\Delta}, \ \Delta \ll c$$

Keller;
Belin, Keller, Maloney

The index will let us get around this, and distinguish between supergravity and stringy growth. To see how, let us first recall basic facts about indices.

Consider a supersymmetric quantum mechanics theory with a supercharge satisfying

$$Q^2 = 0, \quad \{Q, Q^\dagger\} = H$$

Assume the theory also has a fermion # symmetry, and Q is odd.

From these facts, one can easily prove:

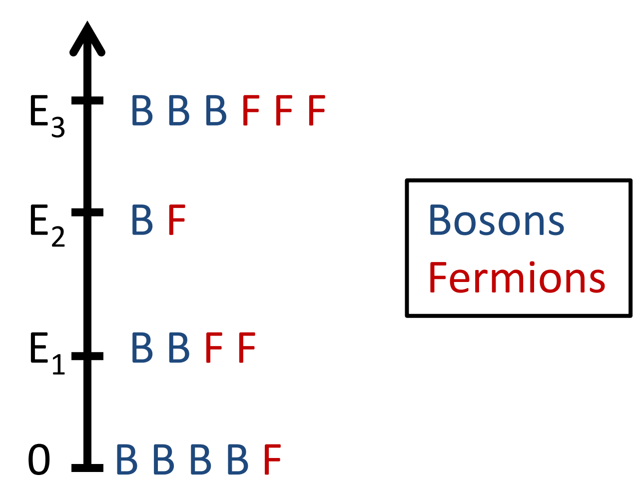
-- all states have non-negative energy

-- states at positive energy are paired by the action of Q

Now, one can define an index:

- The Witten index

$$
\begin{aligned}
Z_{\text{Witten}} &= \mathrm{Tr}(-1)^F \\
&= n_B - n_F \\
&= \mathrm{Tr}\left((-1)^F q^H\right)
\end{aligned}
$$



The Witten index is just a number. A quantity with more information -- an entire q-series -- is available in supersymmetric 2d QFTs.

We'll mostly focus on theories with at least (2,2) supersymmetry.

This means that each chirality has generators

$$T, G^{+}, G^{-}, J \ .$$

In any such theory we can define the elliptic genus:

$$Z_{\mathrm{EG}}(\tau, z) = \mathrm{Tr}_{RR}\left((-1)^{J_0+F_R} q^{L_0} y^{J_0} \bar{q}^{\bar{L}_0}\right)$$

Unpacking the right-moving stuff, we see it is a right-moving Witten index!

It gives rise to a holomorphic modular object, a sort of supersymmetric analogue of the partition function.

Technically speaking, the elliptic genus is a "weak Jacobi form" associated to a moduli space of SCFTs:

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^w e^{2\pi i m\frac{cz^2}{c\tau+d}}\phi(\tau,z)\ ,\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})$$

$$\phi(\tau,z+\ell\tau+\ell')=e^{-2\pi i m(\ell^2\tau+2\ell z)}\phi(\tau,z)\ ,\ \ell,\ell'\in\mathbb{Z}\ .$$

It admits a q,y expansion:

$$\phi(\tau,z)=\sum_{n,\ell\in\mathbb{Z}} c(n,\ell)q^n y^\ell$$

We will be interested in bounding the coefficients.

Defining the "polarity"

$$p(n,l) = 4mn - l^2$$

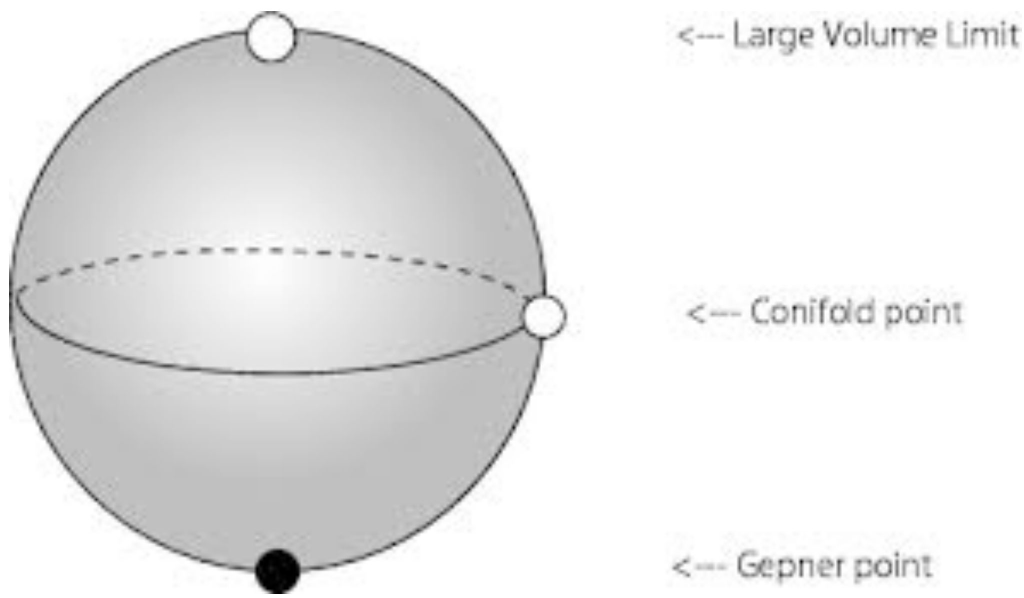
one can in fact show that:

* States with p > 0 are (charged) black holes in AdS

* States with p < 0 are perturbative particles

We will bound the negative polarity coefficients. Happily, these determine the full Jacobi form, by modularity.

Importantly, the coefficients are constant on the moduli space, and so are governed by points with small numbers of states.



If there is a "sugra point" with small numbers of states, the index must behave that way even at "stringy" points due to constancy on the moduli space.

So we can try to find theories where our conditions 1 and 2 are satisfied, but where

$$\rho(\Delta) \sim e^{\Delta^\alpha}, \alpha < 1$$

sub-Hagedorn growth

So, we will consider the elliptic genus of a 2d SCFT with a gravity dual. Lets start with the Bekenstein-Hawking entropy criterion.

The genus gets contributions from extremal spinning black holes in the 3d bulk:

$$\bar{L}_0 = 0,\ L_0 \text{ large}$$

These BTZ black holes have:

$$r_+ = r_- = 2\sqrt{GM}, \quad S = \tfrac{\pi r_+}{2G}$$

As the Brown-Henneaux Virasoro algebra has

$$c = \frac{3}{2G}$$

this entropy can be re-written as:

$$S = 2\pi\sqrt{\frac{cM}{6}}$$

Corrections which are fractional powers of M_{Planck} are absent, but there can be logs. So we expect coefficients in the elliptic genus that go as:

$$c_n = e^{2\pi\sqrt{\frac{cn}{6}} + O(\log\ c)}$$

We could then estimate the elliptic genus, in the regime where black holes dominate, as being:

$$Z(\tau) = \int \ dn \ e^{2\pi\sqrt{\frac{cn}{6}}} e^{2\pi i \tau n}$$

Evaluating by saddle point would give:

$$F = -\pi^2 \frac{m}{\beta^2} + O(\log\ m)$$

This has been heuristic; we need to refine to include the U(1) charge. In the bulk, the U(1) symmetry corresponds to a Chern-Simons gauge field:

$$S_{\mathrm{gauge}}^{\mathrm{boundary}} = -\frac{k}{16\pi}\int_{\partial AdS} d^2x\sqrt{g}g^{\alpha\beta}A_\alpha A_\beta$$

There can be Wilson lines around the non-contractible circle in the black-hole geometry.

The result, for theories with (2,2) supersymmetry, is then:

$$F = -m\frac{\pi^2}{\beta^2} - m\mu^2 + O(\log\, m)$$

This can again be derived from a saddle point argument, using the entropy of the appropriate black holes.

Now, using the S modular transformation, and requiring that the transformed expression (dominated by a sum over low energy states) reproduce the desired answer:

$$\log Z = m\frac{\pi^2}{\beta} + m\beta\mu^2 + O(\log\, m) \quad (\beta < 2\pi)$$

gives a constraint. The basic answer should be

$$\log Z = \frac{c}{24}\beta \qquad\qquad (\beta > 2\pi)$$

and bounding the corrections leads to a constraint on the growth rate of polar coefficients:

$$\boxed{|c(n,\ell)| \leq e^{2\pi\left(n+\frac{m}{2}-\frac{|\ell|}{2}\right)+O(\log m)}.}$$

We had also discussed the Hawking-Page criterion: a good dual to conventional gravity, ought to have a partition function that exhibits a first order phase transition at

$$\beta = 2\pi$$

with known free energies (up to small corrections):

$$\log Z_{EG,NS}(\tau = i\frac{\beta}{2\pi}) = \frac{c_L}{24}\beta + O(1), \;\; \beta > 2\pi.$$

$$\log Z(\tau = i\tfrac{\beta}{2\pi}) = \pi^2 \tfrac{m}{\beta} + \cdots \, , \quad \beta < 2\pi \; .$$

Turns out, this is implied by the Bekenstein-Hawking criterion.

We have explored several classes of examples. The main tool is the DMVV formula.

$$\mathcal{Z}(p,\tau,z) = \sum_{N\geq 0} p^N Z_{EG}(\mathrm{Sym}^N(M);q,y) \ .$$

$$\mathcal{Z}(p,\tau,z) = \prod_{n>0,m\geq 0,l} \frac{1}{(1-p^n q^m y^l)^{c(nm,l)}}.$$

Dijkgraaf, Moore, Verlinde, Verlinde

$$Z_{EG}(M;q,y) = \sum_{m\geq 0,l} c(m,l) q^m y^l.$$

This allows determination of genera of symmetric products, in terms of that of the "seed" CFT.

* For the CFT arising in the D1-D5 system, with target $\mathrm{Sym}^N(K3)$, the growth is indeed sub-Hagedorn:

$$c_n < e^{N^\alpha}, \;\; \alpha < 1 \; .$$

* For the CFTs arising in "generic" symmetric products, one finds Hagedorn behavior.

* But, we have been able to find "new" cases where the genus is consistent with sub-Hagedorn growth.

Benjamin, SK, Keller, Paquette

First class of examples:

It is useful to remember some facts about the space of weak of Jacobi forms.

The ring of Jacobi forms of even weight is a polynomial algebra with four generators:

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + \cdots$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 + \cdots$$

$$\varphi_{-2,1}(\tau, z) = \frac{\theta_1(\tau, z)^2}{\eta^6(\tau)}$$

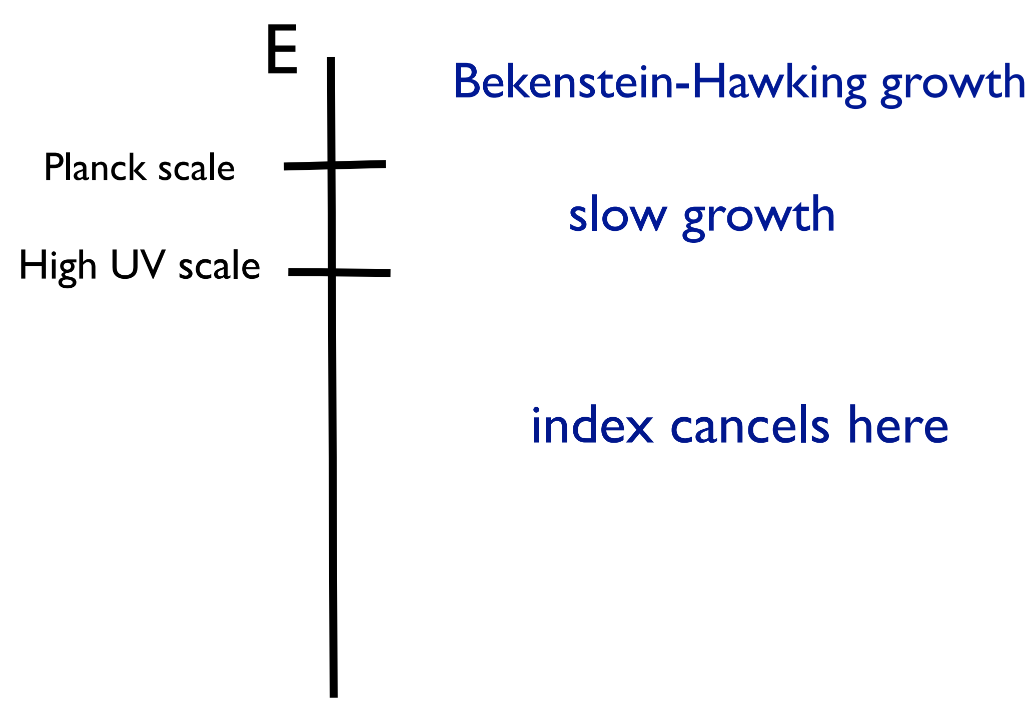
$$\varphi_{0,1}(\tau, z) = 4 \left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau)^2} \right) .$$

We're interested in weight zero and index m = c/6.

Our first class of examples with sub-Hagedorn growth of the index is fairly trivial. If one chooses a "seed" CFT C with

$$Z_{NS}^{EG}(q) = q^{-\frac{1}{4}} + \cdots$$

by tuning the polar pieces, the symmetric product will have sub-Hagedorn growth:

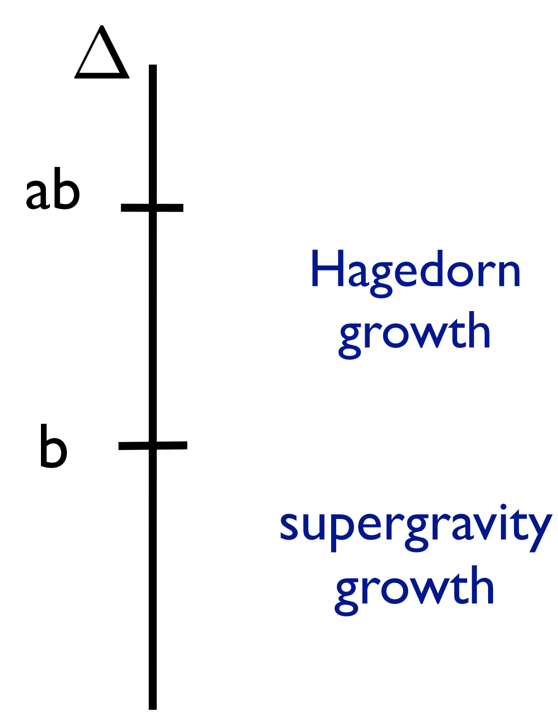


Second class of examples:

The second class of examples comes from adding a parameter to allow a tunable difference between the string and Planck scales, visible in the DMVV formula.

Consider:

$$\mathrm{Sym}^a(\mathrm{Sym}^b(\mathcal{C})) \qquad 1 \ll a \ll b$$



“Data” confirming the saddle point analysis is easily generated. For instance, for a seed theory with genus $\varphi_{0,1}$:

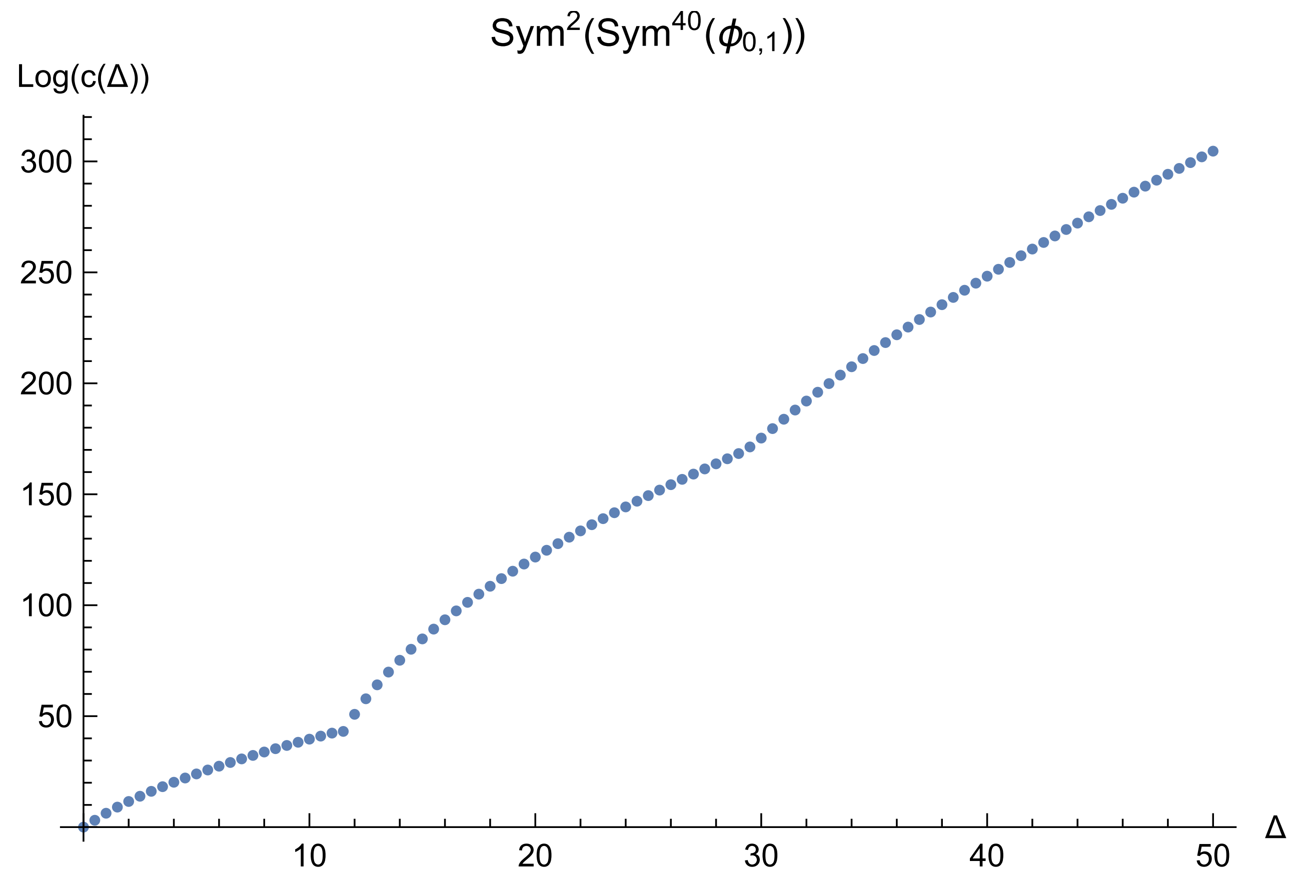


Figure 2: Growth of coefficients in NS sector of $\mathrm{Sym}^2(\mathrm{Sym}^{40}(\phi_{0,1}))$. Note the three very distinctive regions of $a < \Delta < \frac{b}{4}$, $\frac{b}{4} < \Delta < \frac{ab}{4}$, and $\frac{ab}{4} < \Delta$. (The region $\Delta < a$ is too small here to notice.)

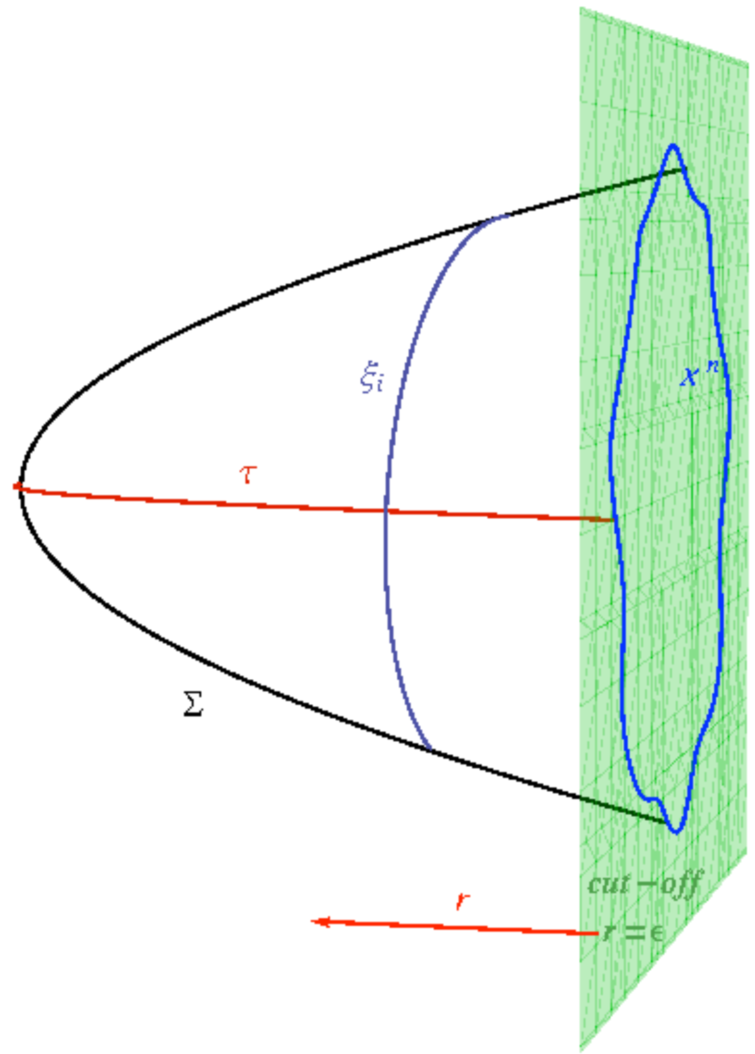
We feel that the index and its refinements may provide a useful order parameter for emergence of macroscopic space-time from an abstract (family of) 2d CFT(s).

IV. Other bounds and equivalences

But obviously, instead of bounds on the density of states, one could try to use other physical observables as an order parameter for the emergence of macroscopic space-time.

Here, I mention two ideas:

i. Entanglement entropy as an order parameter



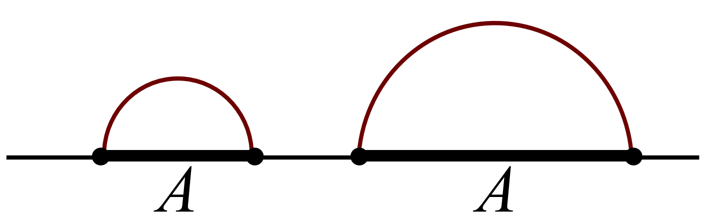
Ryu and Takayanagi famously proposed that the entanglement entropy between a region A and its complement in the boundary QFT, is given by the area of an appropriate bulk minimal surface.

We could require that the CFT entanglement entropy in an appropriate geometry, match the RT formula.

The entanglement entropy for a single segment of length l in 2d CFT is universal at large c

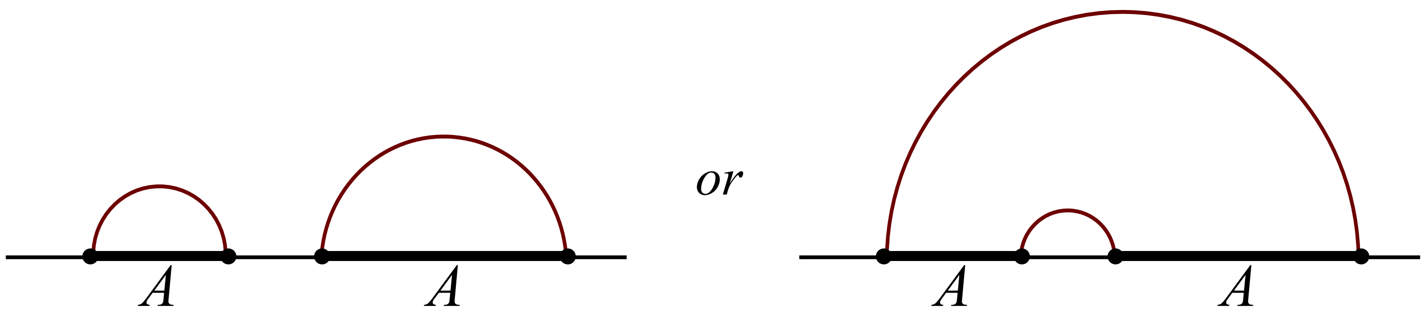
$$S_E = \frac{c}{3}\, \log\, (l/a)$$

But for two strips, it becomes a sensitive probe of the theory:



c.f. Cardy, Calabrese

In fact, in holographic systems, the two interval entanglement is governed by a phase transition between distinct minimal surfaces:



It was studied in some detail by Headrick and, later, Hartman. The requirement that the two interval 2nd Renyi entropy match gravity is essentially identical to the density of states bound we have studied.

(Recall $S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$)

ii. Hard scattering

Heemskerk, Penedones, Polchinski and Sully studied:

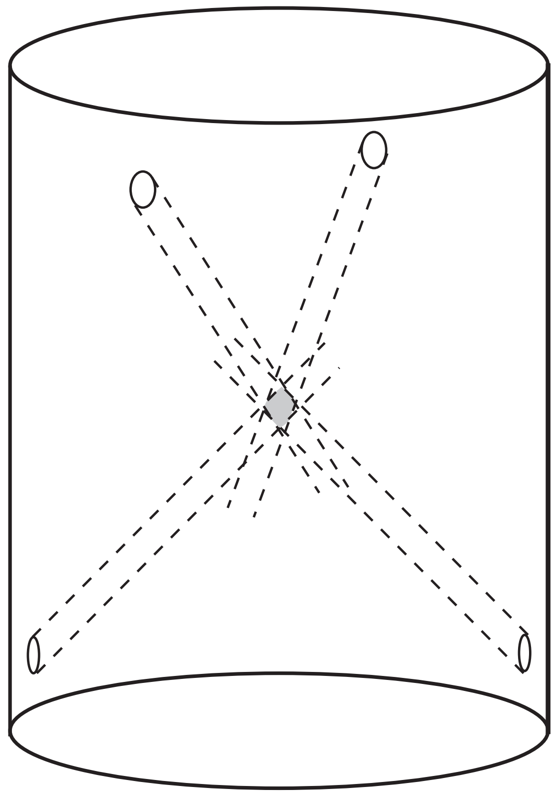


Figure 1: Four-point correlator with wavepackets aligned to intersect in the bulk.

Basic intuition: in a local theory with sub-AdS scale geometry, one can do hard scattering.

This should translate into special singularities in correlation functions of CFTs dual to AdS gravity with a separation of scales between the AdS and string lengths.

Conjecture: These singularities occur precisely for theories where

$$\rho_\infty(\Delta) \sim e^{\Delta^\alpha}, \ \alpha < 1$$

Cleaning up proofs of the statements in this section, and extending to a discussion of the chaos bound, is work in progress with Benjamin, Dyer, Zimet.