Presusy Summer School Lectures on Scattering Amplitudes: Aug 20-22, 2015



## Lecture 2

Yesterday, we introduced the spinor helicity formalism. Today: we'll put it to good use!

One of the prime examples in the field is the n-gluon tree amplitude. Factoring out the color-structure $\mathrm{Tr}(T^{a_1} \cdots T^{a_n})$ one needs

$n = 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$

$1 \quad 3 \quad 10 \quad 38 \quad 154 \quad \cdots$

Feynman diagrams.

However, the answer is remarkably simple and is given by the Parke-Taylor formula:

$$A_n(1^+ \cdots i^- \cdots j^- \cdots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

only neg. helicity
→ rest pos.

This is also known as the MHV gluon tree amplitude (MHV = Maximally Helicity Violating)

NMHV = $- - - + + \cdots +$

$\langle + + \cdots + \rangle = 0$

N$^2$MHV = $- - - - + + \cdots +$ etc. $\langle - + \cdots + \rangle = 0$

The example illustrates

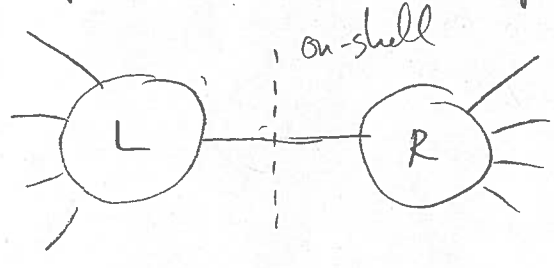
- Feynman diagram calculations become very difficult (and sometimes impossible) when many particles are involved, even at tree level.
- Nonetheless, the answer for the amplitude can be surprisingly simple.

→ why simple?

→ calculate in a better way?

↓ yes! for example on-shell recursion relations

idea: when an internal line in a tree amplitude goes on-shell, the amplitude factorizes into lower pt on-shell amplitudes:




Using complex analysis in a clever way, we can exploit this knowledge of the analytic structure to write an n-particle amplitude in terms of lower-pt amplitudes

3pt → 4pt, then 5pt, then 6pt etc.

So in this sense, the 3-particle amplitudes become the building blocks of all the higher pt (tree) amplitudes
[how and why and when this is possible is something I'll discuss in the last lecture].
Therefore it makes sense for us to first make sure we understand 3-particle amplitudes well.

$\boxed{\text{3-pt amplitudes}}$

We'll focus on massless particles: $p_1^2 = p_2^2 = p_3^2 = 0$

Also, $p_1^\mu + p_2^\mu + p_3^\mu = 0$

Y'day, we learned that

(✱) $\langle 12\rangle[12] = 2p_1\cdot p_2 = (p_1+p_2)^2 = p_3^2 = 0$

So if the momenta are real $\langle 12\rangle^* = [21]$ $\Rightarrow$ all kinematic invariants vanish $A_3 = 0$.

Well-known: a massless particle cannot decay to two massless particles.

But if the momenta $p_{1,2,3}$ are complex-valued, then $\langle 12\rangle$ & $[12]$ are independent.

(✱) then implies that $\langle 12\rangle = 0$ or $[12] = 0$

In fact, studying the other combinations of $p_{1,2,3}$, one finds that either

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{OR} \quad |1] \propto |2] \propto |3]$$

This is called Special (3-particle) kinematics

It implies that $A_3$ can only depend on either $[ij]$ or $\langle ij\rangle$.

Let us suppose we pick $\langle ij\rangle$'s to be $\neq 0$.

Then

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\,\langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 31\rangle^{x_{31}}$$

↑ some constant.

Now recall $p_i = |i\rangle[i|$ when $p_i^2=0$

$p_i^\mu \to 3$ d.o.f, $|i\rangle$, $|i]$ have 2 each $2\times 2=4\neq 3$.

Redundancy: $\boxed{|i\rangle \to t_i|i\rangle \quad \& \quad |i] \to t_i^{-1}|i]}$

then $p_i$ invariant ↖ little group scaling.

Under little grp scaling $A_n \to t_i^{-2h_i}A_n$

for each particle $i=1,\dots,n$. $h_i$ = helicity of particle $i$.

(massless)

Test PT ampl: $k^+ \to t_k^{-2\cdot 1}$ ✓ ; $i,j \to t_{i,j}^{-2\times(-1)} = t_{i,j}^{2}$

Why true?

B/C: ext wave fct's $1$ scalar $(h=0)$

$|\rangle$ or $|]$ fermion $(h=-\frac{1}{2}$ or $h=+\frac{1}{2})$

vector $\varepsilon_-^\mu(p) = \dfrac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]}$ ; $\varepsilon_+^\mu(p) = \dfrac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}$ $h=\pm 1$

$q$ is a reference spinor $q\neq p$.

→ encodes gauge redundancy $\varepsilon^\mu \to \varepsilon^\mu + \alpha p^\mu$

And nothing else in the Feynman diagram can scale under little grp ($p$ inv!)

So starring at our $A_3$, we see that under little grp scaling on particle 1, we must have

$$t^{-2h_1} = t^{x_{12}+x_{31}} \Rightarrow x_{12}+x_{31} = -2h_1$$

likewise ⟶ $x_{12}+x_{23} = -2h_2$

$x_{23}+x_{31} = -2h_3$

$$\Rightarrow \quad x_{12} = h_3 - h_1 - h_2$$
$$x_{23} = h_1 - h_2 - h_3$$
$$x_{31} = h_2 - h_3 - h_1$$

So the little grp scaling determines the 3pt amplitude (up to a constant).

Test gluons:

$$A_3(g_1^- g_2^- g_3^+) = c\ \langle 12\rangle^3 \langle 23\rangle^{-1} \langle 31\rangle^{-1} = c\ \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

→ agrees w/ PT for $n=3$ ✓.

Also dim analysis: $[A_n] = 4-n$

n-particle amplitude in $D=4$ has mass-dim $4-n$ ($\Rightarrow \sigma$ = area).

$p_i \cdot p_j = \langle ij\rangle [ij]$ so $\langle\rangle$ & $[]$ have mass dim 1.

So $[A_3] = 1$ ✓ ok w/ $c = g_{YM}$ = dim'less.

Caviat? We assumed dependence on $\langle ij\rangle$'s only... what if $[ij]\neq 0$ instead?

Run same argument: $A_3(g_1^- g_2^- g_3^+) = c\,\underbrace{\dfrac{[23][31]}{[12]^3}}_{\text{dm}}$

Looks bad, has pole & "wrong" mass dim $c = 2$

So it does not arise from $gAA\partial A \subset F_{\mu\nu}F^{\mu\nu}$

Instead only from some non-local $c\,AA\frac{\partial}{\Box}A$ term, $c \sim (\text{mass})^2$

$\rightarrow$ that's not YM theory.

So we see that insisting on a local theory, this rules out that $A_3(g_1^- g_2^- g_3^+)$ depends on $[ij]$'s only. The correct answer is the one in terms of $\langle ij\rangle$'s.

Let's do another example: gluon & pair of squarks (massless (for now))

$$A_3(g_1^- \tilde{q}_2 \tilde{q}_3^*) = g\,\frac{\langle 12\rangle\langle 13\rangle}{\langle 23\rangle}$$

$$A_3(g_1^+ \tilde{q}_2 \tilde{q}_3^*) = g\,\frac{[12][13]}{[23]}$$

note: conjugates!

We'll use these later!

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Another example: $A_3(g_1^- g_2^- g_3^-) = g'\,\overbrace{\langle 12\rangle\langle 23\rangle\langle 31\rangle}^{\text{dim } 3}$

(but isn't $A_n^{\text{tree}}(-\,-\,\cdots\,-) = 0$ in YM? yes... $g^3$?

$g'$ has mass dim $-2$ $\frac{1}{M^2}\sim g'$. In fact this matrix element arises from $g'\,\mathrm{tr}(F_\mu{}^\nu F_\nu{}^\lambda F_\lambda{}^\mu)$.

## Recursion Relations

As mentioned, the idea is to exploit the analytic structure of amplitudes & factorization. Complex momenta are helpful for this, but doing complex analysis in $n$ vector-variables (as opposed to one single complex variable $z$) is a pain. So we'll introduce a simple complex variable $z$ in the following way.

Suppose particles 1 & 2 are massless. Then define a shift of their spinor helicity variables as follows

$$|\hat{1}] = |1] + z|2] \quad , \quad |\hat{1}\rangle = |1\rangle$$
$$|\hat{2}] = |2] \quad , \quad |\hat{2}\rangle = |2\rangle - z|1\rangle$$

Then $\hat{p}_1^2 = 0$ & $\hat{p}_2^2 = 0$ &

$$\hat{p}_1 + \hat{p}_2 = -\left(|\hat{1}\rangle[\hat{1}| + |\hat{2}\rangle[\hat{2}|\right) = -|1\rangle(|1] + z|2]) - (|2\rangle - z|1\rangle)[2| = p_1 + p_2$$

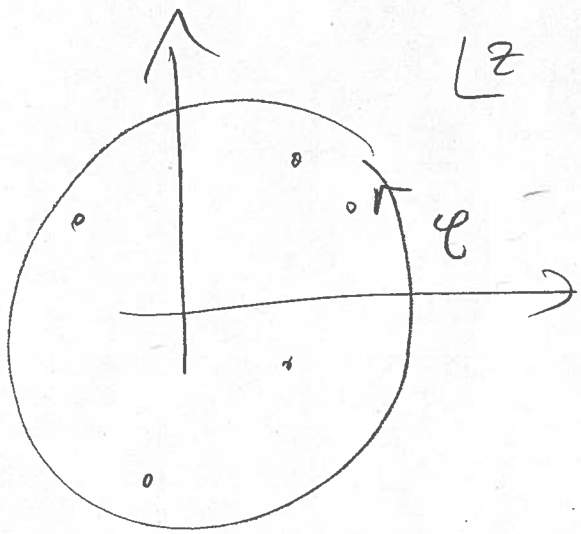
so $\sum_{i=1}^{n} \hat{p}_i = 0$ b/c $\sum_{i=1}^{n} p_i = 0$.

So now in terms of these variable, $\hat{A}_n(z)$.

For a tree amplitude $\hat{A}_n(z)$ can only have simple poles, located away from the origin.

Consider

$$\oint_{\mathcal{C}} \frac{\hat{A}_n(z)}{z} dz = 0$$



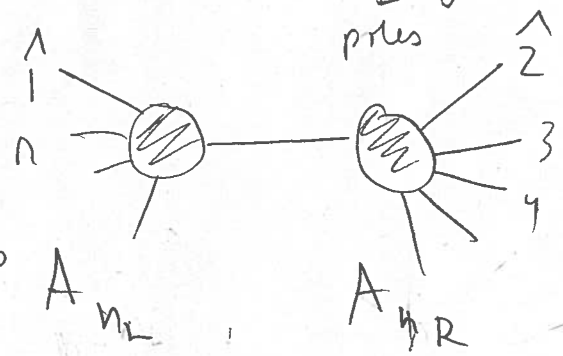

w/ $\mathcal{C}$ surrounding all poles incl. the $\frac{1}{z}$ one.

If there is no pole @ $z=\infty$, then the integral vanishes. This is ensured by $\hat{A}_n(z) \to 0$ for $z \to \infty$. (sufficient, but not a necessary condition).

Cauchy's theorem therefore tells us that

$$A_n = \hat{A}_n(z=0) = - \sum_{\substack{z_I \neq 0 \\ \text{poles}}} \text{Res}\left(\frac{\hat{A}_n(z)}{z}\right)$$

$$= \sum_{\text{diagrams internal states } i}$$




$$= \sum_I \hat{A}_{n_L} \frac{1}{P_I^2} \hat{A}_{n_R}$$

factorization

$\to$ total $n+2$

$$n_L + n_R = n+2 \Rightarrow 3 \leq n_L, n_R < n$$

This is the BCFW (Britto, Cachazo, Feng, Witten 2004) recursion relation.

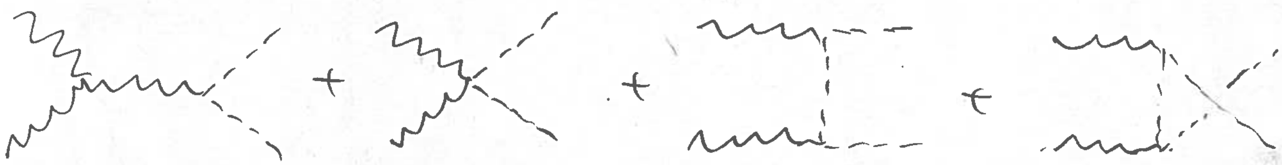
(There are other ones too, also applicable in d-dim)

It is standard at this point in lectures on scattering amplitudes to apply the BCFW recursion relations to derive the PT amplitude by induction on n. You can find that derivation in many of the references I have listed.

So we will not be standard!

Instead we will use BCFW to derive the amplitude for gluon + gluon → squark + squark.

In Feynman diagrams, that process involves four diagrams:



Then once these are calculated, one needs to square $A_4$ to $|A_4|$ and do the usual color + state sum. This is not fun, and it is not just $\gamma$-traces that makes this complicated.

Recall in QED $\quad \sum_h \varepsilon_h^{*\mu} \varepsilon_h^{\nu} \to \eta^{\mu\nu}$ (by ward id)

But not in QCD, either

- $\sum_h \varepsilon_h^{*\mu} \varepsilon_h^{\nu} = \eta^{\mu\nu} +$ ugly.

or

- $\sum_h \varepsilon_h^{*\mu} \varepsilon_h^{\nu} = \eta^{\mu\nu}$ & do separate ghost subtraction.

Neither is fun. L2(

Instead, we will use

- BCFW to calc. $A_4$ (4 diagrams $\to$ 1)
- Exploit spinor helicity to avoid ghost $\varepsilon^* \varepsilon$ complic

Pts
- little grp determines 3pt amplitudes $\leftrightarrow$ massless
- recursion determine n-pt trees from 3pt;