Presusy Summer School Lectures on Scattering Amplitudes : Aug 20 - 22.

UC Davis 2015

Lecture 1:

Over the past ~12 years, there has been remarkable progress in understanding the structure of scattering amplitudes in gauge theory and gravity. The progress is both on the calculational "practical front" (better, more efficients ways to calculate) and on the mathematical front (elucidating the mathematical structure of amplitudes).

Examples of new developments include

- spinor helicity formalism.
- on-shell recursion relations
- generalized unitarity (trees → loops)
- geometrization of amplitudes (polytope picture, Grassmanians, amplituhedron)
- "gravity = (gauge theory)^2"

and much more.

In these lectures, I will take a practical approach to introduce you to formalism that you may find useful to know in your work on QFT, pheno, SUSY.

This also serves as the background needed for delving into the more advanced subjects (which I may or may not have time to tell you more about).

Since this is, after all, the pre-SUSY summer school, I will take some SUSY relevant amplitudes as the primary examples of these lectures. Thus, today we will consider quarks to squarks

$$q + \bar{q} \rightarrow \tilde{q} + \tilde{q}^{*},$$ spin-0 w/ mass m.

and later we will compute gluons to squarks

$$g + g \rightarrow \tilde{q} + \tilde{q}^{*}.$$

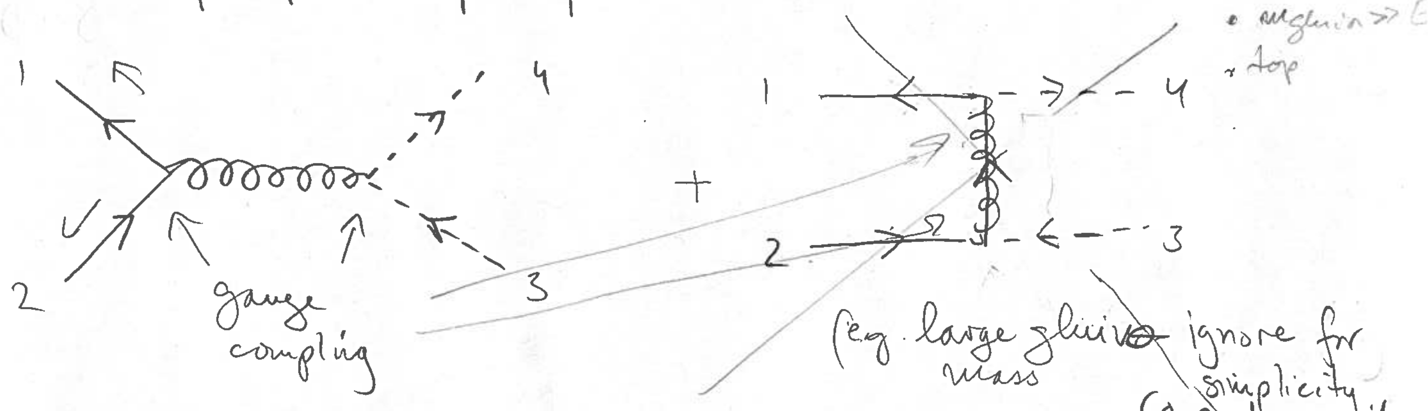
We'll develop the tools needed as we go.

So let's start w/ quarks $\rightarrow$ squarks. We'll assume a high-energy limit so that

- partons are considered fundamental particles & we can use perturbative (super) QCD
- quarks can be approximated as massless. $m_q \ll E$.
- However, we'll keep the squark masses $m \neq 0$

I will be introducing the necessary formalism to you in terms of Feynman diagrams. Then, tomorrow we'll learn about recursion relations.

For $q + \bar{q} \to \tilde{q} + \bar{\tilde{q}}$ we have

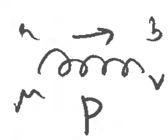


We will (for simplicity) set the Yukawa diagram to zero.

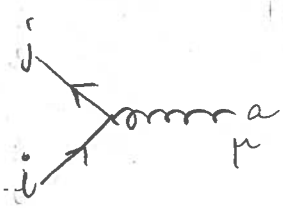
We will consider <u>all</u> particles <u>outgoing</u> (crossing can then takes us back to in, out)

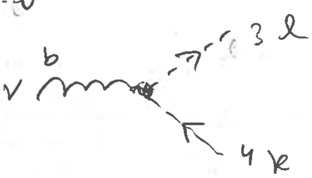
so mom. cons. $p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0$.

<u>Feynman rules</u>: (outgoing)

- external scalar: 1
- external fermion $\bar{u}_\pm(p)$
  external antif. $v_\pm(p)$ ~ massless: helicity ±
- internal gluon (Feynman gauge) $\dfrac{\delta^{ab}\eta_{\mu\nu}}{p^2}$ 

<u>Vertices</u>:

$\bar{q}^a (\not{A} q)^a \to \dfrac{ig\gamma^\mu}{\sqrt{2}} T_{ij}^a$  $\mathrm{Tr}(T^a T^b) = \delta^{ab}$

$|D\tilde{q}|^2 \to \dfrac{ig}{\sqrt{2}}(p_4 - p_3)^\nu T^b_{kl}$ 

So $A_4(q_1^{h_1}\, q_2^{h_2}\, \tilde{q}_3\, \tilde{q}_4^{*})$

$$= \bar{u}_{h_1}\left(\frac{ig\gamma^\mu}{\sqrt{2}}\right) v_{h_2} T^a_{ij}\, \frac{\delta^{ab}\eta_{\mu\nu}}{(p_1+p_2)^2}\cdot\frac{ig}{\sqrt{2}}(p_4-p_3)^\nu T^b_{kl}$$

Now all that is completely standard textbook QFT. The normal procedure is now to compute $\sum|A_4|^2$ by working out some typically lengthy and fairly un-fun γ-matrix traces (recall $\sum_h \bar{u}_h u_h \to -\not{p}$ etc) to get the cross-section $\frac{d\sigma}{d\Omega} \propto \sum|A_4|^2$.

(helicities, colors)

Here is where we deviate from "standard" practice

The fermion wave functions are solutions to the Dirac eq. For the massless case

$$\not{p}\, v_\pm(p) = 0 \quad \& \quad \bar{u}_\pm(p)\,\not{p} = 0$$

$\pm$ ← helicity labels the 2 indep solutions

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}$$

$$\bar{u}_-(p) = \left(0,\ \langle p|_{\dot{a}}\right) \qquad \bar{u}_+(p) = \left([p|^a,\ 0\right)$$

w/ $a, \dot{a} = 1, 2$.

These kets & bra's are commuting 2-component spinors... they are the basis of the Spinor Helicity formalism.

Now let's look at their properties.

Define $p_{a\dot{b}} \equiv p_\mu \sigma^\mu_{a\dot{b}}$. Then, since $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

we have $\boxed{p_{a\dot{b}} |p\rangle^{\dot{b}} = 0 \,, \quad [p|^a p_{a\dot{b}} = 0}$ etc, by the Dirac eq.

Also, $\boxed{|p\rangle^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}} \langle p|_{\dot{b}}}$ etc.

Can form Lorentz inv. (spinor) products:

$$\boxed{\begin{aligned} \langle pq\rangle &\equiv \langle p|_{\dot{a}} |q\rangle^{\dot{a}} = -\langle qp\rangle \\ [pq] &\equiv [p|^a |q]_a = -[qp] \end{aligned}}$$

* often will write $\langle 12\rangle \equiv \langle p_1 p_2\rangle$ etc.

And we can form objects like (w/ a bit of abuse of notat

$$\boxed{\langle p|\gamma^\mu|q] \equiv \langle p|_{\dot{a}} (\bar{\sigma}^\mu)^{\dot{a}b} |q]_b}$$

But note

$$\langle p|\gamma^\mu|q\rangle = 0 = [p|\gamma^\mu|q]$$

$$\bar{u}_- \gamma^\mu u_- = (0, \langle p|) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} = (0, \langle p|) \begin{pmatrix} \sigma^\mu |p\rangle \\ 0 \end{pmatrix} = 0$$

Now recall the spinor completeness relation:

$$\sum_{s=\pm} u_s \bar{u}_s = -\not{p} \quad (m=0) \quad . \quad \text{"Massless crossing": } u_\pm = v_\mp \text{ etc.}$$

this becomes $\boxed{|p\rangle[p| + |p]\langle p| = -\not{p}}$

ie

$$|p\rangle_a [p|_{\dot{a}} = -p_{a\dot{a}} \text{ etc.}$$

So $p^\mu \leftrightarrow (p_{a\dot{b}}, p^{\dot{a}b}) \leftrightarrow |p\rangle[p| \ \& \ |p]\langle p|$

iff. $p^2=0$

Actually, $p_{a\dot{b}} = -|p\rangle[p|$ says that the 2×2 matrix has rank 2 (written in terms of 2 2-component "vectors").

True b/c if $p^2=-m^2 \Leftrightarrow \det p_{a\dot{b}} = m^2$.

> $\langle 12\rangle[12] = 2p_1\cdot p_2$

Well, now we are in business for studying our quarks → squarks amplitude:

Immidiately we see that the amplitude vanishes if the quarks have the SAME helicity

$$\boxed{A_4(q_1^\pm\, q_2^\pm\, \tilde{q}_3\, \tilde{q}_4^*) = 0} \quad \text{b/c} \quad \begin{array}{l}[1|\gamma^\mu|2]=0 \\ \langle 1|\gamma^\mu|2\rangle = 0.\end{array}$$

Consider now opposite helicity.

$$A_4(q_1^- \, q_2^+ \, \tilde{q}_3 \, \tilde{q}_4^*) = -\frac{g^2}{2} \langle 1|\gamma^\mu|2] \frac{1}{(p_1+p_2)^2} (p_4-p_3)_\mu \, T^a_{ij} T^a_{kl}.$$

$0 \quad 0$

- $(p_1+p_2)^2 = \cancel{p_1^2} + \cancel{p_2^2} + 2p_1 \cdot p_2 = \langle 12\rangle[12]$

why? $\langle 12\rangle[12] = -\mathrm{tr}\left[|1\rangle[1|2]\langle 2|\right] = -p_{1\mu}p_{2\nu}\underbrace{\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu)}_{-2\eta^{\mu\nu}}$ ($p_1^{\dot{a}a} \, p_{2b\dot{a}}$)

$= 2p_1 \cdot p_2$. ✓

- $\langle 1|\gamma^\mu|2] = \langle 1|(\cancel{p}_4 - \cancel{p}_3)|2]$

mom. cons. $= \langle 1|(-\cancel{p}_1 - \cancel{p}_2 - 2\cancel{p}_3)|2]$

$= -2\langle 1|\cancel{p}_3|2]$

So

$$\boxed{A_4(\ldots) = g^2 \frac{\langle 1|3|2]}{\langle 12\rangle[12]} T^a_{ij} T^a_{kl}.}$$

nice & compact.

Now, for a moment assume that the squarks are massless too: $m_{\tilde{q}} = 0 \Rightarrow p_3^2 = 0 \Rightarrow \cancel{p}_3 = -|3\rangle[3|$ etc.

Then $\langle 1|\overset{\cancel{p}_3}{3}|2] = -\langle 13\rangle[32]$

$$A_4 = +g^2 \frac{\langle 13\rangle[23]}{\langle 12\rangle[12]} T^a_{ij} T^a_{kl}.$$

- $|A_4|^2$?

p real : $|p\rangle^* = [p|$ & $|p]^* = \langle p|$.

So $\langle pq\rangle^* = [qp]$.

Then $$\sum_{\text{color}} |A_4|^2 = g^4 \frac{\langle 13\rangle[13]\langle 23\rangle[23]}{\langle 12\rangle^2[12]^2} \underbrace{\mathrm{tr}(T^aT^b)}_{\delta^{ab}}\underbrace{\mathrm{tr}(T^aT^b)}_{\delta^{ab}}$$

$$\left.\begin{aligned} s &= -(p_1+p_2)^2 \\ t &= -(p_1+p_3)^2 \\ u &= -(p_1+p_4)^2 \end{aligned}\right\} \quad = g^4 \frac{t \cdot u}{s^2}(N^2-1)$$

Done. No nasty γ-matrix traces!

Was it simple just because we chose the squark mass $= 0$?

Well, let's put it back in! So we'll have

$$|A_4|^2 \supset \langle 1|3|2]\langle 2|3|1] = ?$$

$$\langle 1|\gamma^\mu|2]\langle 2|\gamma^\nu|1] = A\eta^{\mu\nu} + B(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)$$

Dot in p_1^ν or p_2^ν :

$$0 = Ap_1^\mu + Bp_1^\mu(p_1\cdot p_2) \Rightarrow A = -B(p_1 p_2)$$

Fierz id: $\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24]$

so contract w/ $\eta_{\mu\nu}$

$$2\langle 12\rangle[21] = 4A + B\cdot 2(p\cdot p_2) = -2B(p_1 p_2)$$
$$= -2B\langle 12\rangle[12]$$

$\Rightarrow$ $\boxed{B = 2}$

So

$$\langle 1|\gamma^\mu|2]\langle 2|\gamma^\nu|1] = -2(p_1\cdot p_2)\eta^{\mu\nu} + 2(p_1^\mu p_2^\nu + p_2^\nu p_1^\mu).$$

Hence

$$2\langle 1|3|2]\langle 2|3|1] = +4(p_1 p_3)(p_2\cdot p_3) - 2(p_1 p_2)p_3^2$$
$$= \left[(p_1+p_3)^2 - p_3^2\right]\left[(p_2+p_3)^2 - p_3^2\right] - \underbrace{(p_1+p_2)^2}_{-s}\underbrace{p_3^2}_{-m^2}$$
$$= (-t+m^2)(-u+m^2) - sm^2 \qquad p_3^2 = -m^2$$
$$= t_1 u_1 - s m^2.$$

$$\boxed{\sum_{\substack{\text{color}\\\text{helicity}}} |A_4|^2 = 2\times(N^2-1)g^4\,\frac{u_1 t_1 - m^2 s}{s^2}}$$

So here we go ... I spent the first lecture on modern amplitude techniques calculating one amplitude for you... using Feynman diagrams. *SIGH* You thought this would be modern, right?

Anyway. This served to introduce to you the spinor helicity formalism, which we will use heavily in the next two lectures. And you've seen it now in a concrete & relevant example, so hopefully $|p\rangle$ & $|p]$ are a little less scary than an hour ago...

The plan is this:

Lecture 2: recursion relations & the power of spinor helicity.

Lecture 3 Apply to gluon → squarks.
(Outlook, loops, geometry & all that.)