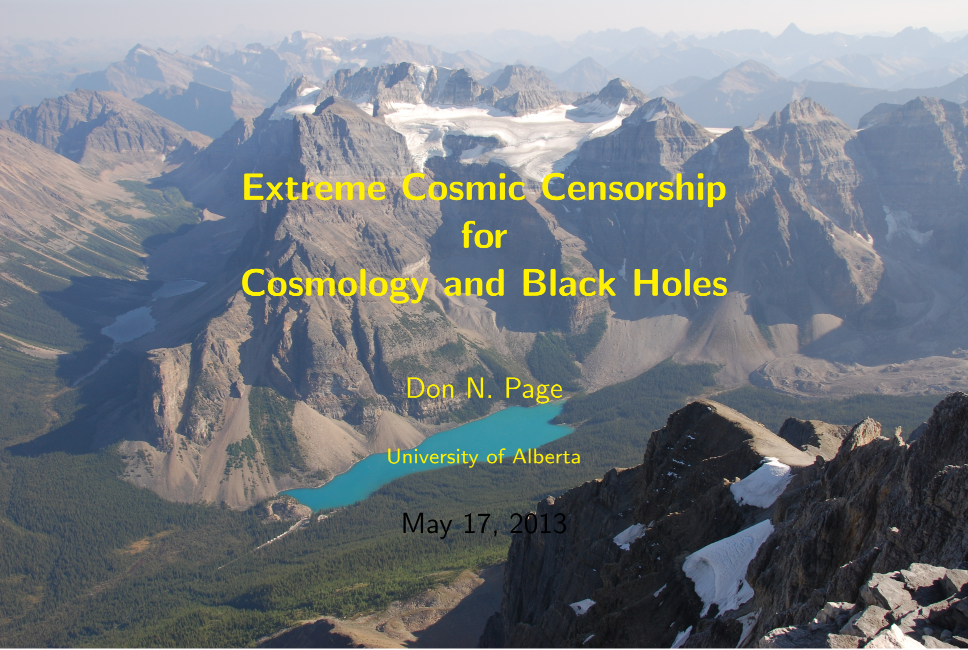

Extreme Cosmic Censorship for Cosmology and Black Holes

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Introduction

A goal of theoretical cosmology is to find a quantum state of the universe that explains our observations.

A goal of black hole physics is to find a quantum description of the evolution of black holes.

In both cases it may be useful to find a restriction that eliminates unphysical states.

Extreme Cosmic Censorship (ECC):

The universe is entirely nonsingular (except for singularities inside black holes which go away when the black holes evaporate).

Classical Models of the Universe

Expansion + gravity + energy conditions $\rightarrow$ big bang singularity

Homogeneity + isotropy $\rightarrow$ Friedmann-Lemaître-Robertson-Walker

Planck + WP + highL + BAO (+ Ali Narimani) $\rightarrow$

$\Lambda = (1.0028 \pm 0.0375) \times 10^{-35}\ \mathrm{s}^{-2} = (10.0065\ \mathrm{Gyr})^{-2}$
$\approx$ ten square attohertz $\approx (10\ \mathrm{Gyr})^{-2}$.

De Sitter entropy $S = 3\pi/\Lambda \approx 5^3 2^{400} \approx 5 t_0^2 \approx 3.23 \times 10^{122}$,

$t_\Lambda \equiv \sqrt{3/\Lambda} = (17.341 \pm 0.325)$ Gyr
$= (1.015 \pm 0.019) \times 10^{61} \approx 2^{608/3} = 1.020 \times 10^{61} \approx 10^{61}$,

$t_0 = 13.798 \pm 0.037$ Gyr
$= (8.0760 \pm 0.0217) \times 10^{60} = (0.796 \pm 0.013) t_\Lambda$
$\approx 5 \times 2^{200}$ mynucuma $= 5 \times 2^{200}(196/195) = 8.0759 \times 10^{60}$,

$H_0 t_0 = 0.9568 \pm 0.0089 \sim n_s = 0.9608 \pm 0.0054$.

CMB energy density when $Ht = 1$ will be about $3^5/2^{22}$ times the dark energy density, this approximation giving $T_0 = 2.72607$ K.

Quantum Models of the Universe

A. Vilenkin, “Creation of Universes from Nothing,” Phys. Lett. B **117**, 25 (1982) (tunneling wave function)

J. B. Hartle and S. W. Hawking, “Wave Function of the Universe,” Phys. Rev. D **28**, 2960 (1983) (no-boundary wave function).

A. D. Linde, “Quantum Creation of the Inflationary Universe,” Lett. Nuovo Cim. **39**, 401 (1984) (Linde’s wave function)

D. N. Page, “Symmetric-Bounce Quantum State of the Universe,” JCAP **0909**, 026 (2009) (symmetric-bounce wave function)

Infinite or Finite Numbers of Quantum States?

An ultimate goal would be to find the actual quantum state of the universe.

An intermediate goal might be to find a restriction to a finite number of states.

String theory is supposed to give an infinite number of states.

It has been argued that $\Lambda > 0$ allows only a finite number of states, such as $\exp{(A/4)}$ where $A = 12\pi/\Lambda$ is the area of the cosmological event horizon of pure de Sitter spacetime.

However, $\Lambda > 0$ by itself does not seem to be sufficient.

Infinitely Many Quantum States with Fixed $\Lambda > 0$

$\Lambda > 0$ de Sitter at very late times has an arbitrarily large volume, and it would seem one could get arbitrarily many perturbed states.

If one evolves backward in time de Sitter with arbitrarily many perturbations added at arbitrarily late times, it would probably lead to a big-bang singularity.

One could alternatively add infinitely many perturbations to the unwrapped Nariai metric (the covering space of $S^2 \times dS_2$, which has an infinitely long time-symmetric throat).

However, this would also probably evolve backward and forward to big bang and big crunch singularities.

Nevertheless, there can be asymptotically locally de Sitter regions to the past and future, so for a finite number of states it is not sufficient to require that there exist one or more asymptotically de Sitter regions in both the past and future.

No-Bang Quantum States

In "No-Bang Quantum State of the Cosmos," Class. Quant. Grav. **25**, 154011 (2008), for getting a finite number of states I proposed excluding states that have a big bang or big crunch or which split into multiple asymptotic de Sitter spacetimes as the Nariai metric would with a large class of perturbations.

I conjecture that this single-nonsingular-de Sitter restriction would lead to a finite number of quantum states.

In "Finite Canonical Measure for Nonsingular Cosmologies," JCAP **1106**, 038 (2011), I showed that the total canonical (Liouville-Henneaux-Gibbons-Hawking-Stewart) measure is finite for completely nonsingular Friedmann-Lemaître-Robertson-Walker classical universes with a minimally coupled massive scalar field and a positive cosmological constant. This suggests that the number of nonsingular quantum states may also be finite.

The Equal-Mixture No-Bang Quantum State

I proposed a no-bang quantum state of the cosmos which is the equal mixture of the Giddings-Marolf states that are asymptotically single de Sitter spacetimes in both past and future and are regular on the throat or neck of minimal 3-volume.

However, after my proposal of "Cosmological Measures without Volume Weighting," JCAP **0810**, 025 (2008), partially solving the Boltzmann-brain problem by weighting by the spatial density of observations rather than by the total number on a hypersurface, I found that the no-bang state then appears to suffer qualitatively from the same problem as the no-boundary state of being dominated by thermal perturbations of nearly empty de Sitter spacetime, so that almost all observers would presumably be Boltzmann brains.

Symmetric-Bounce Quantum State

Therefore, I went to "Symmetric-Bounce Quantum State of the Universe," JCAP **0909**, 026 (2009), a quantum state of the universe that has an initial state that is macroscopically time symmetric about a homogeneous, isotropic bounce of extremal volume and that at that bounce is microscopically in the ground state for inhomogeneous and/or anisotropic perturbation modes.

When combined with volume averaging and "Agnesi Weighting for the Measure Problem of Cosmology," JCAP **1103**, 031 (2011) for damping the weighting of hypersurfaces at late times to avoid a divergence from an infinite lifetime of the universe, the symmetric-bounce state seems to be consistent with observations.

However, it is not so elegant as I would wish and also is not yet unambiguously defined for nonlinear perturbations of the initial bounce state.

Extreme Cosmic Censorship

Although I do not have the final answer as to what the quantum state of the universe is, I suggest that a reasonable state might obey

Extreme Cosmic Censorship (ECC):

The universe is entirely nonsingular (except for singularities inside black holes which go away when the black holes evaporate).

Although I came to this idea independently, it is a strong extension to the future as well as to the past of Anthony Aguirre's proposal in "Eternal Inflation, Past and Future," arXiv:0712.0571:

Consistent Cosmic Censorship (CCC):

Without exception, no physical observer can physically observe a past singularity.

Details of Extreme Cosmic Censorship

I want to exclude big-bang and big-crunch singularities but not those inside black holes.

If one has a spacelike curve running through a black hole but joining two timelike curves that stay outside, by distorting the spacelike curve one can bring it outside the black hole and then move it and its endpoints arbitrarily far forward in time, past the evaporation of the hole.

However, if one has a perturbed Nariai metric that evolves to multiple asymptotically de Sitter regions, a spacelike curve joining one to another cannot be pushed arbitrarily far forward in time without running into a big crunch separating the different asymptotic de Sitter regions.

Thus I am proposing to exclude singularities that a sequence of spacelike curves cannot go around.

Extreme Cosmic Censorship and Black Hole Firewalls

Although I have proposed Extreme Cosmic Censorship mainly for cosmology, it seems to apply to the firewall problem for black holes.

A. Almheiri, D. Marolf, J. Polchinski and J. Sully, "Black Holes: Complementarity or Firewalls?," JHEP **1302**, 062 (2013) (AMPS) give a provocative argument that suggests that an "infalling observer burns up at the horizon" of a sufficiently old black hole, so that the horizon becomes what they called a "firewall."

Unitary evolution suggests that at late times the Hawking radiation is maximally entangled with the inside of the remaining black hole. This further suggests that what is just inside cannot be significantly entangled with what is just outside.
But without this latter entanglement, an observer falling into the black hole should be burned up by high-energy radiation.

Excluding Firewalls with Extreme Cosmic Censorship

Allowing quantum states without the entanglement across a black hole horizon is allowing states that are singular just inside the black hole or would rapidly become singular when evolved back in time.

Such quantum states would be excluded by Extreme Cosmic Censorship.

Without such an exclusion, the number of quantum states for a black hole could be unbounded, greatly exceeding the Bekenstein-Hawking $\exp(A/4)$.

I suggest that it is impossible to form firewall states from nonsingular initial conditions (or from sending in regular data from a boundary of AdS in AdS/CFT).

When firewall states are excluded, the early Hawking radiation can be maximally entangled with the physically allowed black hole states without violating quantum monogamy.

When Is Nonlocality Important

The solution to the firewall problem probably also involves nonlocality, the fact that one does not have localized operators that commute in quantum gravity. For example, generically changing the quantum state in the bulk changes the mass and angular momentum recorded in the gravitational field at infinity.

Y. Kiem, H. L. Verlinde, and E. P. Verlinde, "Black-Hole Horizons and Complementarity," Phys. Rev. **D52**, 7053-7065 (1995):

"**Space-time complementarity (kinematical)**.

Different microscopic observables that are spacelike separated on a Cauchy surface Σ*, but have support on matter field configurations that, when propagated back in time, have collided with macroscopically large center of mass energies, are not simultaneously contained as commuting operators in the physical Hilbert space. Instead such operators are complementary.*"

Possible Criterion for When Nonlocality Is Important

Double Stress-Tensor Criterion:

Relevant operators at two events X and Y do not commute if there is no third event Z in the spacetime connected by causal geodesics to both X and Y such that when the stress-energy tensor at X is parallel propagated along the XZ geodesic to give $T_{\mu\nu}(X,Z)$ at Z and the the stress-energy tensor at Y is parallel propagated along the YZ geodesic to give $T^{\mu\nu}(Y,Z)$ at Z, the trace of the contraction, $T_{\mu\nu}(X,Z)T^{\mu\nu}(Y,Z)$, is less than one in Planck units.

Applying the Double Stress-Tensor Criterion

Consider a $k = 0$ FLRW universe with points X and Y at the present time t_0 and present Hubble constant H_0.

In many cases the minimum for $T_{\mu\nu}(X,Z)T^{\mu\nu}(Y,Z)$ is for point Z at $a/a_0 = 1/(1+z)$ from which null geodesics sent in opposite directions reach X and Y. Then the gamma-factor between the comoving frames parallel-transported back from X and Y to Z is

$\gamma = \frac{1}{2}[(1+z)^2 + (1+z)^{-2}]$.

$T_{\mu\nu}(X,Z)T^{\mu\nu}(Y,Z) = (\rho+P)^2\gamma^2 - 2P(\rho-P) \approx \left[\frac{3}{16\pi}H_0^2(1+z)^2\right]^2$.

If our past has inflation with inflationary Hubble expansion rate H, this reaches unity when X and Y are separated by a distance

$D \approx \frac{8}{H_0 H}\sqrt{\frac{\pi}{3\Omega_m}} \sim \frac{15 t_0}{H}$,

which is a few times the present horizon scale multiplied by the ratio of the inflationary horizon scale to the Planck scale.

Boost Measure Weighting

Choose a fiducial point X in the spacetime, and count only observations at points Y such that there exists a point Z in the spacetime causally related to both X and Y and for which $T_{\mu\nu}(X, Z) T^{\mu\nu}(Y, Z) < 1$.

This **Boost Measure** is somewhat similar to the causal patch measure, but it is not quite so restrictive.

In typical nonvacuum cosmologies with comoving matter, this boost measure criterion will keep the spatial region of the observations finite, though it will not restrict the temporal separation along the matter worldlines.

One would still need something, such as Agnesi weighting, to regulate the time, or else assume that any vacuum in the landscape will decay faster than the formation of Boltzmann brains, as is typically done in certain other measures such as the causal patch measure.

Summary

Extreme Cosmic Censorship (ECC) *(The universe is entirely nonsingular, except for transient black holes.)* may be useful for restricting quantum states in cosmology and in black holes and in solving the firewall problem.

The **Double Stress-Tensor Criterion** *(Operators at X and Y may not commute unless there is a causally related Z such that $T_{\mu\nu}(X,Z)T^{\mu\nu}(Y,Z) < 1$.)* may be useful for suggesting when nonlocality is important in quantum gravity.

The related **Boost Measure** *(Only include observations at points Y such that there exists a point Z causally related to both Y and a fiducial point X for which $T_{\mu\nu}(X,Z)T^{\mu\nu}(Y,Z) < 1$.)* might be useful as another alternative for the measure problem of cosmology.

There are certainly many deep open problems in cosmology,

Précis

Assumptions:

- Initial pure state of large Schwarzschild black hole
- Unitary evolution (no loss of information)
- Complete evaporation into photons and gravitons
- Rapid scrambling, so von Neumann entropy near maximum

Results for von Neumann entropy of emitted Hawking radiation:

- Initially increases and then decreases
- Goes back to zero when black hole decays away at time $t_{\rm decay} \approx 8895\, M_0^3 \approx 1.159 \times 10^{67} (M_0/M_\odot)^3 \rm yr$
- Peaks at $t_* \approx 0.5381\, t_{\rm decay} \approx 4786\, M_0^3$
- $S_{\rm vN}(t_*) \approx 0.5975\, \tilde{S}_{\rm BH}(0) \approx 6.268 \times 10^{76}\, (M_0/M_\odot)^2$

D. N. Page, "Time Dependence of Hawking Radiation Entropy," arXiv:1301.4995 [hep-th].

Introduction

Interest in black hole information has surged recently with A. Almheiri, D. Marolf, J. Polchinski and J. Sully, "Black Holes: Complementarity or Firewalls?," JHEP 1302 (2013) 062.
They give a provocative argument that suggests that an "infalling observer burns up at the horizon" of a sufficiently old black hole, so that the horizon becomes what they called a "firewall."

Unitary evolution suggests that at late times the Hawking radiation is maximally entangled with the inside of the remaining black hole.
This further suggests that what is just inside cannot be significantly entangled with what is just outside.
But without this latter entanglement, an observer falling into the black hole should be burned up by high-energy radiation.

Time Dependence of Hawking Radiation Entropy

One cannot externally observe entanglement across the horizon. However, it should eventually be transferred to the radiation. Therefore, we would like to know the retarded time dependence of the von Neumann entropy of the Hawking radiation.

A. Strominger, "Five Problems in Quantum Gravity," Nucl. Phys. Proc. Suppl. **192-193**, 119 (2009) [arXiv:0906.1313 [hep-th]], has emphasized this question and outlined five candidate answers:

- bad question
- information destruction
- long-lived remnant
- non-local remnant
- maximal information return

I shall assume within proof maximal information return.

Assumptions

- Unitary evolution (no loss of information)
- Initial approximately pure state
  (e.g., $S_{\mathrm{vN}}(0) \sim S(\mathrm{star}) \sim 10^{57} \ll \tilde{S}_{\mathrm{BH}}(0) \sim 10^{77}$)
- Nearly maximal entanglement between hole and radiation
- Complete evaporation into just final Hawking radiation
- Nonrotating uncharged (Schwarzschild) black hole
- Initial black hole mass large, $M_0 > M_\odot$
- Massless photons and gravitons; other particles $m > 10^{-10}$ eV
- Therefore, essentially just photons and gravitons emitted

Arguments for Nearly Maximal Entanglement

D. N. Page, "Average Entropy of a Subsystem," Phys. Rev. Lett. **71**, 1291 (1993) [gr-qc/9305007].
"There is less than one-half unit of information, on average, in the smaller subsystem of a total system in a random pure state."

D. N. Page, "Information in Black Hole Radiation," Phys. Rev. Lett. **71**, 3743 (1993) [hep-th/9306083].
"If all the information going into gravitational collapse escapes gradually from the apparent black hole, it would likely come at initially such a slow rate or be so spread out . . . that it could never be found or excluded by a perturbative analysis."

Y. Sekino and L. Susskind, "Fast Scramblers," JHEP **0810**, 065 (2008) [arXiv:0808.2096 [hep-th]], conjecture:

- The most rapid scramblers take a time logarithmic in the number of degrees of freedom.
- Black holes are the fastest scramblers in nature.

These conjectures support my results using an average over all pure states of the total system of black hole plus radiation.

Numerical Calculations

D. N. Page, "Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged, Nonrotating Hole," Phys. Rev. D **13**, 198 (1976).
D. N. Page, "Particle Emission Rates from a Black Hole. 2. Massless Particles from a Rotating Hole," Phys. Rev. D **14**, 3260 (1976).
D. N. Page, "Particle Emission Rates from a Black Hole. 3. Charged Leptons from a Nonrotating Hole," Phys. Rev. D **16**, 2402 (1977).

Photon and graviton emission from a Schwarzschild black hole:

- $dM/dt = -\alpha/M^2 \approx -0.000\,037\,474/M^2$.
- $d\tilde{S}_{\mathrm{BH}}/dt = -8\pi\alpha/M \approx -0.000\,941\,82/M$.
- $d\tilde{S}_{\mathrm{rad}}/dt \approx 0.001\,398\,4/M = -\beta d\tilde{S}_{\mathrm{BH}}/dt$.
- $\beta \equiv (d\tilde{S}_{\mathrm{rad}}/dt)/(-d\tilde{S}_{\mathrm{BH}}/dt) \approx 1.4847$.

Semiclassical Evolution

Black hole mass time dependence:

$$M(t) = (M_0^3 - 3\alpha t)^{1/3} = M_0(1 - t/t_{\rm decay})^{1/3}.$$

Decay time for a large Schwarzschild black hole:

$$t_{\rm decay} = \gamma M_0^3 \equiv \frac{1}{3\alpha} M_0^3 \approx 8895 M_0^3 \approx 1.159 \times 10^{67} \left(\frac{M_0}{M_\odot}\right)^3 \mathrm{yr}.$$

Semiclassical Bekenstein-Hawking black hole entropy:

$$\tilde{S}_{\rm BH}(t) = 4\pi M_0^2 \left(1 - \frac{t}{\gamma M_0^3}\right)^{2/3} \approx 4\pi M_0^2 \left(1 - \frac{t}{8895 M_0^3}\right)^{2/3}.$$

Semiclassical Hawking radiation entropy:

$$\begin{aligned} \tilde{S}_{\rm rad}(t) &= 4\pi\beta M_0^2 \left[1 - \left(1 - \frac{t}{\gamma M_0^3}\right)^{2/3}\right] \\ &\approx 4\pi(1.4847) M_0^2 \left[1 - \left(1 - \frac{t}{8895 M_0^3}\right)^{2/3}\right]. \end{aligned}$$

von Neumann Entropies of the Radiation and Black Hole

Take the semiclassical entropies $\tilde{S}_{\mathrm{rad}}(t)$ and $\tilde{S}_{\mathrm{BH}}(t)$ to be approximate upper bounds on the von Neumann entropies of the corresponding subsystems with the same macroscopic parameters.

Therefore, the von Neumann entropy of the Hawking radiation, $S_{\mathrm{vN}}(t)$, which assuming a pure initial state and unitarity is the same as the von Neumann entropy of the black hole, should not be greater than either $\tilde{S}_{\mathrm{rad}}(t)$ or $\tilde{S}_{\mathrm{BH}}(t)$.

Take my 1993 results as suggestions for the *Conjectured Anorexic Triangle Hypothesis (CATH)*:

Entropy triangular inequalities are usually nearly saturated.

This leads to the assumption of nearly maximal entanglement between hole and radiation, so $S_{\mathrm{vN}}(t)$ should be near the minimum of $\tilde{S}_{\mathrm{rad}}(t)$ and $\tilde{S}_{\mathrm{BH}}(t)$.

Time of Maximum von Neumann Entropy

Since the semiclassical entropy $\tilde{S}_{\rm rad}(t)$ is monotonically increasing with time, and since the semiclassical entropy $\tilde{S}_{\rm BH}(t)$ is monotonically decreasing with time, the maximum von Neumann entropy is at the crossover point, at time

$$t_* = \epsilon t_{\rm decay} \approx 0.5381\, t_{\rm decay} \approx 4786\, M_0^3 \approx 6.236 \times 10^{66} (M_0/M_\odot)^3 {\rm yr},$$

with

$$\epsilon \equiv 1 - [\beta/(\beta+1)]^{3/2} \approx 0.5381,$$

at which time the mass of the black hole is

$$M_* = [\beta/(\beta+1)]^{1/2} M_0 \approx 0.7730\, M_0,$$

and its semiclassical Bekenstein-Hawking entropy $4\pi M^2$ is

$$\tilde{S}_{\rm BH*} = [\beta/(\beta+1)] \tilde{S}_{\rm BH}(0) \approx 0.5975\, \tilde{S}_{\rm BH}(0).$$

Maximum von Neumann Entropy of the Hawking Radiation

At the time t_* when $\tilde{S}_{\mathrm{rad}}(t) = \tilde{S}_{\mathrm{BH}}(t)$, the von Neumann entropy of the radiation and of the black hole is maximized and has the value

$$S_* \equiv S_{\mathrm{vN}}(t_*) = \tilde{S}_{\mathrm{rad}}(t_*) = \tilde{S}_{\mathrm{BH}}(t_*) = \left(\frac{\beta}{\beta+1}\right) 4\pi M_0^2 \approx 0.5975\, \tilde{S}_{\mathrm{BH}}(0)$$
$$= 0.5975(4\pi M_0^2) \approx 7.509 M_0^2 \approx 6.268 \times 10^{76} (M_0/M_\odot)^2.$$

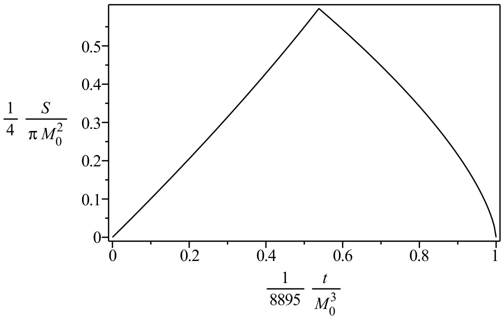
Note that this maximum of the von Neumann entropy is about 19.5% greater than half the original semiclassical Bekenstein-Hawking entropy of the black hole. The time t_* for the maximum von Neumann entropy is about 0.8324 times the time $t_{1/2} = (1 - 2^{-3/2}) t_{\mathrm{decay}} \approx 0.6464\, t_{\mathrm{decay}} \approx 1.201\, t_*$ for the black hole to lose half its area and semiclassical Bekenstein-Hawking entropy.

Time Dependence of the Entropy of the Hawking Radiation

The von Neumann entropy of the Hawking radiation $S_{\mathrm{vN}}(t)$ from a large nonrotating uncharged black hole is very nearly the semiclassical radiation entropy $\tilde{S}_{\mathrm{rad}}(t)$ for $t < t_*$ and is very nearly the Bekenstein-Hawking semiclassical black hole entropy $\tilde{S}_{\mathrm{BH}}(t)$ for $t > t_*$, or, using the Heaviside step function $\theta(x)$,

$$
\begin{aligned}
S_{\mathrm{vN}}(t) &\approx 4\pi\beta M_0^2 \left[1 - \left(1 - \frac{t}{t_{\mathrm{decay}}}\right)^{2/3}\right] \theta(t_* - t) \\
&+ 4\pi M_0^2 \left(1 - \frac{t}{t_{\mathrm{decay}}}\right)^{2/3} \theta(t - t_*) \\
&\approx 4\pi(1.4847) M_0^2 \left[1 - \left(1 - \frac{t}{8895 M_0^3}\right)^{2/3}\right] \theta(4786 M_0^3 - t) \\
&+ 4\pi M_0^2 \left(1 - \frac{t}{8895 M_0^3}\right)^{2/3} \theta(t - 4786 M_0^3).
\end{aligned}
$$

Plot of Hawking Radiation Entropy vs. Time



Corrections to the Approximate Entropy Formula

- Fluctuations in $M(t)$: $\Delta S = O(M_0)$.
- Fluctuations in $\mathbf{x}(t)$: $\Delta S = O(M_0)$.
- Entropy of black hole motion: $\Delta S = O(\ln M_0)$.
- Nonmaximal entanglement: $\Delta S = O(1)$.
- Entropy near black hole: $\Delta S = O(1)$.
- Fuzziness of t boundary: $\Delta S = O(1)$.

Impure Black Hole Initial States

Suppose the black hole forms in a mixed state of von Neumann entropy S_0, nearly maximally entangled with a reference system (X) with total state pure. Let the hole be (Y) and the radiation be (Z), with effective Hilbert-space dimensions X, Y, and Z (last two changing with time).

$X < YZ \to S(X) = S(YZ) \approx \ln X = S_0$ for all time.

(1) $XZ < Y \to Z < XY, S(XZ) = S(Y) \approx S_0 + \ln Z = S_0 + \tilde{S}_{\rm rad}$,
$S(Z) = S(XY) \approx \ln Z = \tilde{S}_{\rm rad}$.

(2) $Y < XZ < X^2 Y \to S(Y) = S(XZ) \approx \ln Y = \tilde{S}_{\rm BH}$,
$S(Z) = S(XY) \approx \ln Z = \tilde{S}_{\rm rad}$.

(3) $XY < Z \to Y < XZ, S(XY) = S(Z) \approx S_0 + \ln Y = S_0 + \tilde{S}_{\rm BH}$,
$S(Y) = S(XZ) \approx \ln Y = \tilde{S}_{\rm BH}$.

Entropy Time Dependence for Impure States

Let $f = S_0/\tilde{S}_{\rm BH}(0) = (\ln X)/(4\pi M_0^2)$.

$\tau = 1 - \tilde{S}_{\rm BH}(t)/\tilde{S}_{\rm BH}(0) = 1 - (1 - t/t_{\rm decay})^{2/3}$.

$\tau_{12} = \frac{1-f}{1+\beta}$, $\tau_{23} = \frac{1+f}{1+\beta}$.

$S_{\rm BH} = S(Y) = \tilde{S}_{\rm BH}(0)[\theta(\tau_{12} - \tau)(f + \beta\tau) + \theta(\tau - \tau_{12})(1 - \tau)]$.

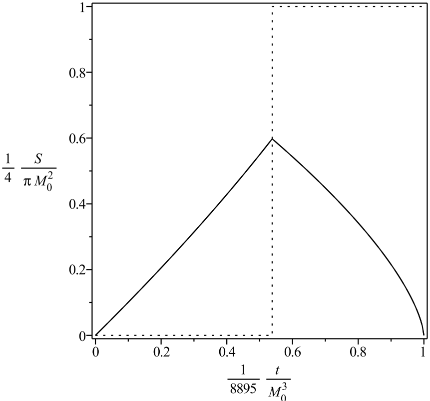
At $\tau = \tau_{12}$ or $t_{12} = t_{\rm decay}\{1 - [(\beta + f)/(1 + \beta)]^{3/2}\}$, the von Neumann entropy of the black hole reaches its peak of $S_{\rm BH}(t_{12}) = [(f + \beta)/(1 + \beta)]\tilde{S}_{\rm BH}(0)$, which is less than the initial Bekenstein-Hawking entropy $\tilde{S}_{\rm BH}(0) = 4\pi M_0^2$ unless the black hole starts maximally mixed, $S_{\rm BH}(0) = \tilde{S}_{\rm BH}(0)$, or $f = 1$.

$S_{\rm vN} = S(Z) = \tilde{S}_{\rm BH}(0)[\theta(\tau_{23} - \tau)(\beta\tau) + \theta(\tau - \tau_{23})(1 + f - \tau)]$.

At $\tau = \tau_{23}$ or $t_{23} = t_{\rm decay}\{1 - [(\beta - f)/(1 + \beta)]^{3/2}\}$, the von Neumann entropy of the Hawking radiation reaches its peak of $S_{\rm BH}(t_{23}) = \beta[(1 + f)/(1 + \beta)]\tilde{S}_{\rm BH}(0)$, which for $f = 1$ is $[2\beta/(1 + \beta)]\tilde{S}_{\rm BH}(0) \approx 1.1951\,\tilde{S}_{\rm BH}(0)$ at $t_{23} \approx 0.9138\,t_{\rm decay}$.

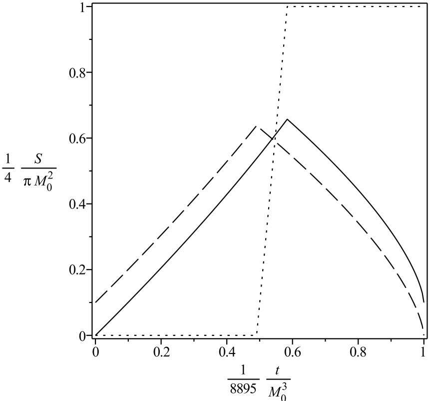
Plot of Hole and Radiation Entropy vs. Time for $f = 0$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



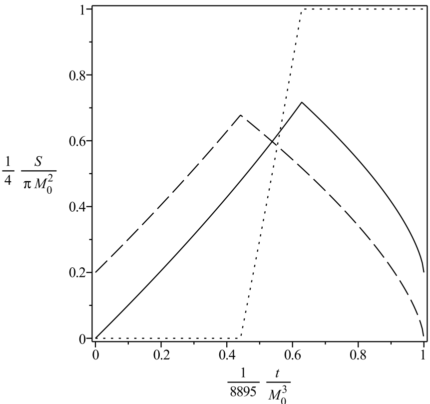
Plot of Hole and Radiation Entropy vs. Time for $f = 0.1$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



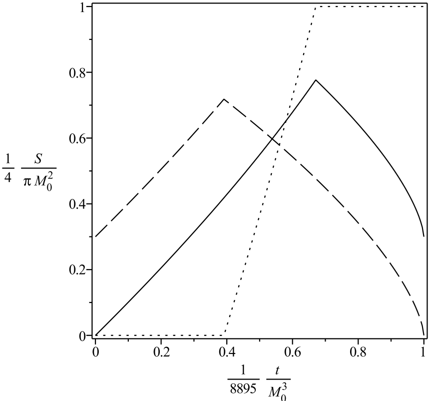
Plot of Hole and Radiation Entropy vs. Time for $f = 0.2$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



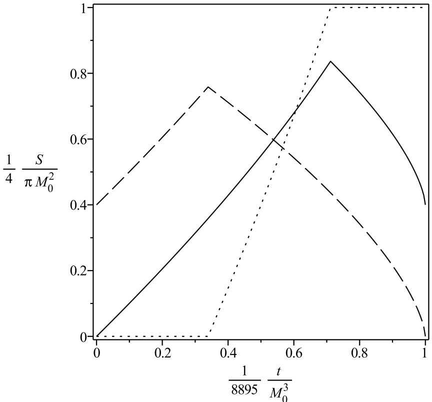
Plot of Hole and Radiation Entropy vs. Time for $f = 0.3$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



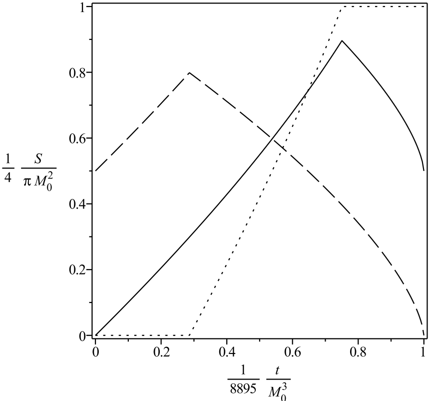
Plot of Hole and Radiation Entropy vs. Time for $f = 0.4$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



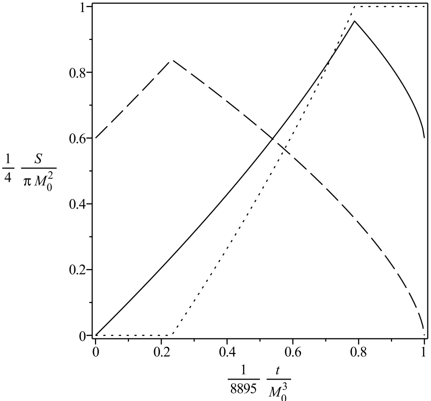
Plot of Hole and Radiation Entropy vs. Time for $f = 0.5$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



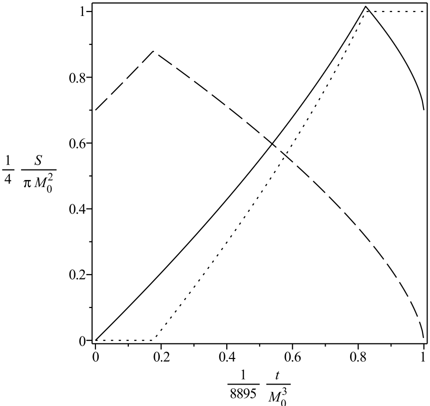
Plot of Hole and Radiation Entropy vs. Time for $f = 0.6$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



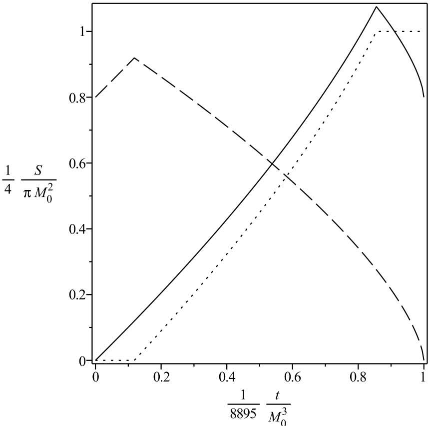
Plot of Hole and Radiation Entropy vs. Time for $f = 0.7$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



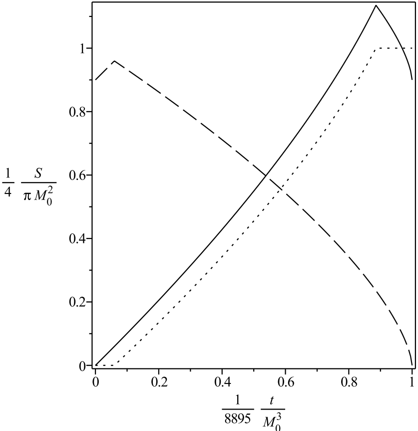
Plot of Hole and Radiation Entropy vs. Time for $f = 0.8$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



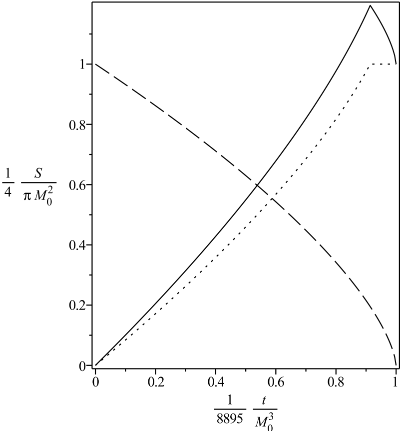
Plot of Hole and Radiation Entropy vs. Time for $f = 0.9$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



Plot of Hole and Radiation Entropy vs. Time for $f = 1$

Solid line is the von Neumann entropy of the Hawking radiation.
Dashed line is the von Neumann entropy of the black hole.
Dotted line is $[S(X) + S(Z) - S(XZ)]/[2S(X)]$.



Conclusions

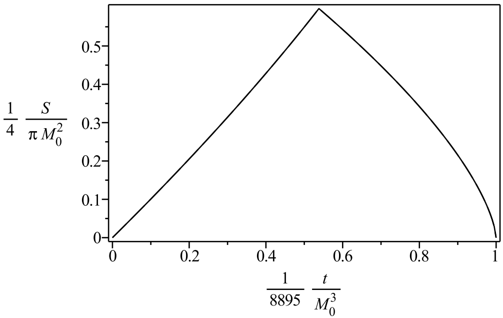
Under the assumptions that a Schwarzschild black hole of initial mass $M_0 > M_\odot$ (too massive to emit anything but photons and gravitons) starts in nearly a pure quantum state and decays away completely by a unitary process while being nearly maximally scrambled at all times, the von Neumann entropy of the Hawking radiation increases up to a maximum of $S_* \approx 0.5975(4\pi M_0^2) \approx 7.509 M_0^2 \approx 6.268 \times 10^{76}(M_0/M_\odot)^2$ at time $t_* \approx 0.53810\, t_{\rm decay} \approx 4786 M_0^3 \approx 6.236 \times 10^{66}(M_0/M_\odot)^3 {\rm yr}$ and then decreases back down to near zero.

If the black hole starts in a maximally mixed state ($f = 1$, so $S_0 \equiv f\tilde{S}_{\rm BH}(0) = \tilde{S}_{\rm BH}(0) = 4\pi M_0^2$), the von Neumann entropy of the Hawking radiation increases from zero up to a maximum of $S_{*'} \approx 1.1951(4\pi M_0^2) \approx 15.018 M_0^2 \approx 1.254 \times 10^{77}(M_0/M_\odot)^2$ at $t_{*'} \approx 0.91384\, t_{\rm decay} \approx 8129 M_0^3 \approx 1.059 \times 10^{67}(M_0/M_\odot)^3 {\rm yr}$ and then decreases back down to $S_0 = 1.049 \times 10^{77}(M_0/M_\odot)^2$.

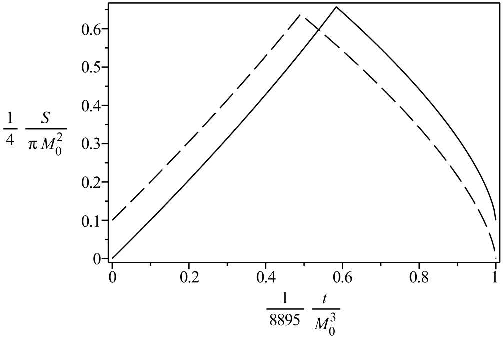
Acknowledgments

- Leonard Susskind: invitation to firewall meeting 2012 Nov. 30
- Andrew Strominger: asking about entropy vs. time
- Other participants: questions and discussions
- Samuel Braunstein: suggestion for using reference system
- NSERC of Canada: financial support

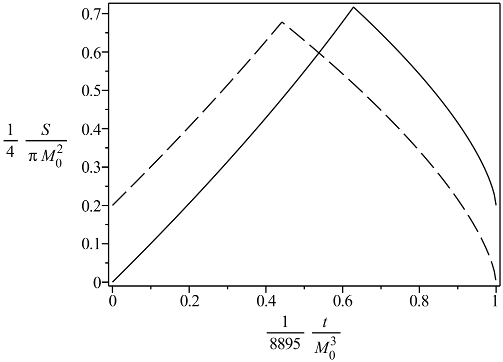
Plot of Hole and Radiation Entropy vs. Time for $f = 0$



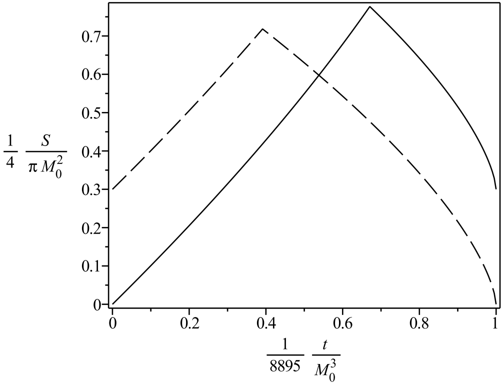
Plot of Hole and Radiation Entropy vs. Time for $f = 0.1$



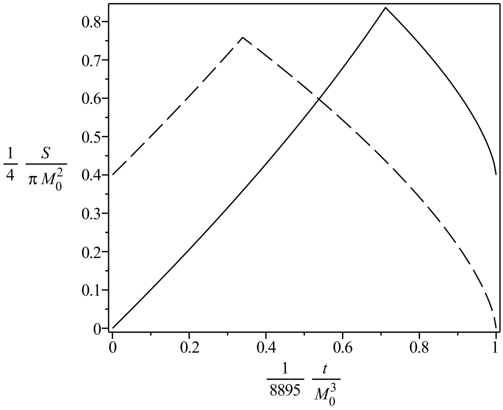
Plot of Hole and Radiation Entropy vs. Time for $f = 0.2$



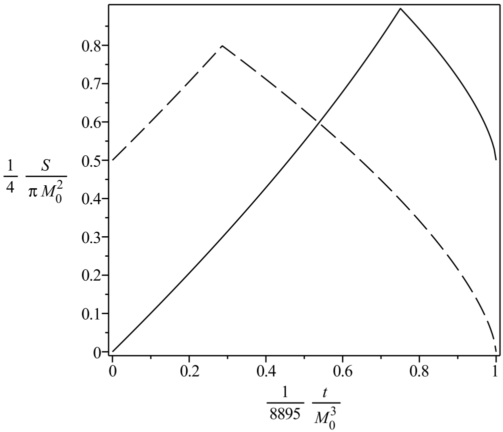
Plot of Hole and Radiation Entropy vs. Time for $f = 0.3$



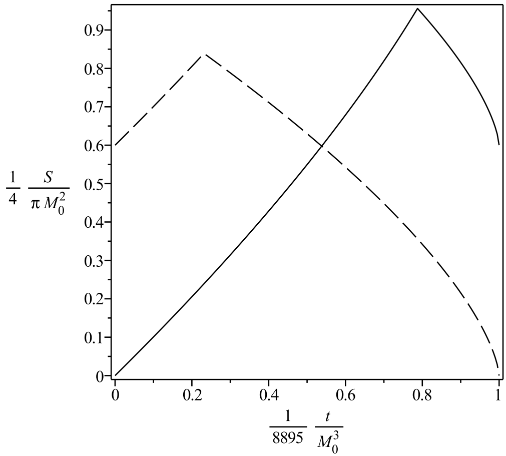
Plot of Hole and Radiation Entropy vs. Time for $f = 0.4$



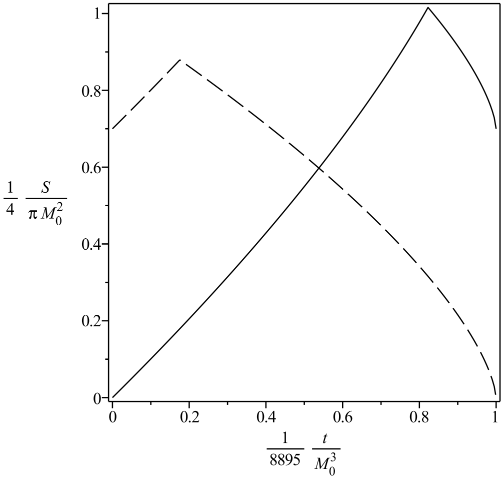
Plot of Hole and Radiation Entropy vs. Time for $f = 0.5$



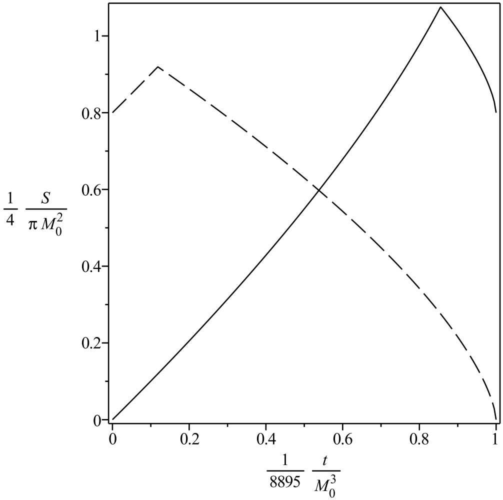
Plot of Hole and Radiation Entropy vs. Time for $f = 0.6$



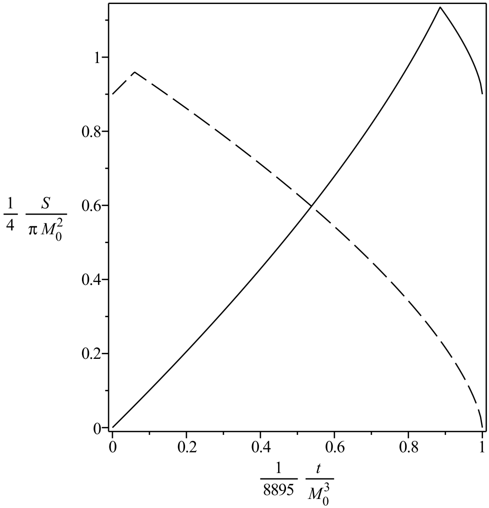
Plot of Hole and Radiation Entropy vs. Time for $f = 0.7$



Plot of Hole and Radiation Entropy vs. Time for $f = 0.8$



Plot of Hole and Radiation Entropy vs. Time for $f = 0.9$



Plot of Hole and Radiation Entropy vs. Time for $f = 1$