

Horizon instability of an extreme Reissner-Nordström black hole

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in collaboration with

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based on **arXiv:1212.2557**

(In)stability of Black holes

- Classical stability of 4D Black holes
 - Stable (mostly)
 - Power-law decay of perturbations
- **Extreme** black holes in 4D
 - à Novel instability on the horizon
 - Scalar fields
 - EM & Gravitational perturbations

Contents

1. 1957 – 2010

BH Stability & Price's law

2. 2011

Instability of extreme horizons

3. 2012

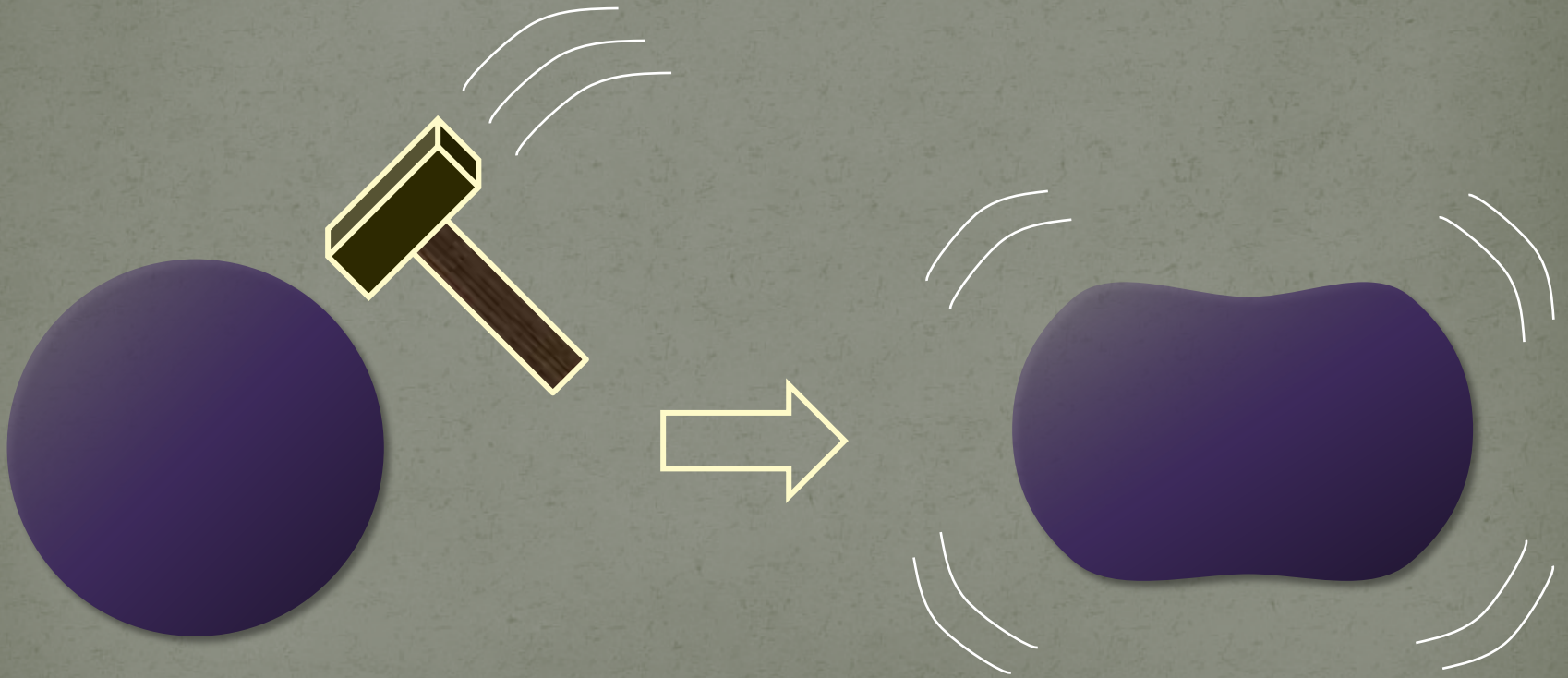
Generalizations

4. 2013

Backreaction

1957 – 2010: BH Stability & Price's law

– BH Stability:



[Regge & Wheeler “Stability of a Schwarzschild singularity” 1957]

1957 – 2010: BH Stability & Price's law

– Price's law (1972):

– Scalar field on BH decays by power law:

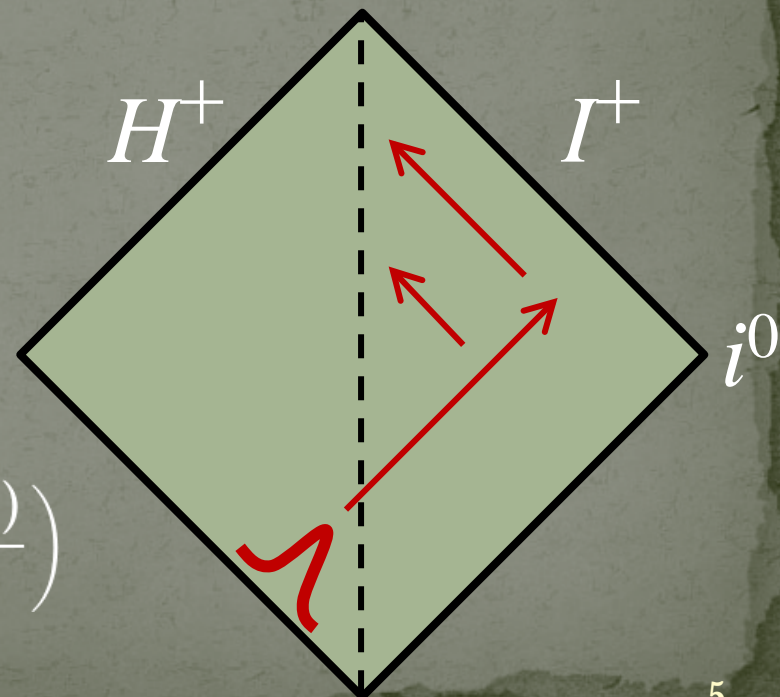
$$\psi_\ell(t, r = r_0) \propto t^{-(2\ell+3)} \text{ or } t^{-(2\ell+2)}$$



Due to backscattering

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial r_*^2} - V(r)\phi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right)$$



1957 – 2010: BH Stability & Price's law

- Price's law (1972):

- Scalar field on BH decays by power law:

$$\psi_\ell(t, r = r_0) \propto t^{-(2\ell+3)} \text{ or } t^{-(2\ell+2)}$$

- By heuristic arguments

- Confirmations
 - Generalizations (Sch. \rightarrow Kerr, RN, ...)
 - Also on H^+ and I^+ [Gundlach+ 1993]
 - Quasi-normal modes
 - ...

1957 – 2010: BH Stability & Price's law

– Rigorous results:

Uniform boundedness [Wald 1979, Kay & Wald 1987]

Regular initial data $\psi(t_0, r)$ on Schwarzschild

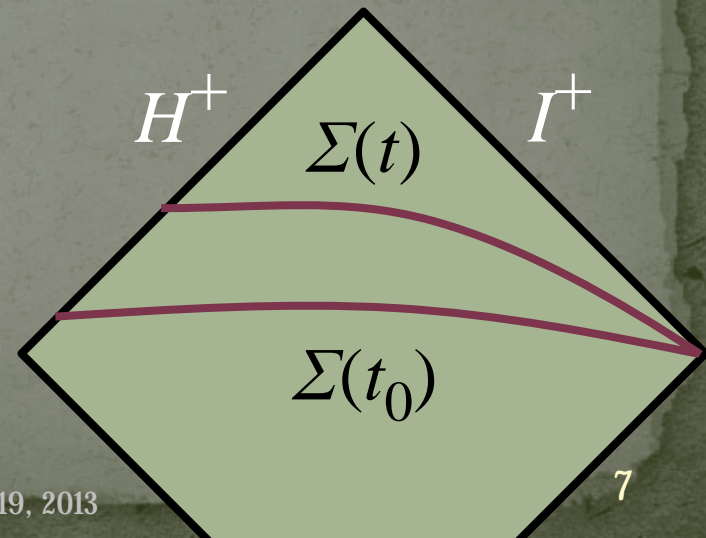
$$\hat{a} \quad |\psi(t, r)| < C[\psi(t_0)]$$

everywhere including H^+

ü Energy estimate

$$E[\Sigma(t)] \leq E[\Sigma(t_0)]$$

$$\approx \int_{\Sigma(t)} (\partial_t \psi)^2 + (\partial_r \psi)^2 + (\partial_\Omega \psi)^2$$



1957 – 2010: BH Stability & Price's law

- Rigorous results:

Decay laws:

- Scalar field on stationary asym. flat spacetime

$$\text{à } |\psi(t, r)| < C[\psi(t_0)] t^{-3}$$

[Tataru 2010]

- Spherically sym. Scalar on Schwarzschild

$$\text{à } |\psi(t)| + |\partial_t \psi(t)| < C[\psi(t_0)] t^{-3}$$

including *backreaction*

[Dafermos & Rodnianski 2004]

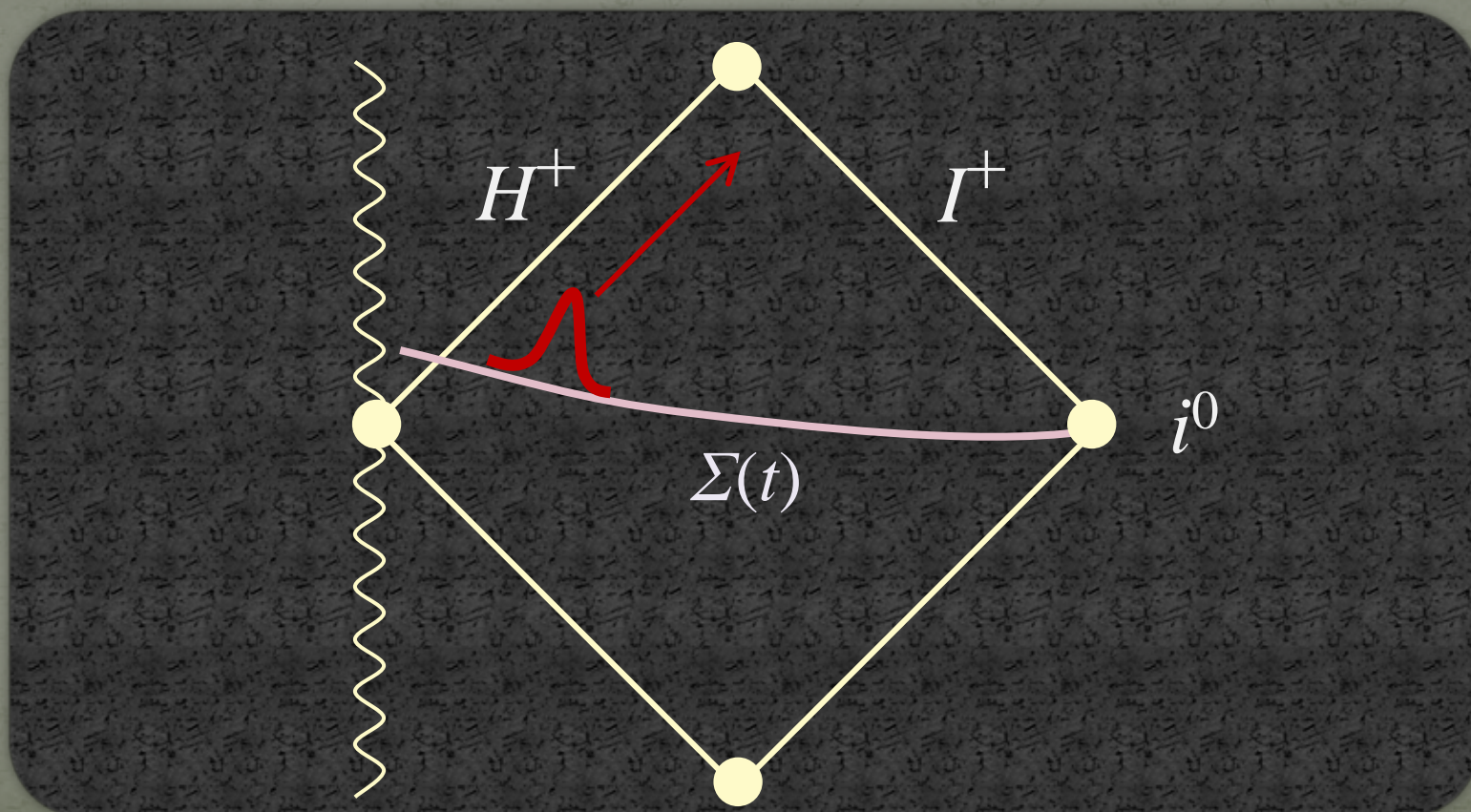
- Many others...

2011: Instability of extreme horizons

- Non-extreme BHs are stable
- **Extreme** BH?
 - ü $\kappa = 0$ à New phenomena on the horizon
- Aretakis' results:
 - ü $\psi(t, r)$ and its derivatives **decay around BH**
 - ü **Non-decay & blow-up on horizon**

2011: Instability of extreme horizons

- Aretakis' argument:
 - Massless scalar on extreme Reissner-Nordström



2011: Instability of extreme horizons

- Aretakis' argument: [Aretakis 2010, 2011]
- Massless scalar on extreme Reissner-Nordström

Main theorems:

- Pointwise decay:

$$- |\psi(v,r)| < C[\psi(v_0)] \cdot v^{-3/5}$$

$$- |\partial_r^{n \leq \ell} \psi(v,r)| < C[\psi(v_0)] \cdot v^{-1/4}$$

- Non-decay & Blow-up on horizon:

$$- |\partial_r^{\ell+1} \psi(v,r)| \rightarrow H_\ell[\psi(v_0)]$$

$$- |\partial_r^{\ell+k} \psi(v,r)| > H_\ell[\psi(v_0)] \cdot v^{k-1} \quad (k \geq 2)$$

2011: Instability of extreme horizons

– Aretakis' argument:

ü \$ Conserved quantities on extreme horizon

$$ds^2 = - \left(1 - \frac{M}{r} \right)^2 dv^2 + 2dvdr + r^2 d\Omega^2$$

$$\square\psi = 0 \quad \text{w/} \quad \psi(v, r, \Omega) = \sum_{\ell=0}^{\infty} \psi_{\ell}(v, r) Y_{\ell}(\Omega)$$

$$2r\partial_v\partial_r(r\psi_{\ell}) + \partial_r(\Delta\partial_r\psi_{\ell}) - \ell(\ell+1)\psi_{\ell} = 0$$

è

$$\left[\Delta \equiv (r - M)^2 \right]$$

2011: Instability of extreme horizons

– Aretakis' argument:

ü \$ Conserved quantities on extreme horizon

$$r = M \quad \& \quad \ell = 0$$

$$\text{P} \quad \partial_v \partial_r (r \psi_0) \Big|_{r=M} = 0$$

$$\text{P} \quad \partial_r (r \psi_0) \Big|_{r=M} = H_0[\psi] = \text{const.}$$

$$2r \partial_v \partial_r (r \psi_\ell) + \partial_r (\Delta \partial_r \psi_\ell) - \ell(\ell + 1) \psi_\ell = 0$$

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2011: Instability of extreme horizons

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$$\partial_r \text{ à } r = M \text{ \& } \ell = 0$$

$$\text{P } \partial_v \partial_r^2 (r\psi_0) + \partial_r \psi_0 \Big|_{r=M} = 0$$

$$\text{P } \partial_r^2 (r\psi_0) \Big|_{r=M} \rightarrow -H_0 v$$

$$2r \partial_v \partial_r (r\psi_\ell) + \partial_r (\Delta \partial_r \psi_\ell) - \ell(\ell + 1)\psi_\ell = 0$$

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2011: Instability of extreme horizons

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For " ℓ ,

$$\partial_r^\ell [r \partial_r (r \psi_\ell)]|_{r=M} = H_\ell[\psi] = \text{const.}$$

$$\text{p} \quad \partial_r^{\ell+k} \psi_\ell \propto v^{k-1}$$

$$2r \partial_v \partial_r (r \psi_\ell) + \partial_r (\Delta \partial_r \psi_\ell) - \ell(\ell + 1) \psi_\ell = 0$$

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2011: Instability of extreme horizons

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- Massless scalar on extreme Reissner-Nordström

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- Non-decay & Blow-up on horizon:

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2012: Generalizations

- Aretakis' argument: [Aretakis 2011, 2012]
 - \$ Conserved quantities on extreme horizons
 - Massless scalar on extreme RN & Kerr
- Instability is ubiquitous
 - Gravitational perturbations on extreme Kerr [Lucietti & Reall 2012]
 - Grav. perturbations on high-D extreme horizons [Murata 2012]
 - $H_\ell = 0$ case
 - Massive scalar
 - Coupled EM & Grav. perturbations [Lucietti, Murata, Reall, NT 2012]

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2012: Generalizations

– Main results:

[Lucietti, Murata, Reall, NT 2012]

Instability for

1. $H_\ell = 0$
2. Massive scalar
3. Coupled EM & Grav. perturbations

on extreme RN background

2012: Generalizations

1. Instability for $H_\ell = 0$

- Horizon instability μH_ℓ
- What if $H_\ell = 0$?
 - a. No Instability
 - b. \$ Instability, same growth rate
 - c. \$ Instability, slower growth rate
- Confirm numerically

2012: Generalizations

1. Instability for $H_\ell = 0$

- Horizon instability μH_ℓ

- What if $H_\ell = 0$?

- a. No Instability

- b. \$ Instability, same growth rate

- c. \$ Instability, slower growth rate**

- Confirm numerically

2012: Generalizations

1. Instability for $H_\ell = 0$

– Numerics:

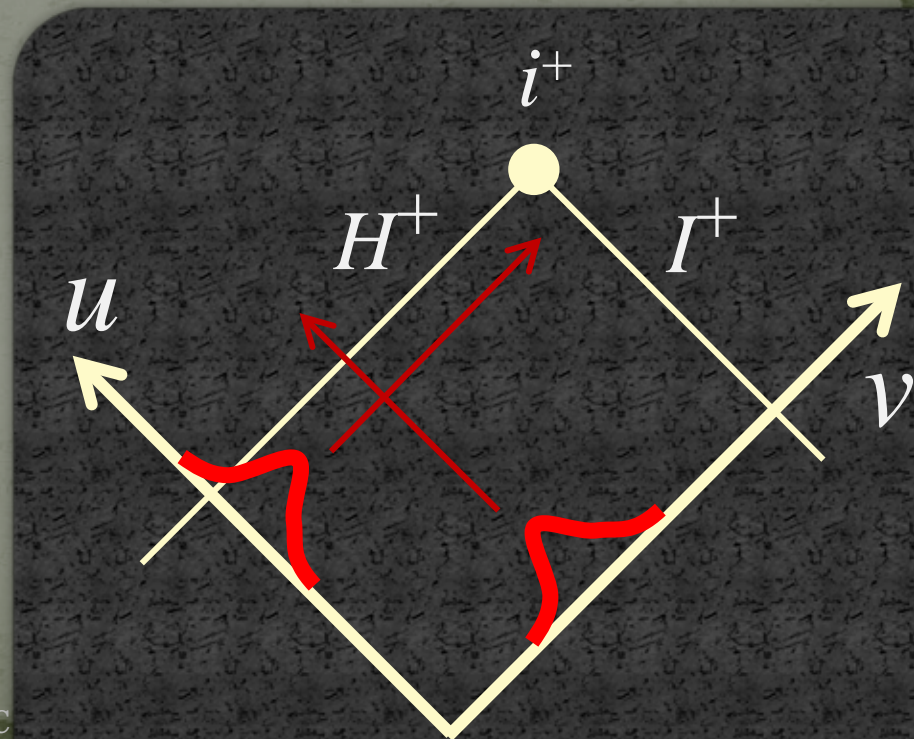
• Massless scalar field on extreme RN

• Double-null coordinates

$$ds^2 = -F(u, v)dudv + r^2 d\Omega^2$$

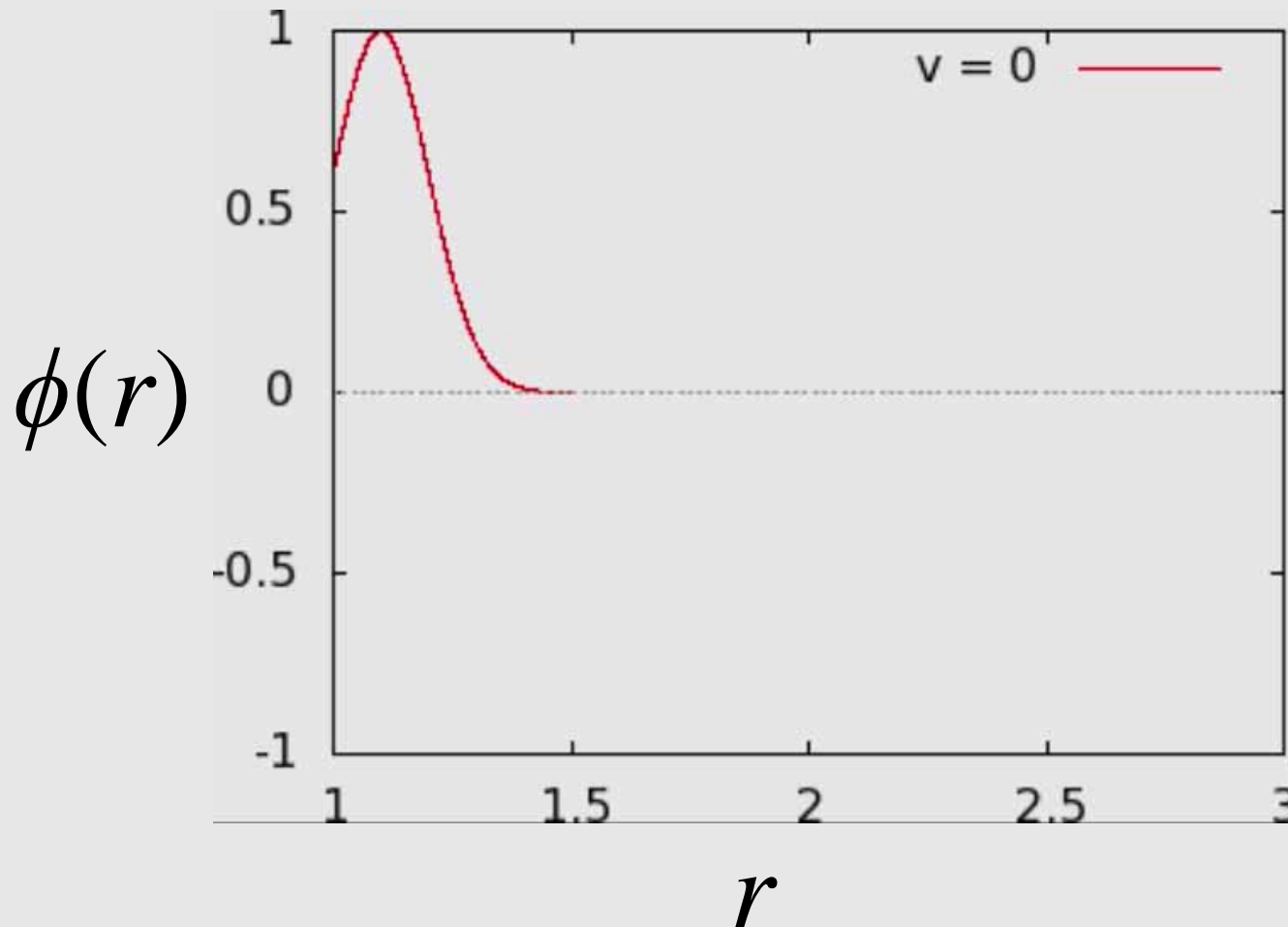
$$-\partial_u \partial_v \phi = V(u, v)\phi$$

$$[\phi \equiv r\psi]$$



2012: Generalizations

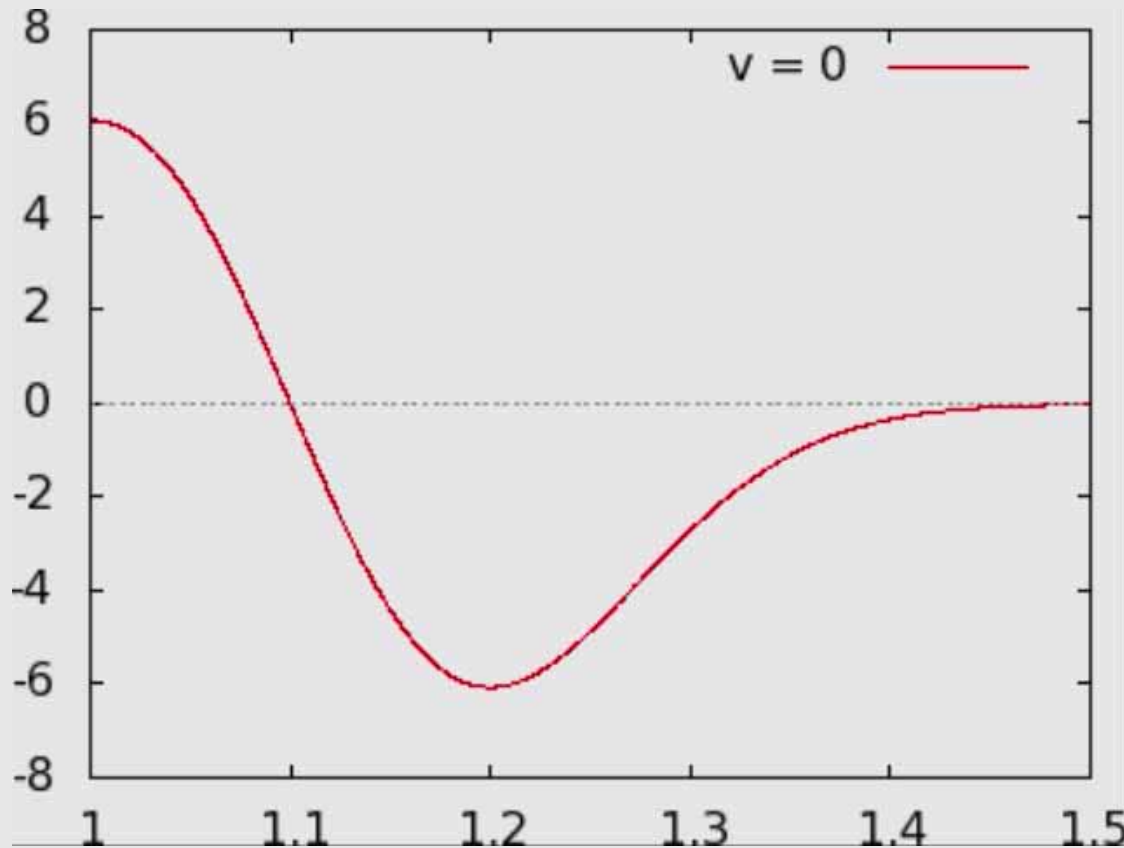
– $H_{\ell=0} \neq 0$



2012: Generalizations

– $H_{\ell=0} \neq 0$

$$\partial_r \phi(r)$$

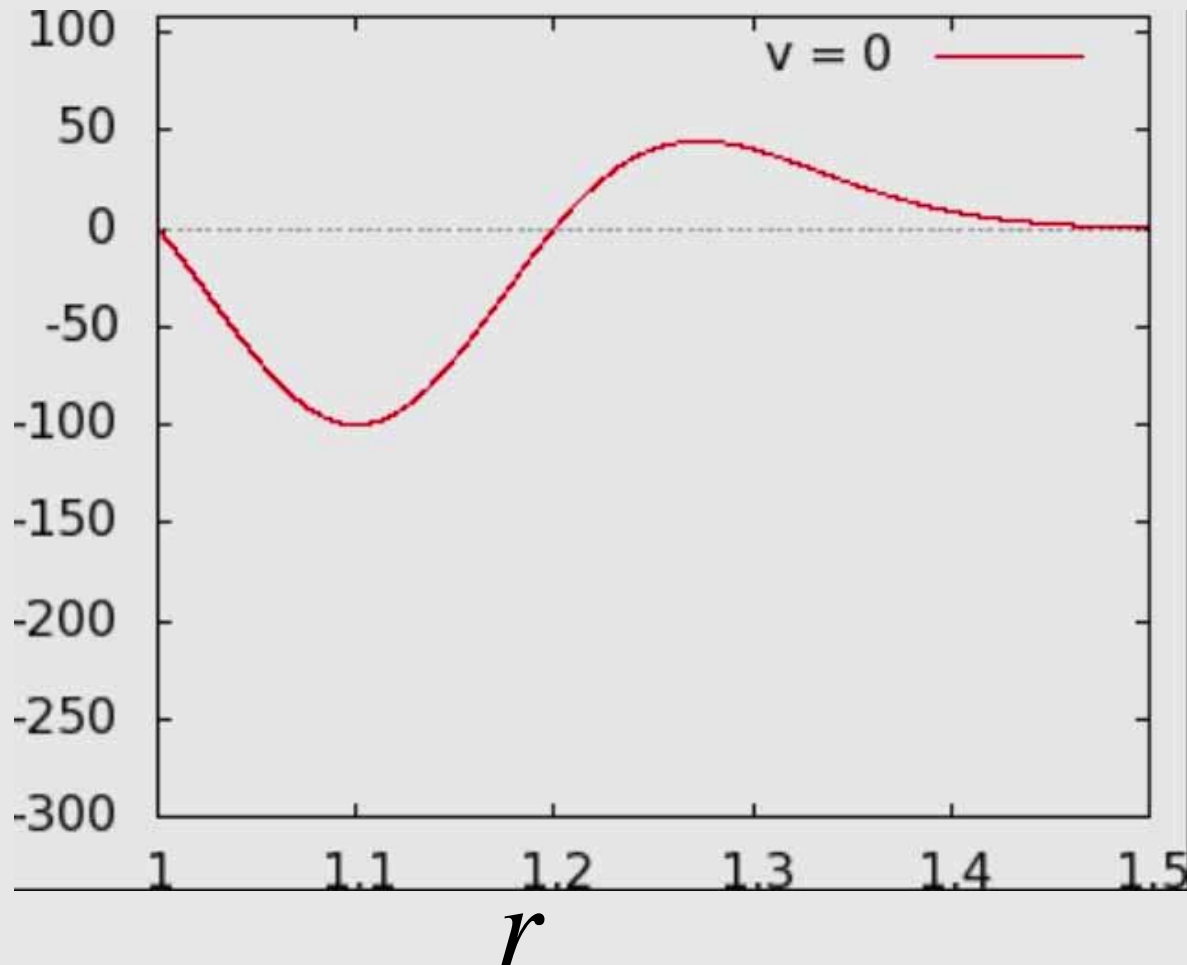


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2012: Generalizations

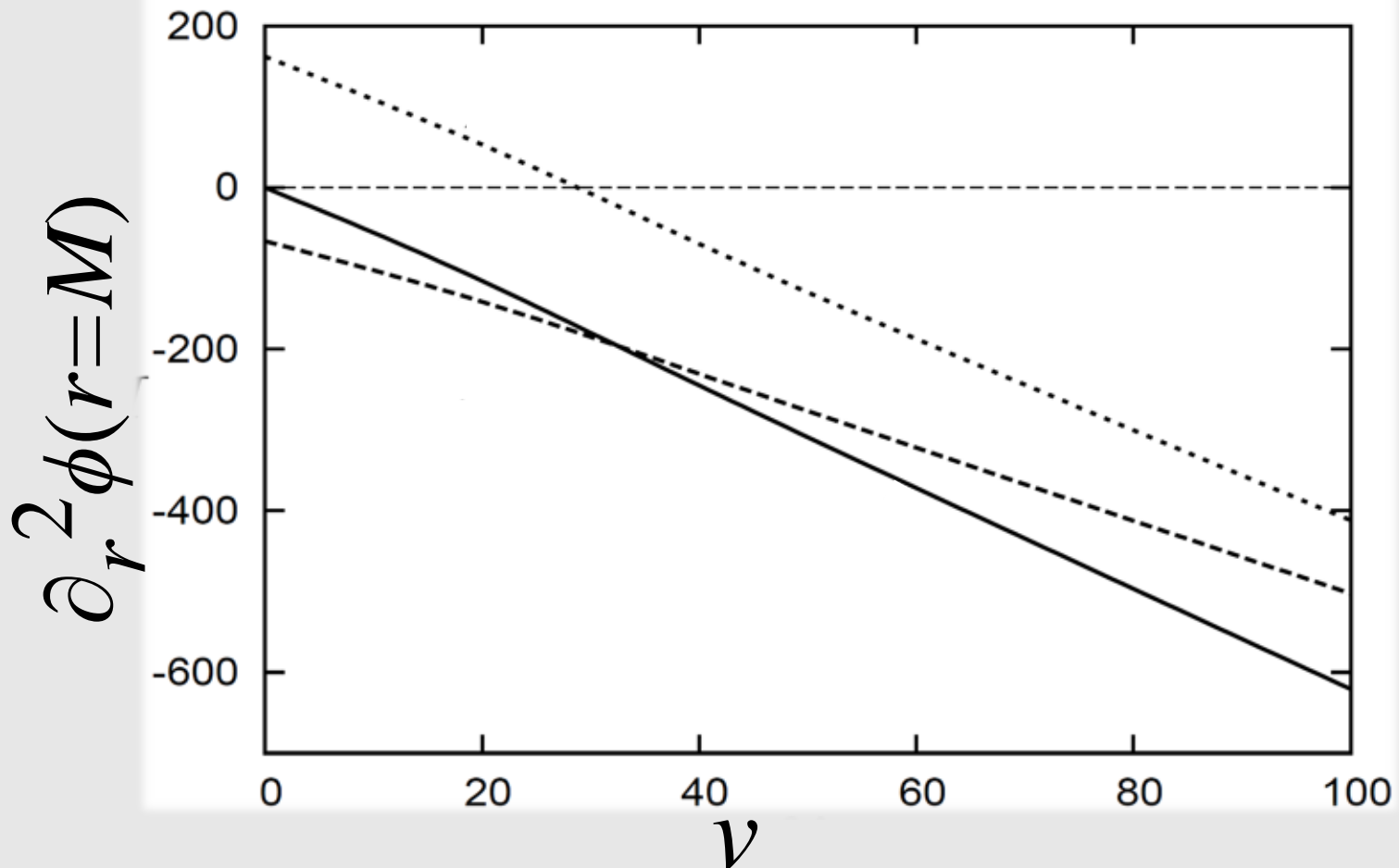
– $H_{\ell=0} \neq 0$

$\partial_r^2 \phi(r)$



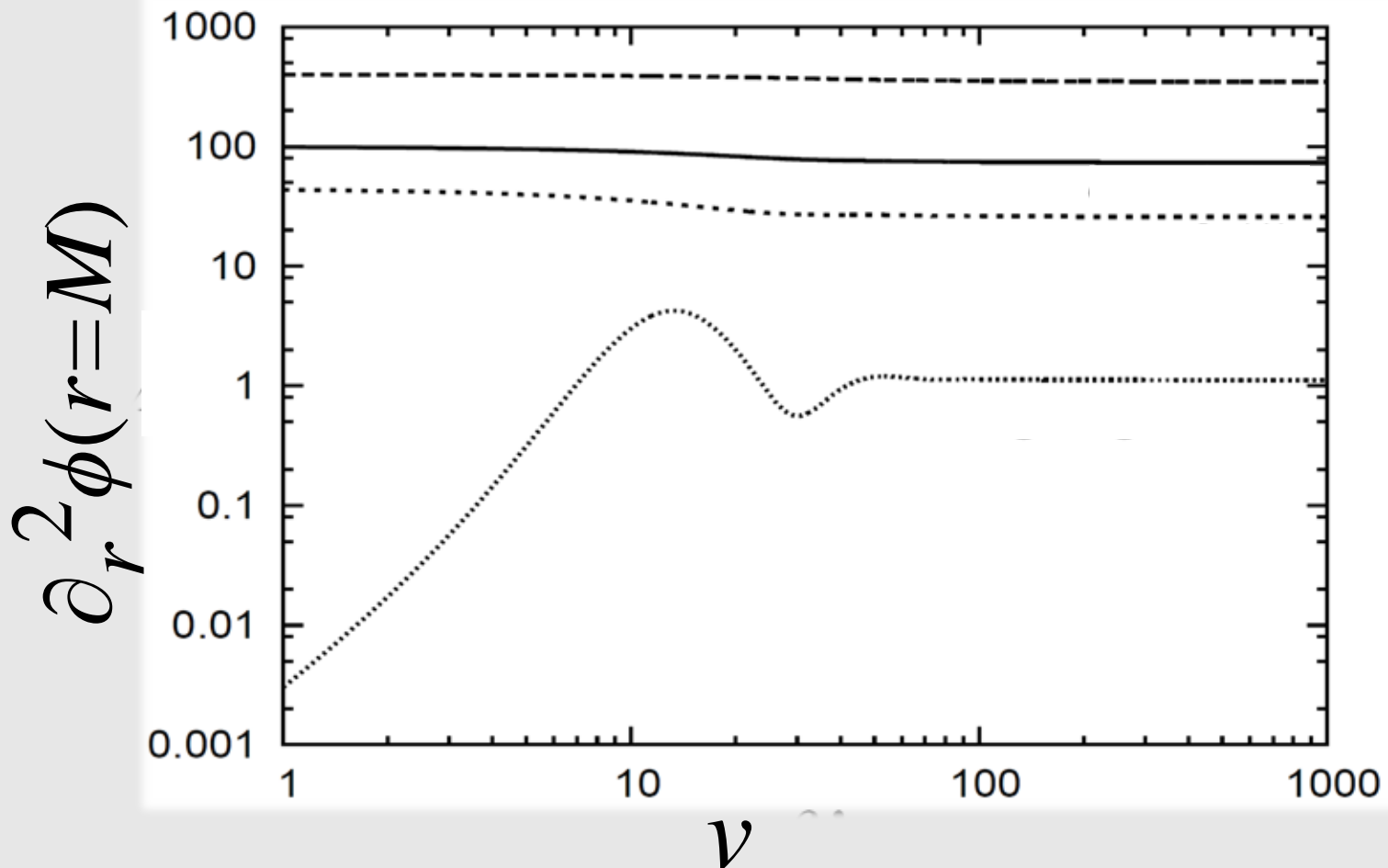
2012: Generalizations

– $H_{\ell=0} \neq 0: \partial_r^2 \phi_0|_{r=M} \rightarrow -H_0 v$



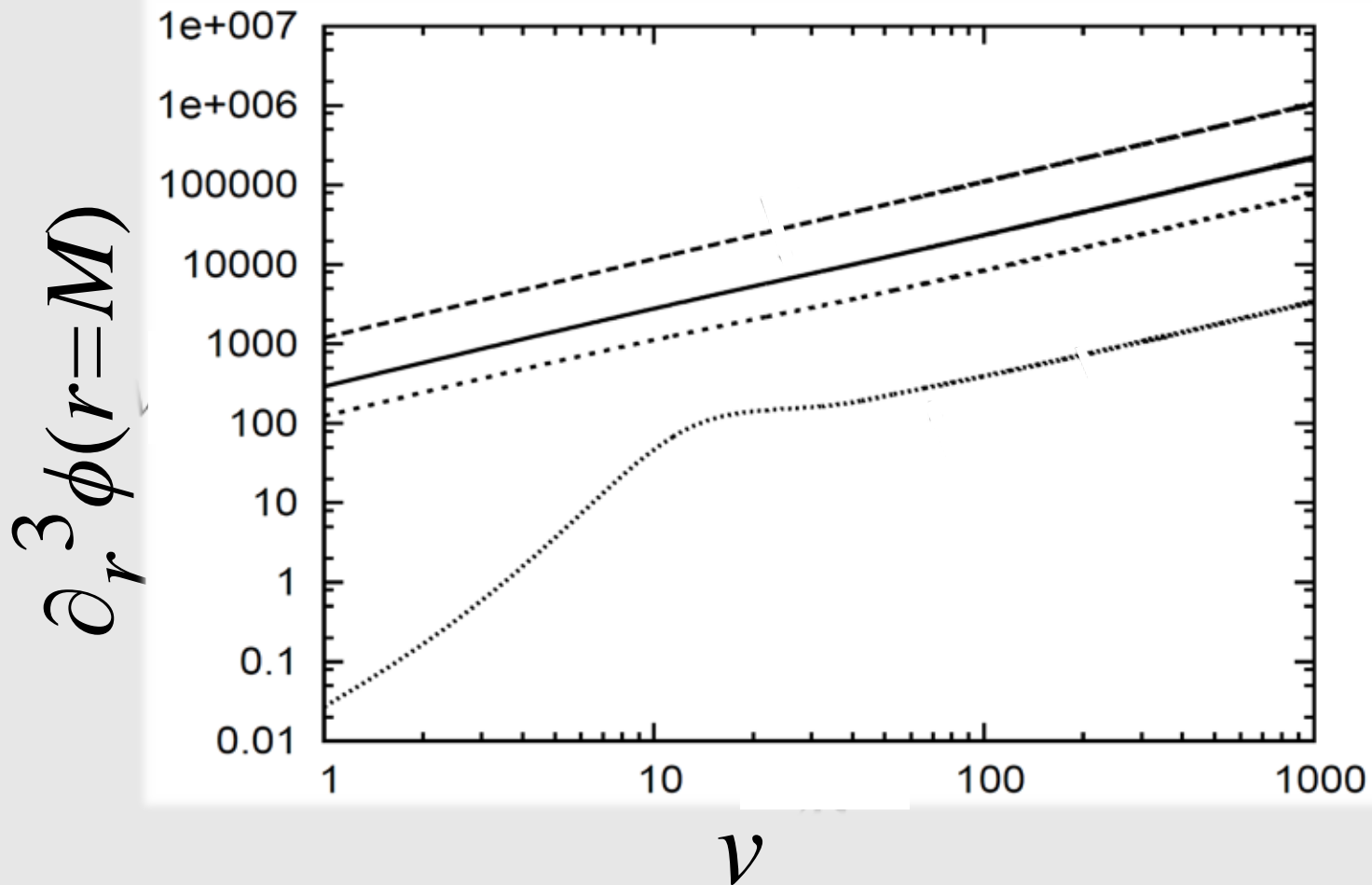
2012: Generalizations

– $H_{\ell=0} = 0$: $\partial_r^2 \phi_0|_{r=M} \rightarrow C v^0$



2012: Generalizations

– $H_{\ell=0} = 0$: $\partial_r^3 \phi_0|_{r=M} \rightarrow C' v^1$



2012: Generalizations

1. Instability for $H_\ell = 0$

– Numerical results:

– $H_{\ell=0} \neq 0$:

$$\phi \Big|_{r=M} \rightarrow -\frac{2H_0}{v}$$

– $H_{\ell=0} = 0$:

$$\phi \Big|_{r=M} \rightarrow \frac{C}{v^2}$$

[Match with Aretakis' proof (2012)]

2012: Generalizations

2. Instability for massive scalar

$$2r\partial_v\partial_r\phi + \partial_r\left(\Delta\partial_r\frac{\phi}{r}\right) - \left(\frac{\ell(\ell+1)}{r^2} + m^2\right)r\phi = 0$$

– Conserved quantity?

i. Set $m^2 = n(n+1)M^{-2}$

ii. Apply $\partial_r^n f_n(r)$, then set $r = M$

$$f_n(r) = \sum_{k=0}^n c_k \left(\frac{r}{M} - 1\right)^k$$

$$\partial_v\partial_r^n [f_n(r)\partial_r\phi] + \sum_{k=0}^n a_k \partial_r^k \phi \Big|_{r=M} = 0$$

2012: Generalizations

iii. Find $c_{k=1,\dots,n}$ to make $a_{k=1,\dots,n} = 0$

è $H_n[\phi] = \partial_r^n [f_n(r) \partial_r \phi] \Big|_{r=M}$ conserved

i. Set $m^2 = n(n+1)M^{-2}$

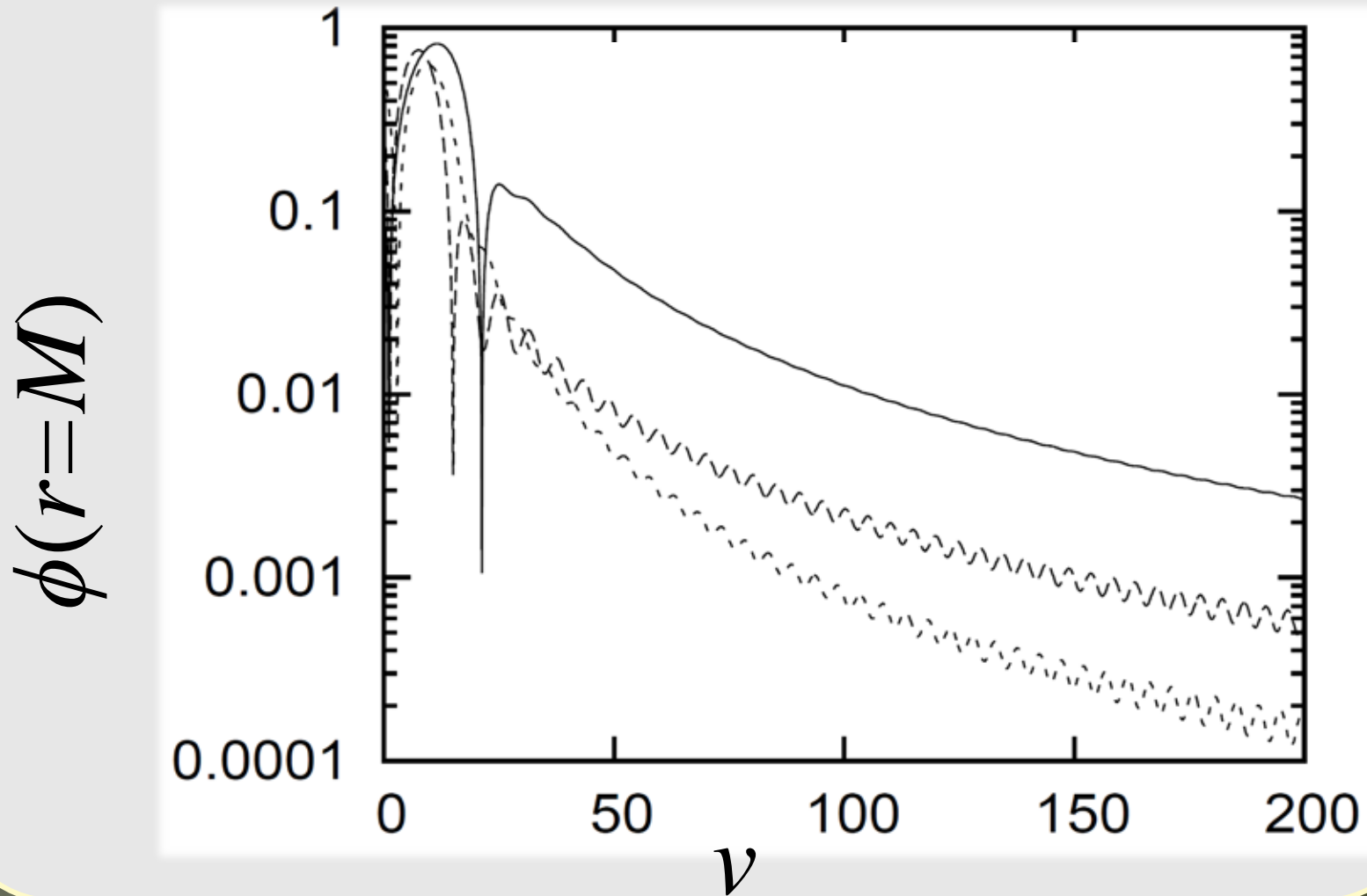
ii. Apply $\partial_r^n f_n(r)$, then set $r = M$

$$f_n(r) = \sum_{k=0}^n c_k \left(\frac{r}{M} - 1 \right)^k$$

$$\text{è } \partial_v \partial_r^n [f_n(r) \partial_r \phi] + \sum_{k=0}^n a_k \partial_r^k \phi \Big|_{r=M} = 0$$

2012: Generalizations

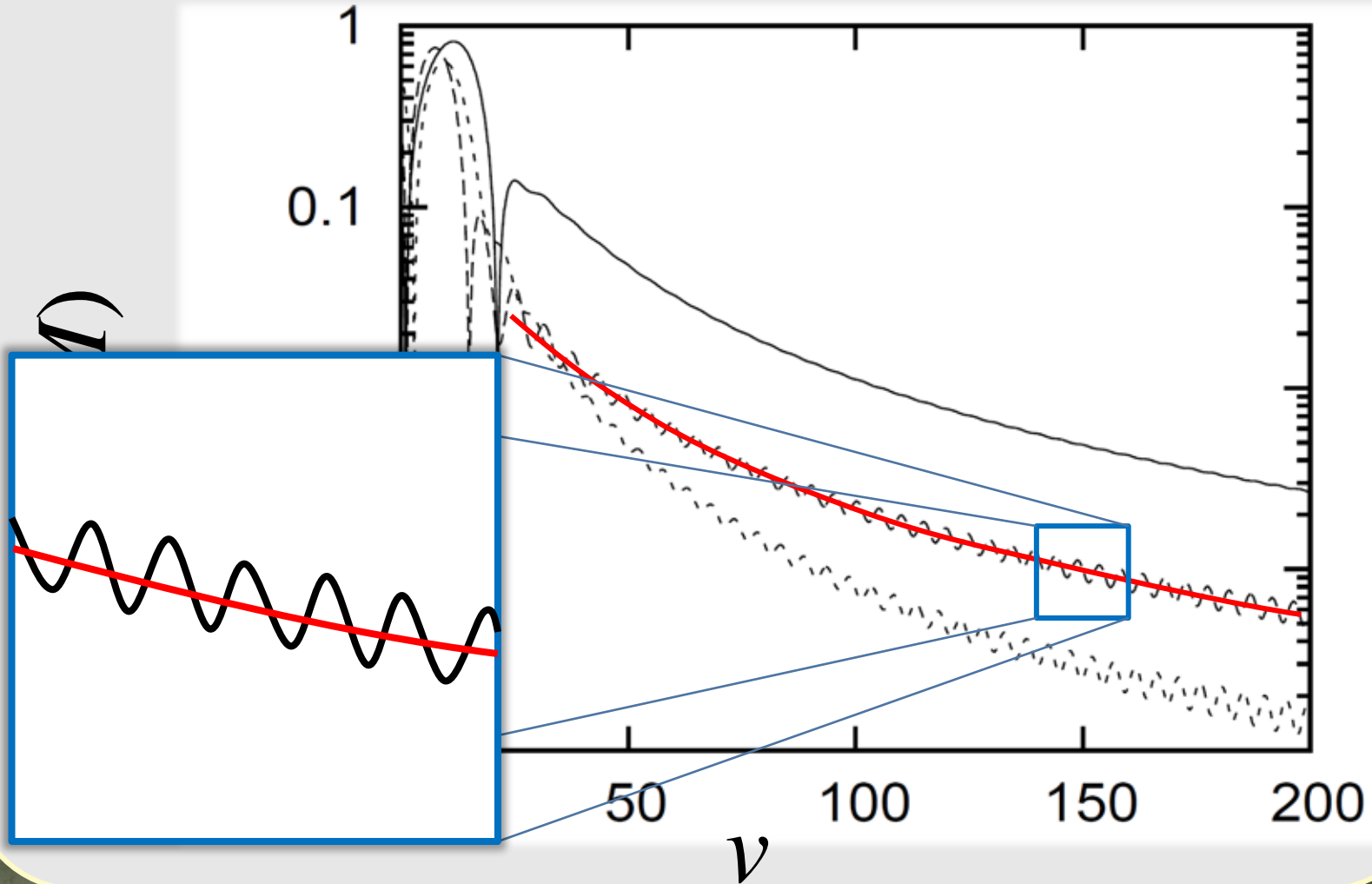
– $m^2 = 2$



2012: Generalizations


– $m^2 = 2$

Decay + Oscillation



2012: Generalizations

$$\phi|_{r=M}(v) = \phi^{\text{mean}}(v) + \phi^{\text{osci}}(v)|_{r=M}$$


$$\left[\begin{aligned} \phi^{\text{mean}}(v)|_{r=M} &\equiv T^{-1} \int_{v-T/2}^{v+T/2} \phi|_{r=M}(v') dv' \\ T &= m/2\pi \end{aligned} \right]$$

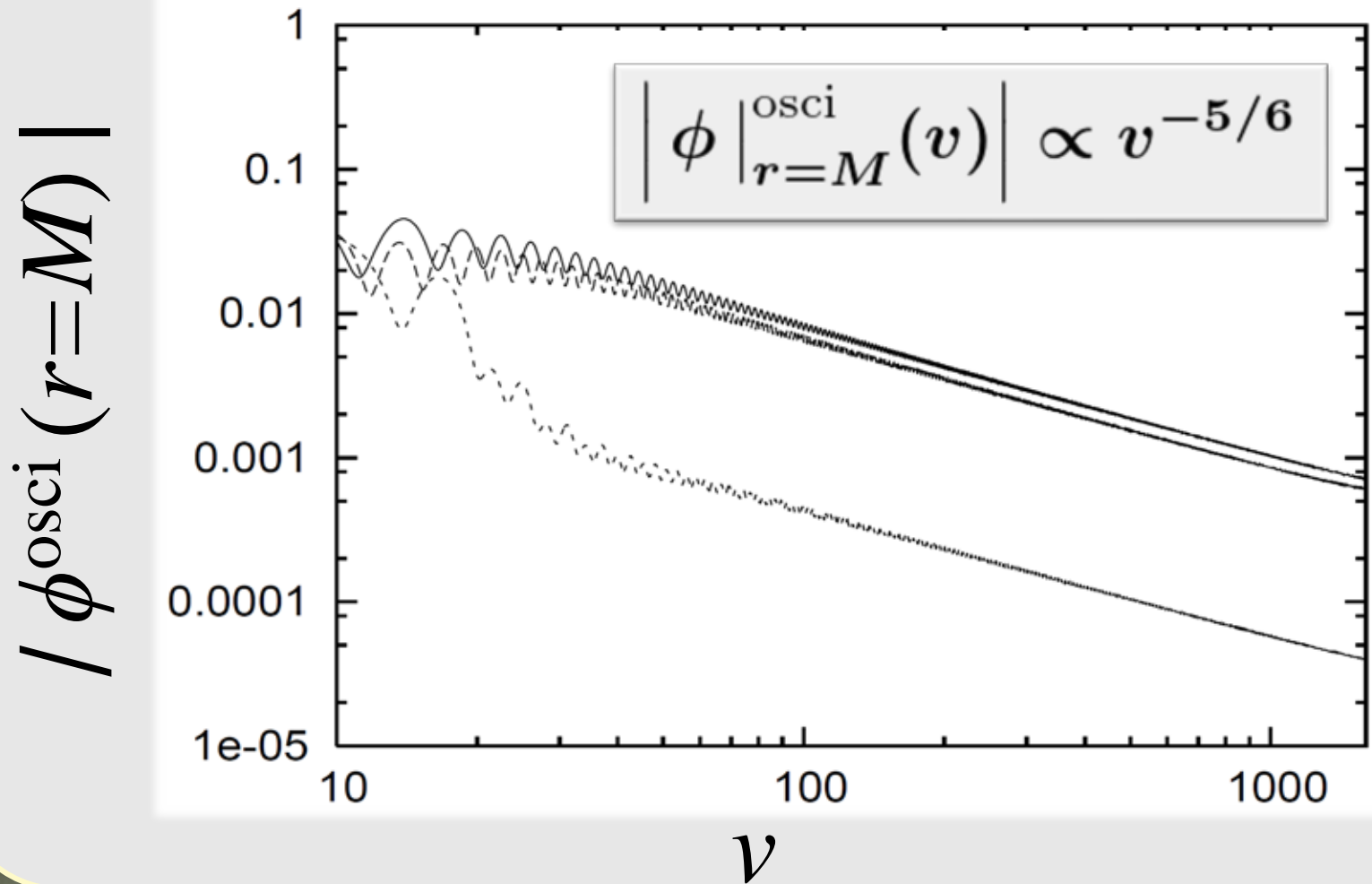
- Damped oscillation on RN background:

$$\left| \phi|_{r=M}^{\text{osci}}(v) \right| \propto v^{-5/6}$$

[Koyama & Tomimatsu 2000]

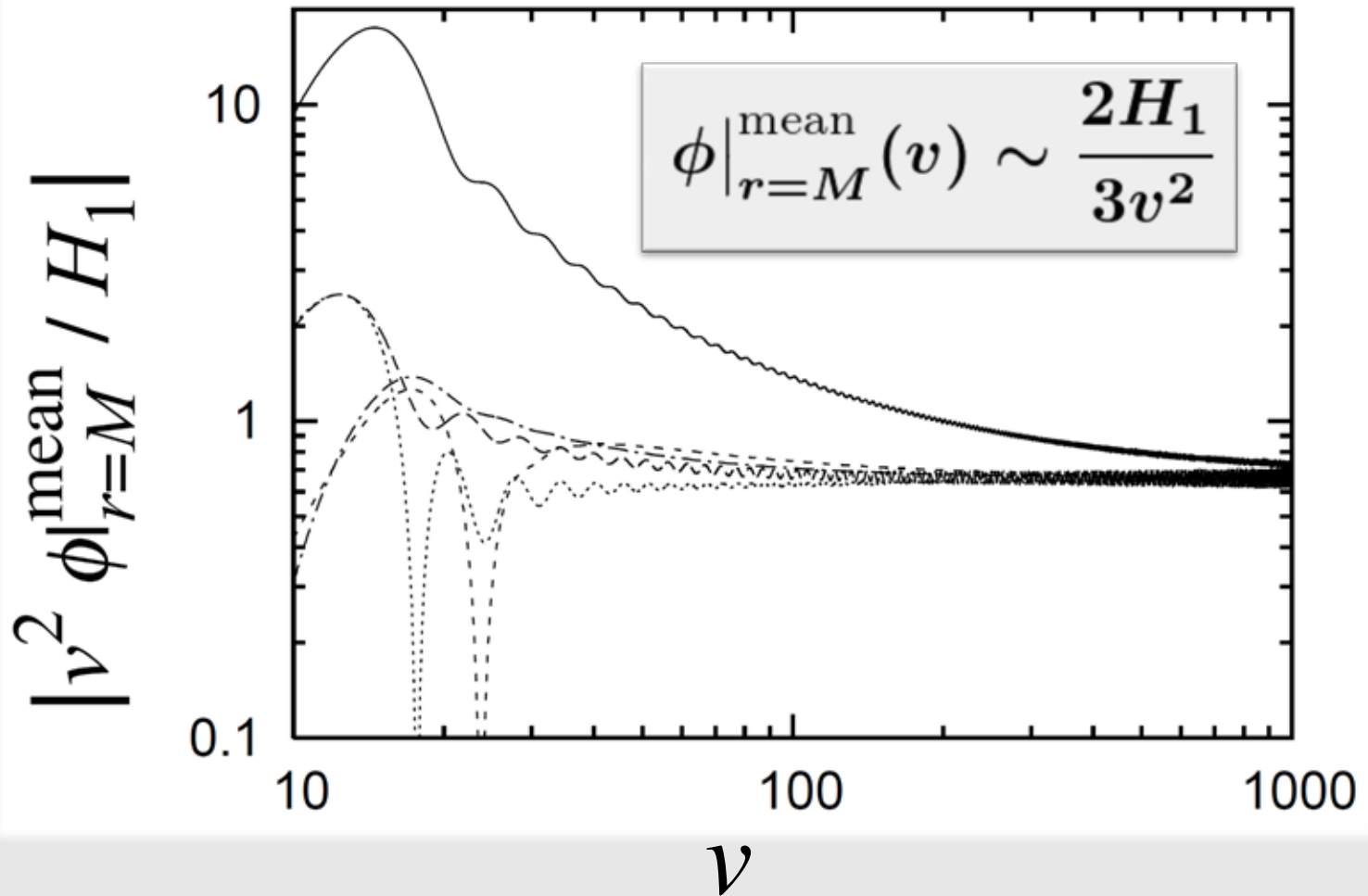
2012: Generalizations

– $m^2 = 2$



2012: Generalizations

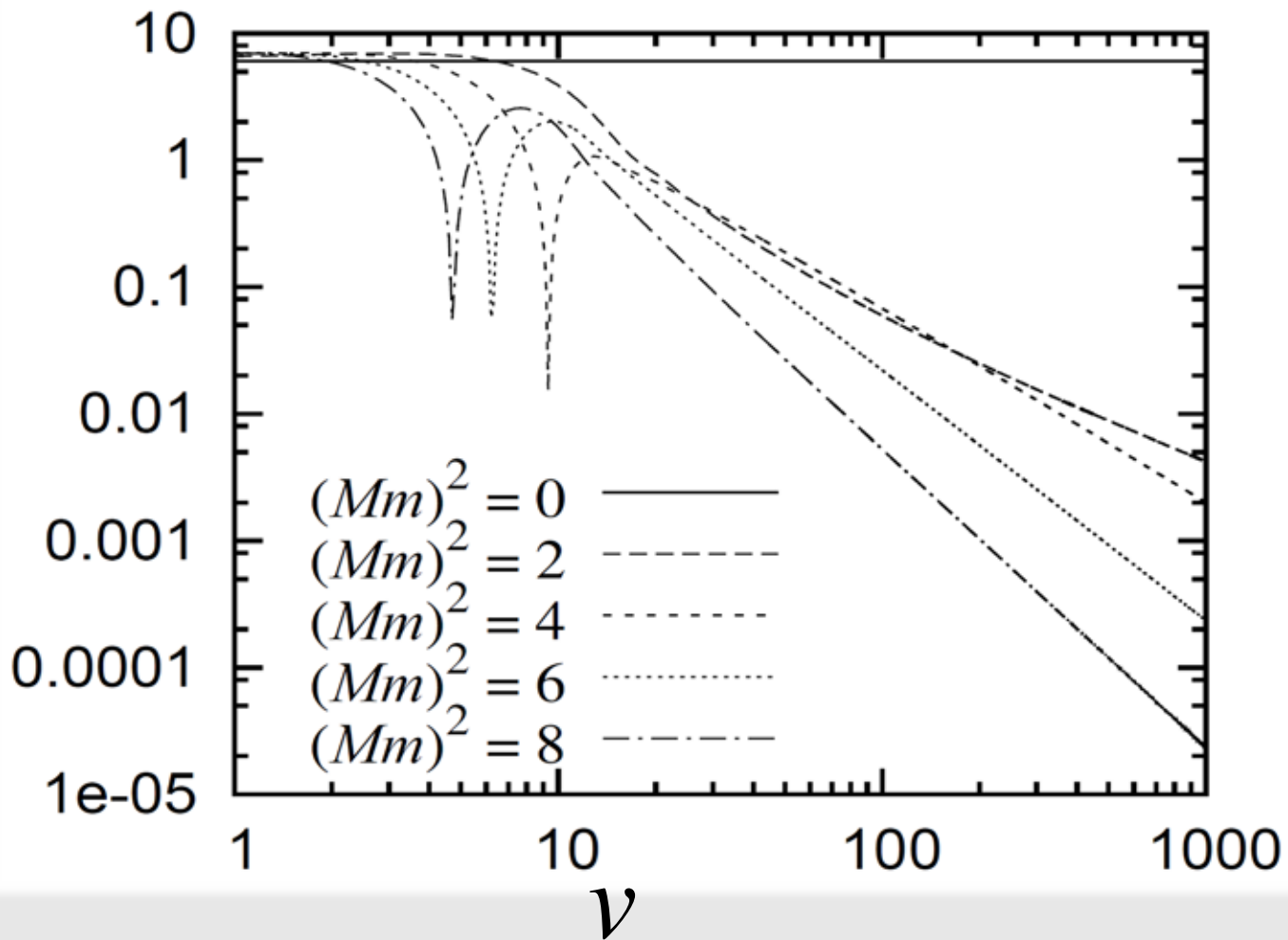
– $m^2 = 2$



2012: Generalizations

- General $m^2 = n(n+1)$: $\partial_r^k \phi \mu \nu^{-(n+1)+k}$

$$H_0(\nu) \sim \partial_r \phi(\nu)$$



2012: Generalizations

3. Instability for coupled EM & Grav. Perturbations

Einstein + Maxwell + ~~Scalar~~

– Moncrief's gauge invariant equations

[Moncrief 1974]

i. Hamiltonian formulation

$$H = \int d^4x \left(\pi^{ij} g_{ij,t} + A_i E_{,t}^i - N C - N_i C^i - A_0 E_{,i}^i \right)$$

ii. Expand $O(\varepsilon^2)$

iii. Take $\delta(\text{constraint})$ as canonical variables

$$(\delta q_i, \delta \pi_i), \quad (\delta C_i, \delta \pi_i^C)$$

$$\{\delta q_i, \delta C_j\} = 0, \quad \{\delta \pi_i, \delta C_j\} = 0, \quad \{\delta C_j, \delta \pi_i^C\} = \delta_{ij}$$

2012: Generalizations

3. Instability for coupled EM & Grav. Perturbations

$$2\partial_v\partial_r\psi + \partial_r(r^{-2}\Delta\partial_r\psi) = r^{-2}W(r)\psi$$

Ex.) $\ell = 1$, odd parity on extreme RN

$$\psi = \left(1 - \frac{2M}{3r}\right)\pi_f + \frac{r}{6}p_1 \quad W(r) = 2\left(1 + \frac{2M^2}{r^2}\right)$$

$$\left(\begin{array}{l} \pi_f = \frac{1}{2} \int_{S^2} d\Omega (\hat{e}^i)^* \delta A_i \\ p_1 = \int_{S^2} d\Omega (\hat{e}^i)^* \left(\partial_v \delta g_{ri} - \partial_r \delta g_{vi} + \frac{2}{r} \delta g_{vi} \right) \end{array} \right)$$

2012: Generalizations

3. Instability for coupled EM & Grav. Perturbations

$$2\partial_v\partial_r\psi + \partial_r(r^{-2}\Delta\partial_r\psi) = r^{-2}W(r)\psi$$

$$\text{p} \quad H_p[\psi] \equiv \partial_r^p(r^2 f \partial_r \psi) \Big|_{r=M} = \text{const.}$$

$$\text{p} \quad \left\{ \begin{array}{l} \partial_r^4 \psi \propto H_2 v \quad (\ell = 1) \\ \partial_r^{\ell+1\pm 1} \psi_{\pm} \propto H_{\ell\pm 1} v \quad (\ell \geq 2) \end{array} \right.$$

2013: Backreaction

- ≤ 2012 :
 - Linear perturbations
 - Non-decay & Blow-up on extreme horizon
- Backreaction?
 - ∅ Non-decay & Blow-up \rightarrow Singularity etc?
 - ∅ Suppression by nonlinearity \rightarrow non-extreme RN?
- Clarify by solving the full Einstein equations

2013: Backreaction

- Gravity + Maxwell + Scalar:

$$ds^2 = -f(u, v)dudv + r(u, v)^2 d\Omega^2$$

$$\phi = \phi(u, v) \quad A = \alpha(u, v)dv$$

- EoMs:

$$f_{,uv} = \frac{1}{2r^4 f} (2r^4 f_{,u} f_{,v} + r^2 f^3 + 4r^2 f^2 r_{,u} r_{,v} - 2Q^2 f^3 - r^4 f^2 \phi_{,u} \phi_{,v})$$

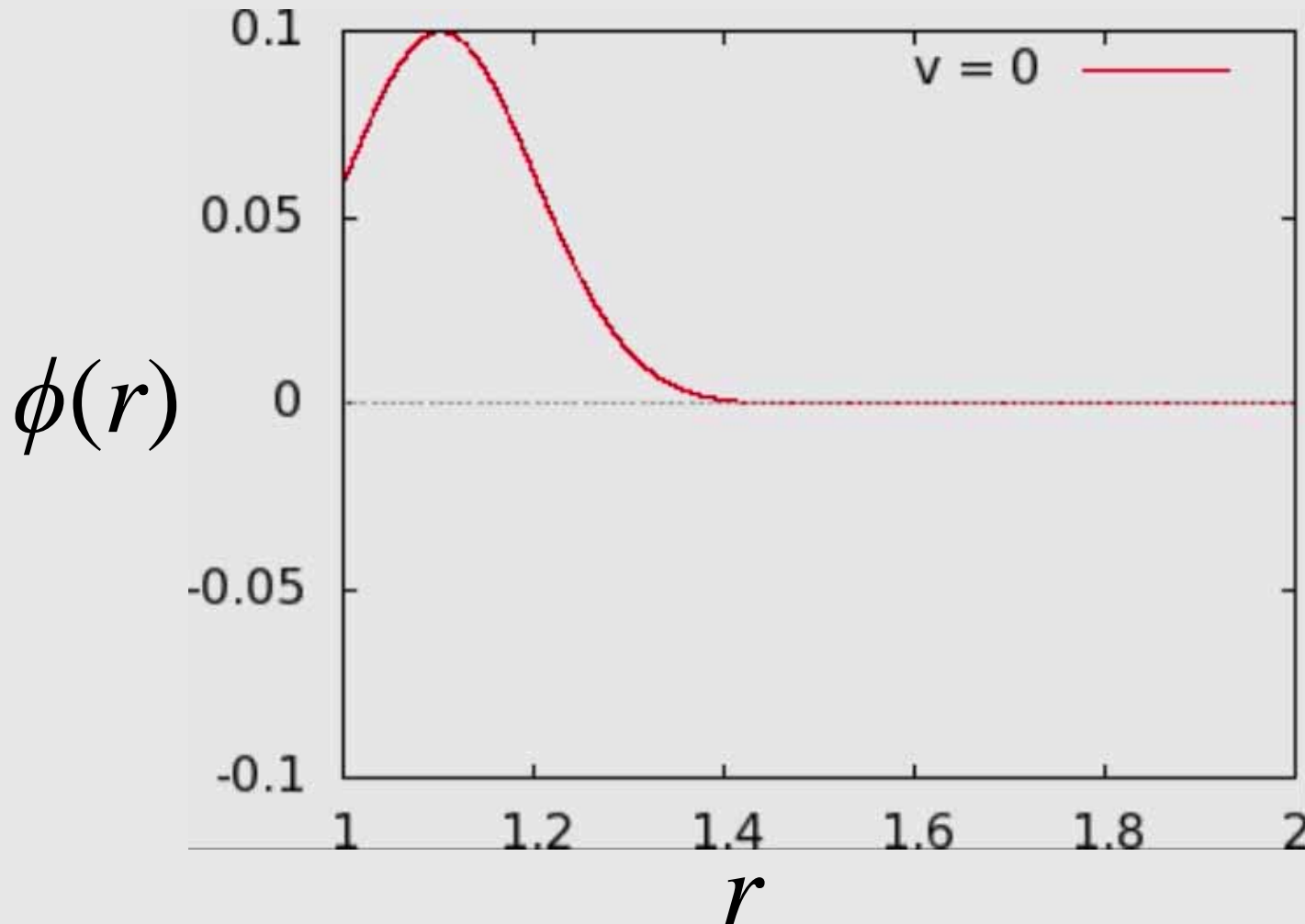
$$r_{,uv} = -\frac{1}{16r^3} (16r^2 r_{,u} r_{,v} + 4r^2 f - 4Q^2 f - m^2 r^4 f \phi^2)$$

$$\phi_{,uv} = -\frac{1}{4r} (4r_{,u} \phi_{,v} + 4r_{,v} \phi_{,u} + m^2 r f \phi) \quad \alpha_{,u} = \frac{Qf}{r^2}$$

- Initial data = Extreme RN + Small wave packet

2013: Backreaction

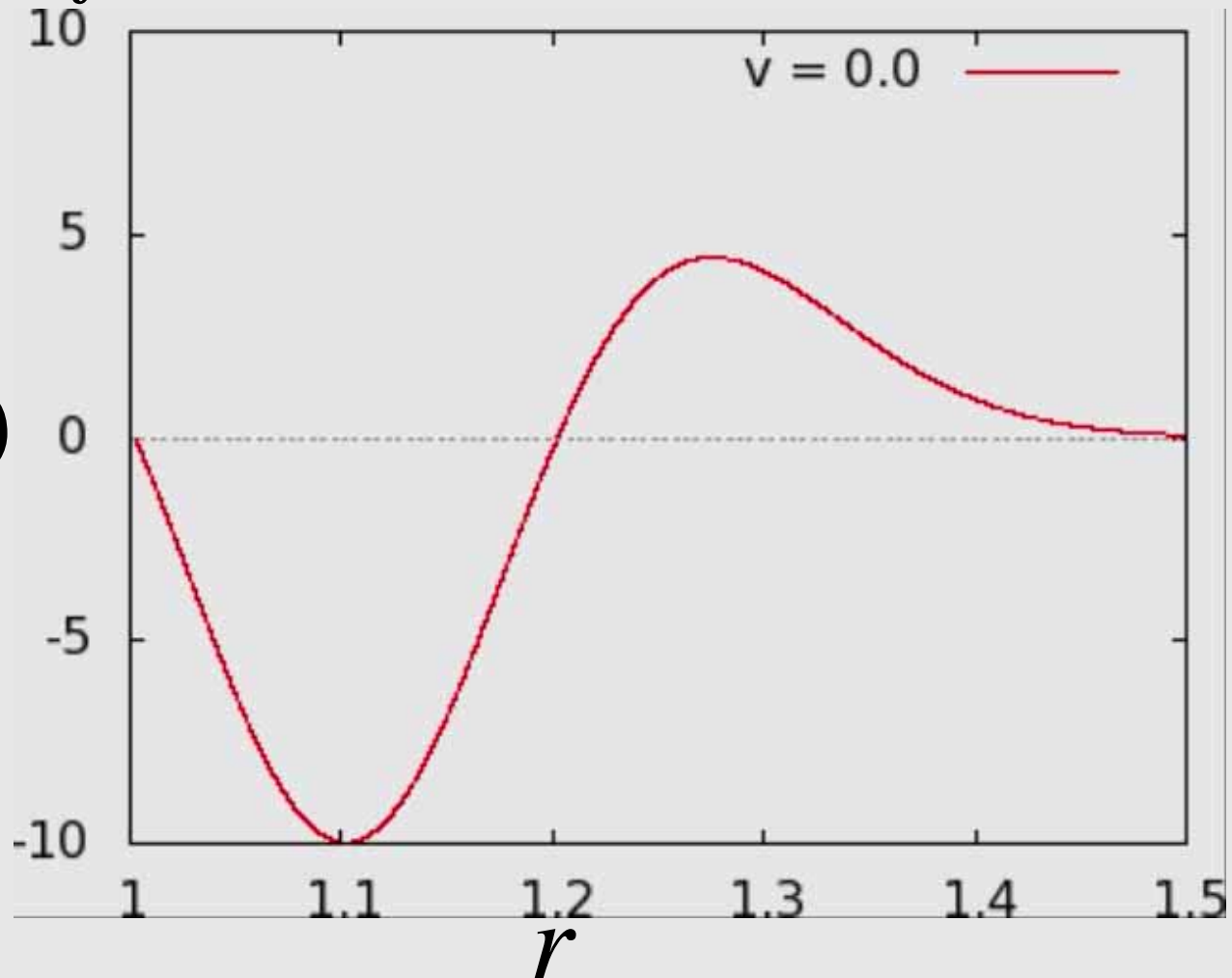
- Preliminary results:



2013: Backreaction

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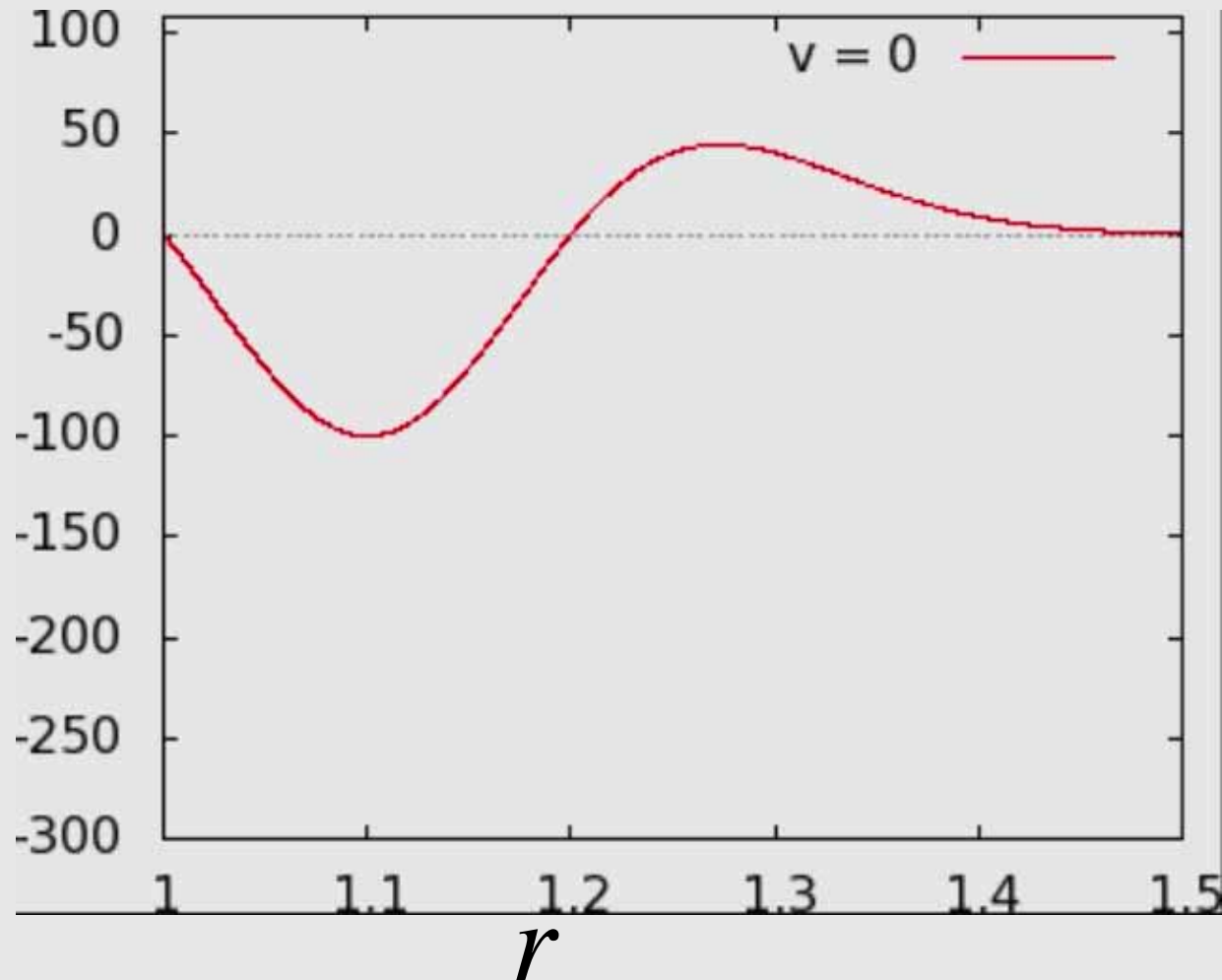
$$\partial_r^2 \phi(r)$$



2013: Backreaction

– *No backreaction*

$$\partial_r^2 \phi(r)$$



Summary

1957 – 2010	BH Stability & Price's law
2011	Instability of extreme horizons
2012	Generalizations
2013	Backreaction

- **Linear Instability** on extreme horizons is universal
- **Backreaction:**
Linear blow-up \rightarrow Non-linear suppression \rightarrow Non-extreme
- Any physical applications?

– Conformal isometry

[Bizon & Friedrich 2012]

[Lucietti, Murata, Reall, NT 2012]

- Extreme RN has **discrete conformal isometry**

$$\Phi : (t, r, \theta, \phi) \rightarrow (t, r' = M + \frac{M^2}{r - M}, \theta, \phi)$$

$$\Phi_*(g) \rightarrow \left(\frac{M}{r - M} \right)^{-2} g$$

$$\tilde{\psi}(u, r, \theta, \phi) = \frac{M}{r - M} \psi(v' = u, r', \theta, \phi)$$

- **Newman–Penrose constants on Γ^+**

$$\psi_0 \sim \frac{f_0(u)}{r} + \frac{C_0}{r^2} + \dots$$

- **NP constants « Aretakis constants by Φ**

– AdS_2 analysis

[Lucietti, Murata, Reall, NT 2012]

– Extremal \rightarrow Near-horizon geometry = AdS_2

– Massless scalar on AdS_2

$$\psi \rightarrow -\frac{2H_0}{v^2}$$

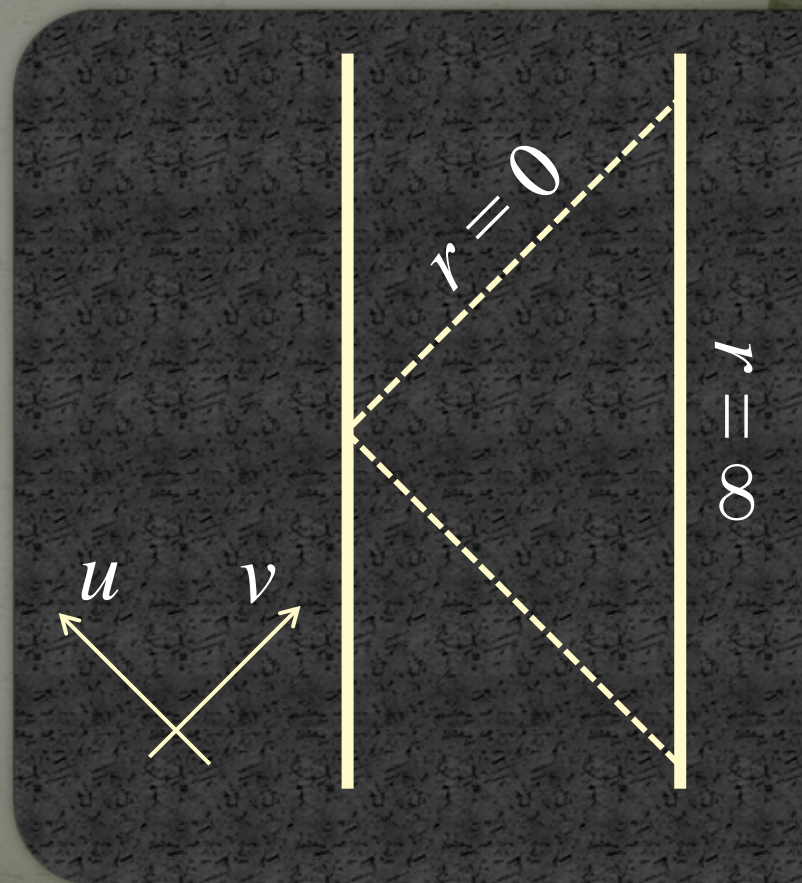
– Massive scalar

$$\psi \rightarrow r^{\Delta_m} F(v)$$

$$\Delta_m = \frac{1}{2} + \sqrt{m^2 + \frac{1}{4}}$$

$$\text{è } \psi \sim v^{-(n+1)}$$

$$\text{for } m^2 = n(n+1)$$



2013: Backreaction

– Preliminary results:

$$\partial_r \phi(r)$$

