Renormalization of Entanglement Entropy and the Gravitational Effective Action

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S_{ent} quantifies the information that one disregards by ignoring a subsystem.

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How is the system $\mathscr{S}$ partitioned into the two subsystems $\mathscr{S}_A$ and $\mathscr{S}_B$?

- Choose a spatial boundary at a fixed time, the entangling surface.
  - S_{ent} measures correlations across the boundary.
  - If ρ is a pure quantum state, then S_{ent} is associated with the boundary.

Why study entanglement entropy?

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Quantum information science

- Entanglement is the primary resource for quantum information processing and quantum computation.

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Black hole physics

- Entanglement entropy may account for the entropy of black holes.

  [Sorkin 1983], [Bombelli *et al* 1986], [Srednicki 1993], [Frolov and Novikov 1993]

Black hole thermodynamics

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Classical laws of black hole mechanics [Bardeen *et al* 1973]

⓿ The surface gravity κ is constant over the horizon.

❶ The mass M, surface gravity κ, and horizon area A satisfy $\delta M = \frac{M_P^2 \kappa}{8\pi} \delta A$.

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- Quantum state of Φ outside of the horizon is thermal with temperature $T_H = \frac{\kappa}{2\pi}$ at spatial infinity.
- Black hole radiates as a black body at temperature T_H.
- $\frac{1}{4} M_P^2 A$ is the entropy for the internal energy M and temperature $\frac{\kappa}{2\pi}$.

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Entanglement entropy associated with quantum fields outside of the horizon has leading contribution $O(\Lambda^2 A)$ for UV cutoff Λ. [Sorkin 1983]

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Proposal of Callan and Wilczek for the entanglement entropy associated with a spatial boundary in quantum field theory [Callan and Wilczek 1994]

$$S_{\text{ent}} = -\lim_{\delta \to 0} \left(2\pi \frac{\mathrm{d}}{\mathrm{d}\delta} + 1 \right) W_\delta$$

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- W_δ is UV divergent, so S_{ent} is UV divergent.

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Gravitational effective action for a smooth spacetime metric with UV cutoff Λ

$$
\begin{aligned}
W \quad = \quad & \int \mathrm{d}^D x \sqrt{g} \, [c_0 \Lambda^D + c_2 \Lambda^{D-2} R + c_{4,1} \Lambda^{D-4} R^2 + c_{4,2} \Lambda^{D-4} R_{\mu\nu} R^{\mu\nu} \\
& \qquad + c_{4,3} \Lambda^{D-4} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_{6,1} \Lambda^{D-6} R^3 + \cdots]
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Conjecture of Susskind and Uglum

- The UV divergences in the entanglement entropy can be absorbed in the renormalization of the couplings of the gravitational effective action W. [Susskind and Uglum 1994]

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- For any entangling surface with no extrinsic curvature in any spacetime dimension
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Qualifications

- Only for a restricted class of quantum states, each expressible as a Euclidean path integral
- Only for spacetimes possessing a bifurcate Killing horizon with boost invariant bifurcation surface
- Not applicable to quantum fluctuations of gravity

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Entanglement entropy associated with the spatial boundary Ω from a UV-divergent term $\mathcal{F}$ in the gravitational effective action

$$S_{\mathrm{ent}}[\mathcal{F},\Lambda] = 2\pi\, c_{\mathcal{F}} \Lambda^{D-\dim \mathcal{F}} \int_{\Omega} \mathrm{d}^{D-2}y \sqrt{\gamma}\, I_1[\mathcal{F}]$$

- $I_1[\mathcal{F}]$ is a geometric invariant constructed from $\mathcal{F}$ for the smooth spacetime metric.
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Contribution to the entanglement entropy from the Einstein-Hilbert term

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Contributions to the entanglement entropy from subleading UV-divergent terms depend on the quantum state $|\Psi\rangle$

Plan for the remainder

1. Computation of the entanglement entropy
   - Definition of the entanglement entropy associated with a spatial boundary in quantum field theory
   - Definition of the applicable class of quantum states
   - Derivation of the Callan-Wilczek formula
   - Calculation of the entanglement entropy
2. Reconciliation with the literature
3. Comparison to the Wald entropy
4. Conclusions
5. Open questions

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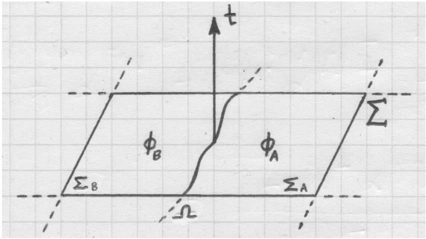

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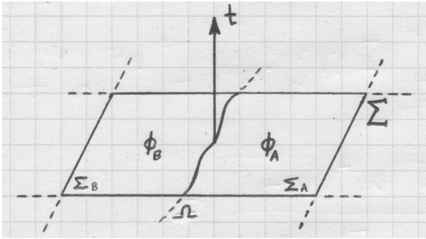

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- Quantum state
  - The quantum state on Σ is $\rho = |\Psi\rangle\langle\Psi|$.
  - In the field operator basis $\{|\phi_A \phi_B\rangle\}$

$$\rho(\phi_A, \phi_B, \phi'_A, \phi'_B) = \langle\phi'_A \phi'_B|\Psi\rangle\langle\Psi|\phi_A \phi_B\rangle.$$

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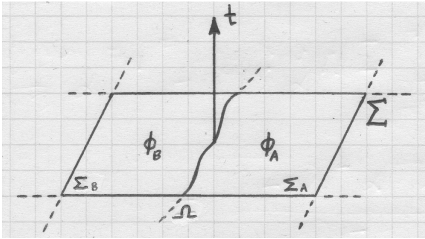

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  - The entanglement entropy is

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$$\langle\phi|\Psi\rangle = \lim_{\epsilon\to 0}\lim_{T\to\infty}\int \mathcal{D}\phi_i\,\langle\phi|U(0,-T(1+i\epsilon))|\phi_i\rangle = \int_{\Phi_<(\tau=0)=\phi}\mathcal{D}\Phi_<\,e^{-S_E[\Phi_<,\mathbf{g}]}$$

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The reduced quantum state on Σ_A is then

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Derivation of the Callan-Wilczek formula

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Consider the case of flat Euclidean spacetime.

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}z^2 + \delta_{ij}\mathrm{d}\sigma^i\mathrm{d}\sigma^j$$

- Spacelike hypersurface Σ at $\tau = 0$
- Entangling surface Ω at $z = 0$

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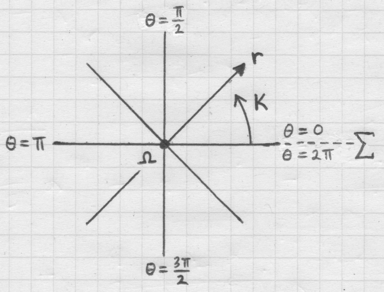

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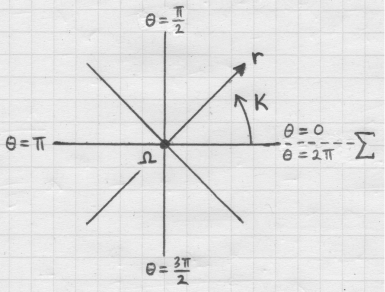

Reduced quantum state on Σ_A

$$\rho_A(\phi_A, \phi'_A) = \int \mathcal{D}\phi_B \int_{\Phi(\theta=0)=(\phi_A,\phi_B)}^{\Phi(\theta=2\pi)=(\phi'_A,\phi_B)} \mathcal{D}\Phi \, e^{-S_E[\Phi,\mathbf{g}]}$$

- Integrate over complete field configurations on a sequence of half planes around Ω, each a hypersurface of constant θ

$$\rho_A(\phi_A, \phi'_A) \propto e^{-2\pi K(\phi_A,\phi'_A)}$$

- Hamiltonian K associated with evolution in Euclidean time coordinate θ

Derivation of the Callan-Wilczek formula

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Entanglement entropy of the normalized reduced quantum state

$$S_{\mathrm{ent}} = \lim_{\epsilon\to 0}\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}+1\right)\ln \mathrm{Tr}\rho_A^{1-\epsilon} = \ln \mathrm{Tr}\rho_A - \frac{\mathrm{Tr}(\rho_A \ln \rho_A)}{\mathrm{Tr}\rho_A}$$

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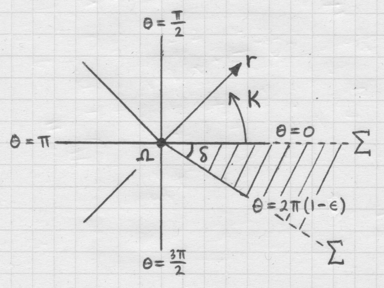
$\mathrm{Tr}\rho_A^{1-\epsilon}$ is given by a Euclidean path integral with Φ identified at $\theta = 0$ and $\theta = 2\pi(1-\epsilon)$.

The spacetime metric $\mathbf{g}$ has a conical singularity at $r = 0$ with deficit angle $\delta = 2\pi\epsilon$.

$$\begin{aligned} \mathrm{Tr}\rho_A^{1-\epsilon} &= \mathrm{Tr}\, e^{-(2\pi-\delta)K} \\ &= \int \mathcal{D}\Phi\, e^{-S_E^{(\delta)}[\Phi,\mathbf{g}]} \\ &= e^{-W_\delta} \end{aligned}$$



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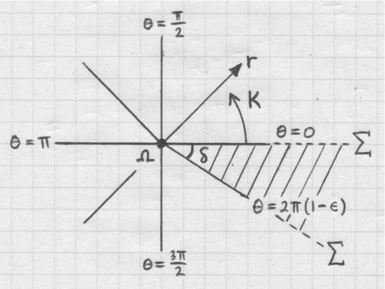

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Applicability of the Callan-Wilczek formula

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Preceding derivation generalizes to a class of Euclidean spacetime metrics

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \alpha^2(r,\sigma)\mathrm{d}\theta^2 + \gamma_{ij}(r,\sigma)\mathrm{d}\sigma^i\mathrm{d}\sigma^j$$

$$\alpha|_{r=0} = 0, \quad \partial_r\alpha|_{r=0} = 1, \quad \partial_r^m\alpha|_{r=0} = 0 \quad \text{for} \quad m = 2,4,6,\ldots$$

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- Rotational symmetry in θ about Ω
- Otherwise arbitrary geometry for spacelike hypersurface Σ expressed in Gaussian normal coordinates

$$\mathrm{d}s_\Sigma^2 = \mathrm{d}r^2 + \gamma_{ij}(r,\sigma)\mathrm{d}\sigma^i\mathrm{d}\sigma^j$$

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Euclidean spacetimes correspond to Lorentzian spacetimes possessing a bifurcate Killing horizon with bifurcation surface Ω

- Reflection symmetry in θ corresponds to time reflection symmetry about Σ
- Rotational symmetry in θ corresponds to boost invariance of Ω

Regularization of the conical singularity

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- Regulated conically singular metric

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Compute the entanglement entropy via the Callan-Wilczek formula for the regulated metric

- Take $l \to 0$ with Λ fixed
- Only $O(\epsilon^0)$ and $O(\epsilon^1)$ terms in the gravitational effective action are well defined as $l \to 0$
- Callan-Wilczek formula only requires $O(\epsilon^1)$ terms

Calculation of the entanglement entropy

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Regulated gravitational effective action for conically singular spacetime metric

$$\begin{aligned}\tilde{W}_\delta &= \int \mathrm{d}^D x \sqrt{\tilde{g}}\,[c_0 \Lambda^D + c_2 \Lambda^{D-2} \tilde{R} + \cdots] \\ &\qquad + \int_\Omega \mathrm{d}^{D-2} y \sqrt{\tilde{\gamma}}\,[\bar{c}_0 \Lambda^{D-2} + \bar{c}_2 \Lambda^{D-4} \tilde{R}(\tilde{\gamma}) + \cdots]\end{aligned}$$

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Entanglement entropy via the Callan-Wilczek formula

$$S_{\mathrm{ent}}[\Lambda] = -\lim_{l \to 0} \lim_{\epsilon \to 0} \left(\frac{\mathrm{d}}{\mathrm{d}\epsilon} + 1 \right) \tilde{W}_\delta$$

Calculation of the entanglement entropy

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- Entanglement entropy determined by $O(\epsilon^1)$ piece $\tilde{\mathcal{F}}^{(\epsilon^1)}$ of $\tilde{\mathcal{F}}$
- UV divergences on the entangling surface Ω do not contribute
- Geometric invariant $I_1[\mathcal{F}]$ constructed from $O(\epsilon^0)$ piece $\tilde{\mathcal{F}}^{(\epsilon^0)}$ of $\tilde{\mathcal{F}}$

Leading UV-divergent term in entanglement entropy

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Einstein-Hilbert term in the gravitational effective action

$$W = \int \mathrm{d}^D x \sqrt{g}\, c_2 \Lambda^{D-2} R$$

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Entanglement entropy from the Einstein-Hilbert term

$$S_{\text{ent}}[R, \Lambda] = 2\pi \int_\Omega \mathrm{d}^{D-2}\sigma \sqrt{\gamma}\, 2c_2 \Lambda^{D-2} = 4\pi c_2 \Lambda^{D-2} A_\Omega$$

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$$W_{CT} = \int \mathrm{d}^D x \sqrt{g}\, \frac{M_{P0}^{D-2}}{16\pi} R$$

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Renormalized entanglement entropy from the Einstein-Hilbert term

$$S_{\mathrm{ent}}[R] = 4\pi c_2 \Lambda^{D-2} A_\Omega + \frac{1}{4} M_{P0}^{D-2} A_\Omega = \frac{1}{4} M_P^{D-2} A_\Omega$$

Renormalized Planck mass

$$M_P^{D-2} = M_{P0}^{D-2} + 16\pi c_2 \Lambda^{D-2}$$

Subleading UV-divergent terms in entanglement entropy

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4-derivative terms in the gravitational effective action

$$W = \int \mathrm{d}^D x \sqrt{g}\, [\cdots + c_{4,1}\Lambda^{D-4} R^2 + c_{4,2}\Lambda^{D-4} R_{\mu\nu} R^{\mu\nu} + c_{4,3} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \cdots]$$

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Entanglement entropy contributions

$$\begin{aligned} S_{\mathrm{ent}}[R^2,\Lambda] &= -2\pi c_{4,1}\Lambda^{D-4}\int_\Omega \mathrm{d}^{D-2}\sigma\sqrt{\gamma}\, 8\left[\alpha''' + \gamma^{ij}\gamma''_{ij} - \frac{1}{2}R(\gamma)\right] \\ S_{\mathrm{ent}}[R_{\mu\nu}R^{\mu\nu},\Lambda] &= -2\pi c_{4,2}\Lambda^{D-4}\int_\Omega \mathrm{d}^{D-2}\sigma\sqrt{\gamma}\, 2\left[2\alpha''' + \gamma^{ij}\gamma''_{ij}\right] \\ S_{\mathrm{ent}}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},\Lambda] &= -2\pi c_{4,3}\Lambda^{D-4}\int_\Omega \mathrm{d}^{D-2}\sigma\sqrt{\gamma}\, 8\alpha''' \end{aligned}$$

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Checks

- Agreement with work of Fursaev and Solodukhin
  [Fursaev and Solodukhin 1995]
- Euler density E_4 in $D = 4$ spacetime dimensions

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Dependence on the quantum state of subleading UV-divergent contributions to the entanglement entropy

Reconciliation with the literature

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Checks of the Susskind-Uglum conjecture at one loop

- Spin 0 (minimally coupled): Confirmed [Susskind and Uglum 1994], *etc.*
- Spin 0 (nonminimally coupled): Disagreement [Fursaev 1995], *etc.*
- Spin $\frac{1}{2}$: Confirmed [Kabat 1995], *etc.*
- Spin 1: Disagreement [Kabat 1995], *etc.*
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- Spin 2: Possibly but inconclusively confirmed [Iellici and Moretti 1996], [Fursaev and Miele 1997]

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Resolution

- Only renormalized entanglement entropy is physically meaningful, not UV-divergent pieces
- Susskind-Uglum conjecture holds for leading and subleading UV-divergent terms in the entanglement entropy to all orders in perturbation theory

Comparison to Wald entropy

Comparison to Wald entropy

Wald entropy for a classical metric theory of gravity

$$S_W = 2\pi \int_\Omega \mathrm{d}^{D-2}\sigma \sqrt{\gamma}\, \left[\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\tau \frac{\partial \mathcal{L}}{\partial \nabla_\tau R_{\mu\nu\rho\sigma}} + \cdots \right] \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$$

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