Resurgence, trans-series and toward a continuum definition of QFT

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In collaboration with
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Motivation: Can we make sense out of QFT? Dyson, 't Hooft (79), When is there a continuum definition of QFT?

Quoting from M. Douglas comments, in Foundations of QFT, talk at String-Math 2011

"A good deal of mathematical work starts with the Euclidean functional integral (as we will). There is no essential difficulty in rigorously defining a Gaussian functional integral, in setting up perturbation theory, and in developing the BRST and BV formulations (see e.g. K. Costello's work).

A major difficulty, indeed many mathematicians would say the main reason that QFT is still "not rigorous," is that standard perturbation theory only provides an asymptotic (divergent) expansion. There is a good reason for this, namely exact QFT results are not (often) analytic in a finite neighborhood of zero coupling.

Fixing this problem has been the goal of great deal of work. A plausible approach is"constructive QFT" (Glimm/Jaffe/Spencer, Brydges,...)"

Recently, few people are attempting to answer this question, whether/when a n.p. continuum definition of QFT may exist and reinvigorate this problem.

Philip Argyres, Gerald Dunne, MÜ: Resurgence in QFTs and path integrals
Ricardo Schiappa, Marcos Marino, ...:Resurgence in string theory and matrix models
Maxim Kontsevich, a recent talk at PI: Resurgence from the path integral perspective
Garoufalidis, Costin, ... Math and Topological QFTs

The common concept, which all these folks seem to be highly influenced by (and which is virtually unknown in physics community) is a "recent" mathematical progress, called **Resurgence Theory, developed by Jean Ecalle (80s)**, and applied to QM by Pham, Delabaere, Voros. I believe that Ecalle's theory changed (will change?) the overall perspective on asymptotic analysis, for both mathematicians and physicists alike.

I will give necessary definitions and background as we proceed. In physics literature, there are some earlier hints that a resurgent structure must underlie QFT.

In particular, I will illustrate various aspects of resurgence in a non-trivial QFT in 2d, the CP(N-1) model. However, the ideas are general.

CP(N-1) model on R2 and standard problems

An asymptotically free non-linear sigma model with a complex projective target space. Large-N, successful. Many problems are still unresolved at finite-N.

1) Perturbation theory is an asymptotic *(divergent) expansion even after regularization and renormalization.* Is there a meaning to perturbation theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R2. (e.g,Affleck). DIG assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption.

3) ``Infrared embarrassment",e.g., large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

4) A resolution of 2) was put forward by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined "regime". (Affleck, 80) So, *no semi-classical approximation for the confined regime* to date is found, (except a supersymmetric version of the theory).

5) Incompatibility of large-N results with instantons. (Witten, Jevicki, 79)

6) The renormalon ambiguity (technical, but deeper, to be explained), ('t Hooft,79).

CP(N-1) model on R2 and standard problems

I think there is no exaggeration in saying that what "our inheritance" from the few generation earlier is a disaster. To be fair, they at least stated the problems of a typical QFT. (all of the above problems also present in 4d QCD.)

After mid-80's, very few serious field theorists worked on these type of problems, primarily due to sociological reasons, not because the problems were uninteresting.

Our goal, by putting sociological considerations aside, is to make progress in this class of theories.

Furthermore, if we are going to make progress in some foundational aspects of QFT, it is, of course, preferable to have a formalism of *practical utility,* whose results can be compared with *numerical experiments,* i.e., lattice field theory. LFT, as is, a black box.

So far, there had been no such useful continuum formulation of general QFTs. The "constructive QFT" approach did only attempt the first problem, but not very successfully.

In this talk, I will report progress in this direction, and argue a useful n.p. definition may underly the resolution of *all* puzzles/problems quoted above.

Simpler question: Can we make sense of the semi-classical expansion of QFT?

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(o,k)} \sim k!$

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

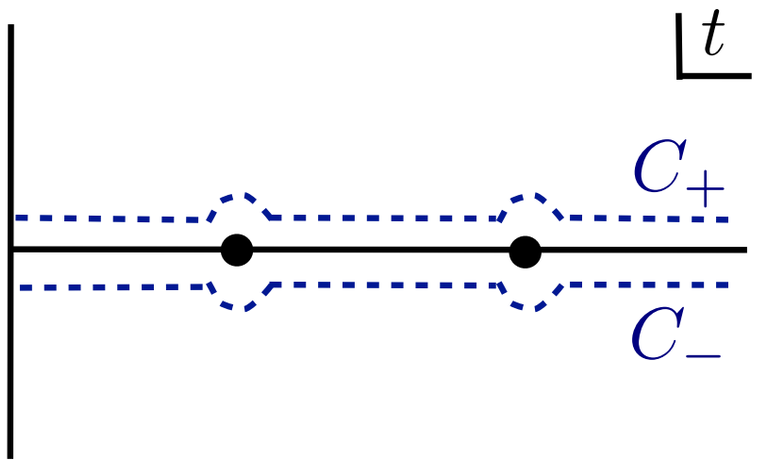
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2}\, dt\ .$$

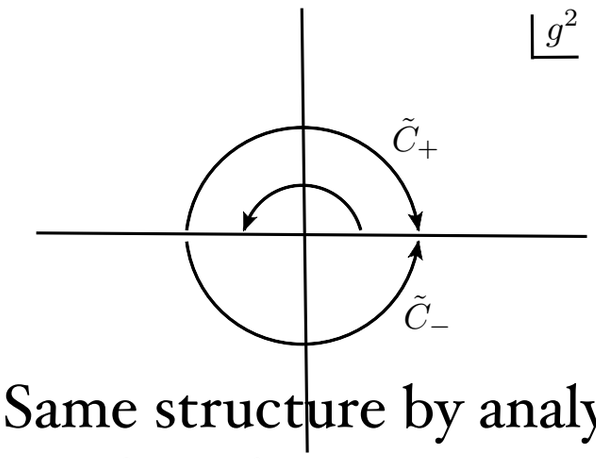
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Borel plane and lateral (left/right) Borel sums

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



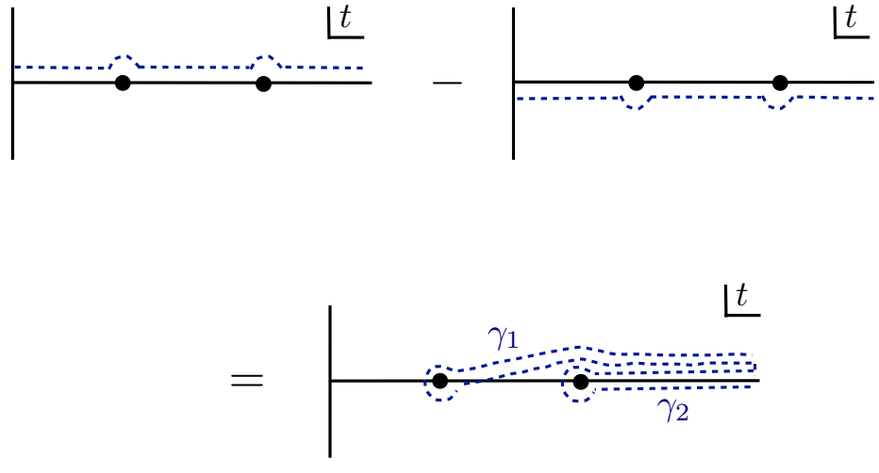
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor. The measure of ambiguity (or Stokes automorphism, or Stokes jump as per g-space interpretation):

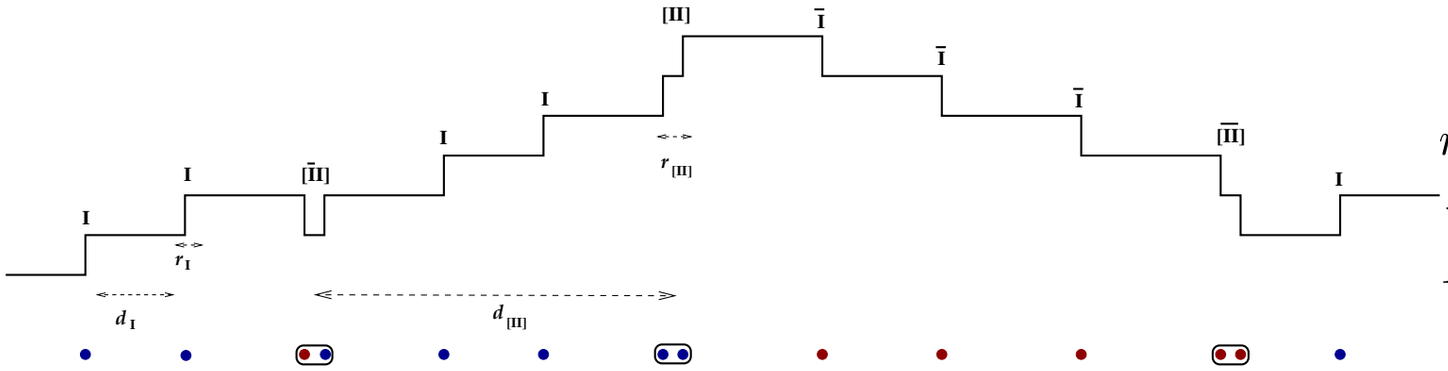


$$\mathcal{S}_{\theta+} = \mathcal{S}_{\theta-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta-})\,,$$

$$\mathrm{Disc}_{\theta-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \ldots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Bogomolny--Zinn-Justin (BZJ) prescription

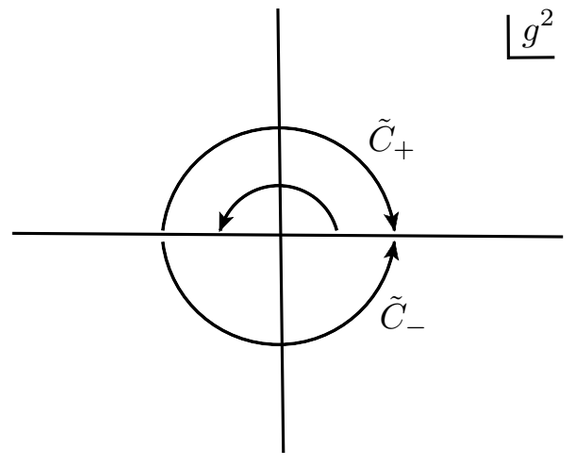
Bogomolny-Zinn-Justin prescription in QM (80s): done for double well potential, but consider a periodic potential. Dilute instanton, molecular instanton gas.



$$r_{\rm I} \ll r_{[{\rm II}]} \sim \ell_{\rm qzm} \ll d_{\rm I} \ll d_{[{\rm II}]},$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$L \ll L\,\log\left(\tfrac{1}{g^2}\right) \ll L\,e^{S_0} \ll L\,e^{2S_0}.$$

How to make sense of topological molecules (or molecular instantons)? Why do we even need a molecular instanton? The answer is above hierarchy!

Naive calculation of I-anti-I amplitude: meaningless result at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. Continue to $g^2 < 0$, evaluate the integral, and continue back to $g^2 > 0$. Result is two fold-ambiguous! $[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0\pm} = \mathrm{Re}\,[\mathcal{I}\overline{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0\pm}$



Remarkable fact: Leading ambiguities cancel. "N.P. CONFLUENCE EQUATION", elementary incidence of Borel-Ecalle summability which I will return:

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0\pm} + \mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0\pm} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

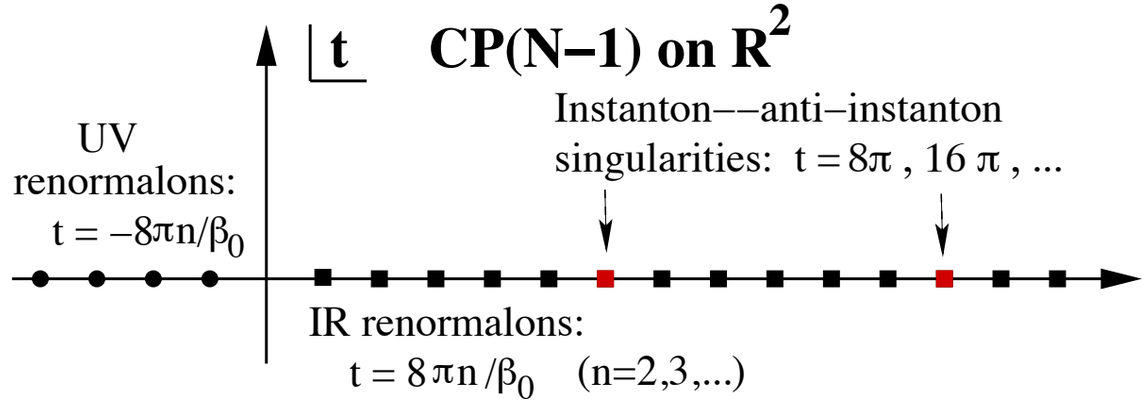
Can this work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft: ***No,*** *on R4, F. David(84), Beneke(93), ...* ***No,*** *on R2.*
Argyres, MÜ: ***Yes,*** *on R3 x S1,* Dunne, MÜ: ***Yes****, on R1 x S1*
There is also a reasonable possibility that it can be extended to decompactified theory.

Why doesn't it work, say for CP(N-1) on R2?

Instanton-anti-instanton contribution, calculated in same way, gives an $\pm i \exp[-2S_I]$.
Lipatov: Borel-transform BP(t) has singularities at $t_n = 2n\, g^2\, S_I$. (Modulo the IR problems with 2d instantons).

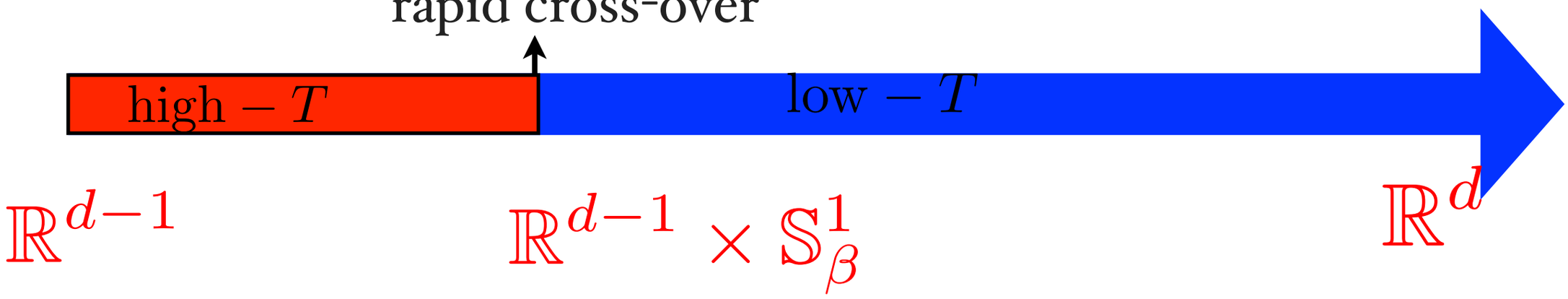
BUT, BP(t) has other (more important) singularities <u>closer</u> to the origin of the Borel-plane. (not due to factorial growth of number of diagrams!)



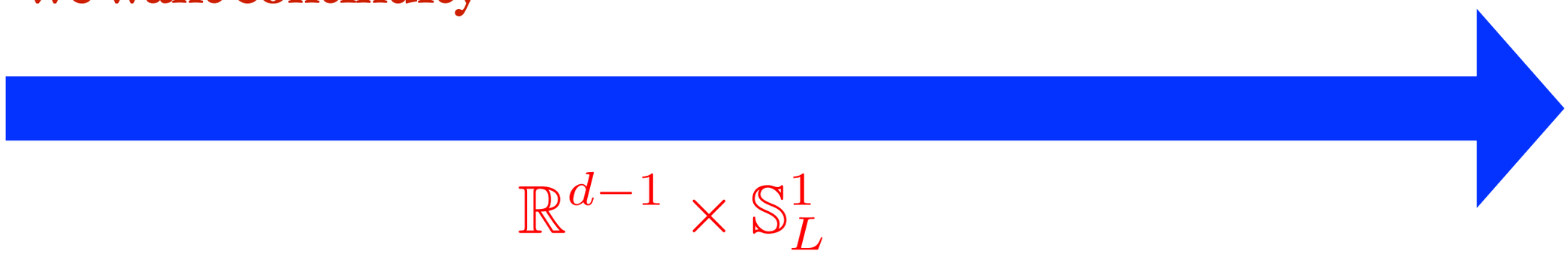
't Hooft called these IR-renormalon singularituies with the hope/expectation that they would be associated with a saddle point like instantons.
No such configuration is known!!

A real problem in QFT, means pert. theory, as is, ill-defined.

$\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1_L$ and Continuity



We want continuity



Thermal finite-N: Rapid crossover at strong scale
Thermal large-N: Sharp phase transition at strong scale

Prevent both by using circle compactification or deformation.

$\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1_L$ and Continuity

Thermal compactification is literally used thousands of times in field theories. For example, Affleck studied the theory on R1 x S1 to tame the instanton size moduli in the small-S1 regime. However, his study is only relevant for the deconfined regime of field theory, and is irrelevant to study the confined regime.

If we want to learn something pertinent to field theory on R2, we have to find a regime of the theory which is weakly coupled and continuously connected to the desired target theory. We call this, "principle of continuity".

Continuity is used in supersymmetry many times, starting with the supersymmetric (Witten) index calculation in early 80s.

In non-supersymmetric theories, the utility of continuity is realized in 2007 (MU). This point of view turns out to be extremely useful. I have explored many remarkable consequences of this simple idea with few collaborators, mainly, Poppitz, Yaffe, Shifman, Argyres, Dunne, Schäfer. I will discuss this idea for CP(N-1) here (with Dunne), along with the resurgence theory to provide an answer to the problems I mentioned earlier.

Sigma-connection holonomy (a new line operator)

Point-wise modulus and phase splitting, derivative of *each* phase transform as "gauge" connection.

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos\frac{\theta_1}{2} \\ e^{i\varphi_2} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \\ e^{i\varphi_3} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \cos\frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \sin\frac{\theta_3}{2} \ldots \sin\frac{\theta_{N-1}}{2} \end{pmatrix}$$

$$\theta_i \in [0, \pi], \qquad \varphi_i \in [0, 2\pi).$$

Build a new line operator, counter-part of the Wilson line, the sigma holonomy:

$$({}^L\Omega)_j(x_1) = \exp\left[i \int_0^L dx_2 \, \mathcal{A}_{2,j}\right] = \exp\left[i(\varphi_j(x_1, 0) - \varphi_j(x_1, L))\right]$$

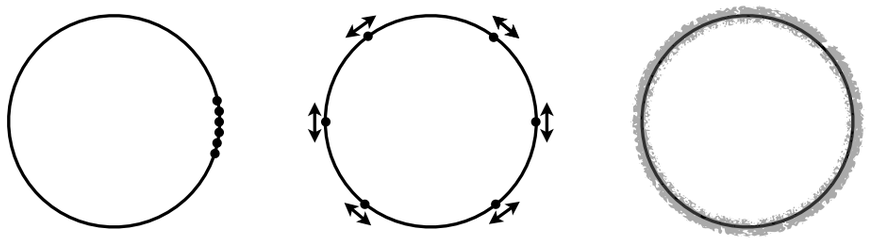
$${}^L\Omega(x_1) = \begin{pmatrix} e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]} & 0 & \ldots & 0 \\ 0 & e^{i[\varphi_2(x_1,0)-\varphi_2(x_1,L)]} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & e^{i[\varphi_N(x_1,0)-\varphi_N(x_1,L)]} \end{pmatrix}$$

One-loop potential for Sigma holonomy

$$V_-[{}^L\Omega] = \frac{2}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1+(-1)^n N_f)(|\mathrm{tr}\,{}^L\Omega^n|-1) \qquad \text{(thermal)}$$

$$V_+[{}^L\Omega] = (N_f-1)\frac{2}{\pi L^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(|\mathrm{tr}\,{}^L\Omega^n|-1) \qquad \text{(spatial)}$$

Three types of holonomy



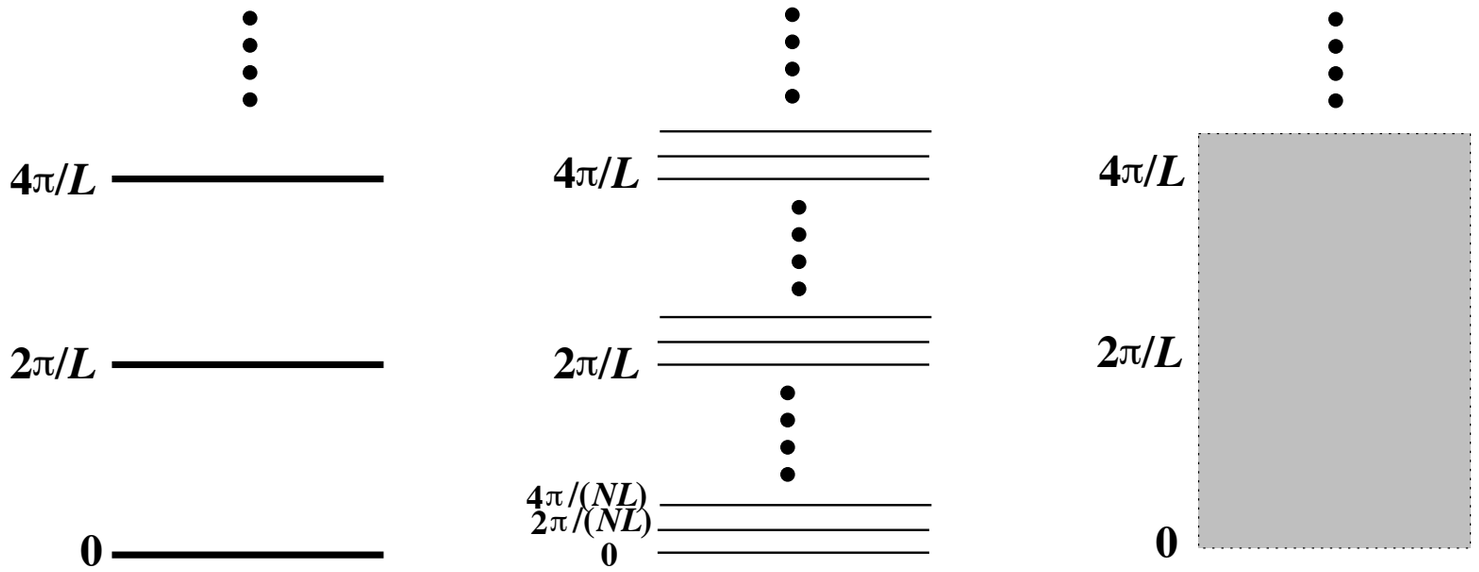
(a) Thermal

(b) Spatial

(c) Strong-coupling non-trivial hol.

To achieve (b) in the $n_f = 0$ case require a deformation of the action. (b) is weak coupling realization of the center-symmetric background.

The dependence of perturbative spectrum to the sigma holonomy background



(a) Center–broken (b) Center–symmetric (c) Center–symmetric $N \to \infty$

Same as gauge theory on R3 x S1, the fact that spectrum become dense in the last case is an imprint of the large-N volume independence (Eguchi-Kawai reduction). This is surprising on its own in a "vector"-model, but I will not talk about it here.

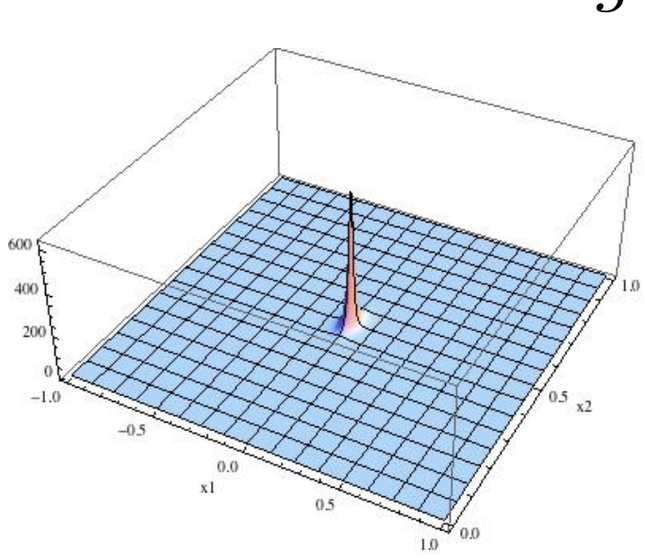
Instead, we will study non-pert. effects in the long-distance effective theory within Born-Oppenheimer approx. in case (b) for finite-N.

Topological configurations, 1-defects

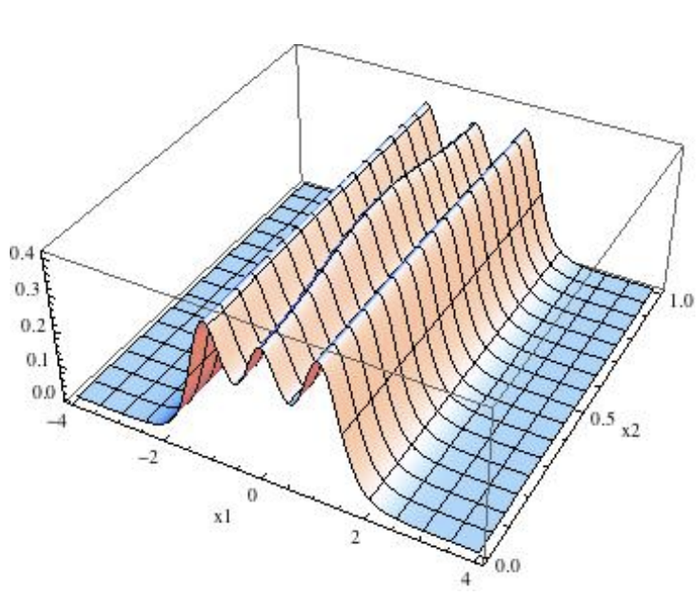
1-defects, Kink-instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The Nth type corresponds to the affine root and is present only because the theory is *locally 2d*! Also see Bruckmann (07), Brendel et.al.(09)

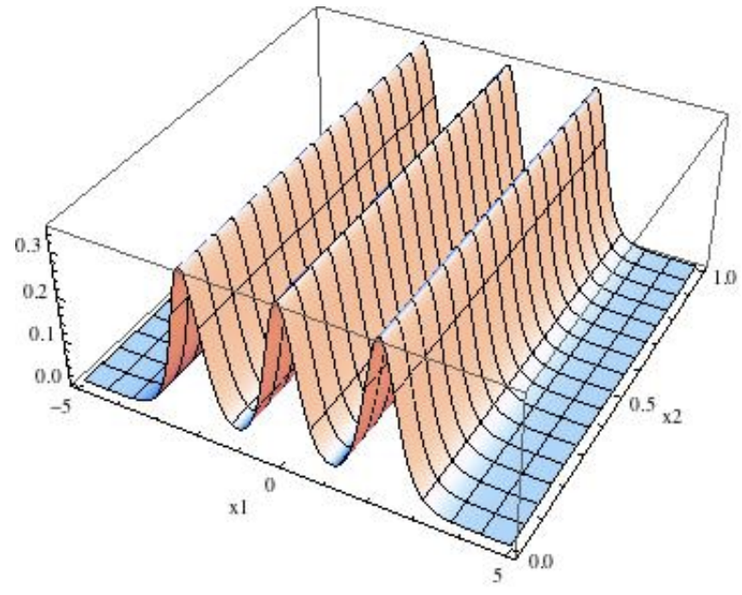
$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee$$

$$\mathcal{K}_k : \qquad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N} \qquad , \qquad k = 1, \ldots, N$$



Small-2d BPST instanton in CP(2)





Large-2d BPST instanton in CP(2) fractionates into 3-types of kink-instantons. (In thermal case, this does not occur at high-T, see Affleck(80s).)

Application: N=(2,2) CP(N-1)

In theories with fermions, each kink-event carries fermionic zero mode as per an index theorem.

In supersymmetric theory, each kink-instanton carries two-zero modes, and there are N-types of elementary kink events. Recall that BPST instanton has 2N zero modes, and this fits nicely with the idea of fractionalization.

The kink-amplitudes generate a superpotential. This superpotential, obtained in the compactified theory, by working out the duality in simple quantum mechanics, is identical to the result by Hori and Vafa (2000), obtained by using mirror symmetry on R2.

Here, I will address the dynamics of most general CP(N-1).

Topological molecules: 2-defects

2-defects are universal: charged and neutral bions

Charged bions: For each negative entry of the extended Cartan matrix $\widehat{A}_{ij} < 0$, there exists a bion $\mathcal{B}_{ij} = [\mathcal{K}_i \overline{\mathcal{K}}_j]$, associated with the correlated tunneling-anti-tunneling event

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_j \qquad \alpha_i \in \Gamma_r^\vee$$

Neutral bions: For each positive entry of the extended Cartan matrix $\widehat{A}_{ii} > 0$, there exists a neutral bion $\mathcal{B}_{ii} = [\mathcal{K}_i \overline{\mathcal{K}}_i]$, associated with the correlated tunneling-anti-tunneling event

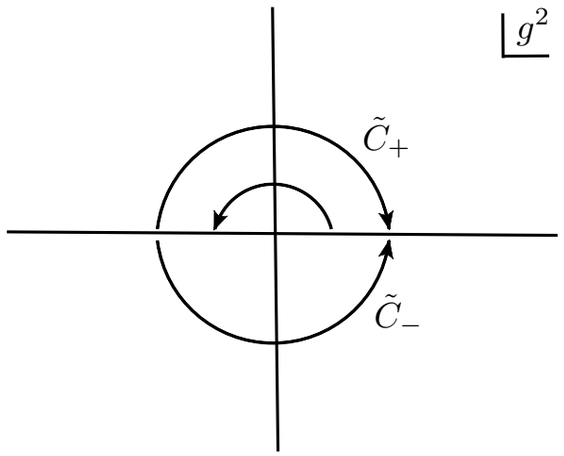
$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_i \qquad \alpha_i \in \Gamma_r^\vee$$

Charged bion is the counter-part of magnetic bion in gauge theory on R3 x S1 (which generates mass gap for gauge fluctuations), MÜ 2007

Neutral bion is the counter-part of neutral bion in gauge theory on R3 x S1 (which generates a center-stabilizing potential), Poppitz-MÜ 2011, Poppitz-Schäfer-MÜ, Argyres-MÜ 2012

Neutral bion and non-perturbative ambiguity in semi-classical expansion

Naive calculation of neutral bion amplitude, as you may guess as per QM example, meaningless at $g^2>0$. The quasi-zero mode integral is dominated at small-separations where a molecular event is meaningless. Continue to $g^2<0$, evaluate the integral there, and continue back to $g^2>0$. Result is two fold-ambiguous!



$$
\begin{aligned}
[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} =& \mathrm{Re}\,[\mathcal{K}_i\overline{\mathcal{K}}_i] + i\,\mathrm{Im}\,[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} \\
=& \left(\log\left(\frac{\lambda}{8\pi}\right)-\gamma\right)\frac{16}{\lambda}e^{-2S_0} \pm i\frac{16\pi}{\lambda}e^{-\frac{8\pi}{\lambda}}
\end{aligned}
$$

As it stands, this is a disaster! It tells us that semi-classical expansion at second order is void of meaning.

This is a general statement valid for many QFTs admitting semi-classical approximation. This also includes, for example, the Polyakov model.... However, in QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects... The truth is far more subtler!

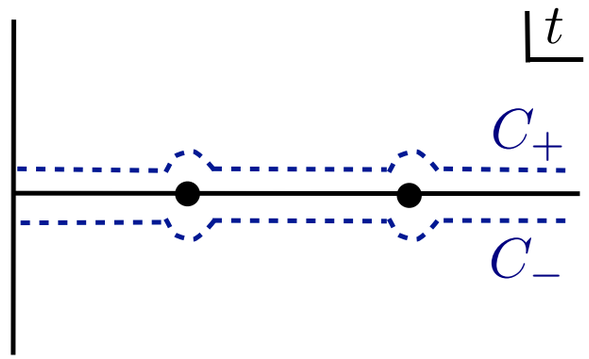
Disaster or blessing in disguise?

Go back to pert. theory, for the compactified center-symmetric CP(N-1) theory. We reduce the long-distance effective theory to simple QM with periodic potentials. Thankfully, the large-order behavior of pert. theory in such QM problems is studied by M. Stone and J. Reeve (78), by using the classic Bender-Wu analysis (69-73).

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \qquad a_q \sim -\frac{2}{\pi}\left(\frac{1}{4\xi}\right)^q q! \left(1 - \frac{5}{2q} + O(q^{-2})\right)$$

Divergent non-alternating series, non-Borel summable, but right and left Borel resummable, with a result:

$$\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\; B\mathcal{E}(t)\; e^{-t/g^2} = \mathrm{Re}\mathcal{S}\mathcal{E}(g^2) \mp i\frac{8\xi}{g^2} e^{-\frac{4\xi}{g^2}}$$
$$= \mathrm{Re}\mathbb{B}_0 \mp i\frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$



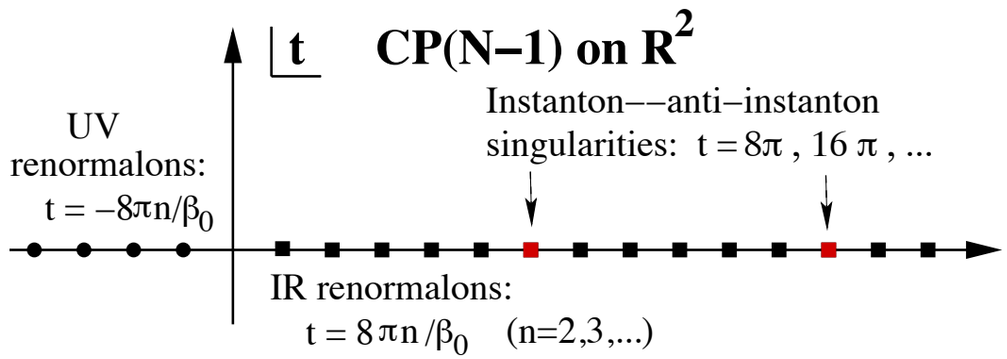
Remarkably,

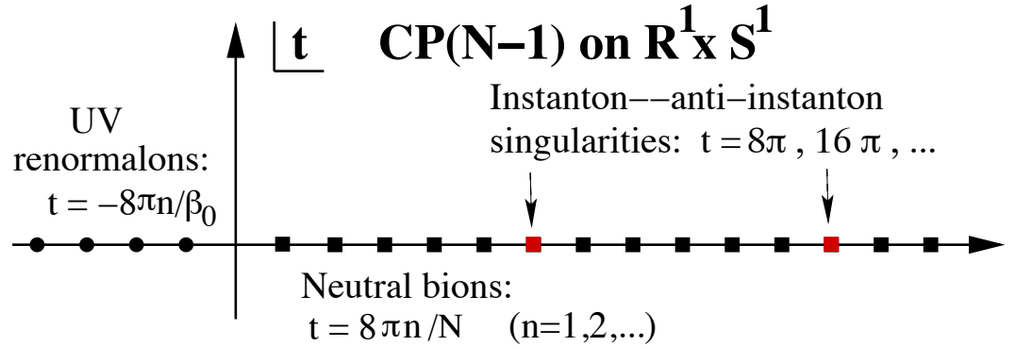
$$\mathrm{Im}\left[\mathcal{S}_\pm \mathcal{E}(g^2) + [\mathcal{K}_i \overline{\mathcal{K}}_i]_{\theta=0\pm}\right] = 0 \qquad \text{up to } e^{-4S_0} = e^{-4S_I/\beta_0}$$

The ambiguities at order exp[-2S_I/N] cancel and QFT is well-defined up to the ambiguities of order exp[-4S_I/N] ! **But** this order for ambiguities is exactly the IR-renormalon territory as per 't Hooft's analysis.

Semi-classical renormalons as neutral bions

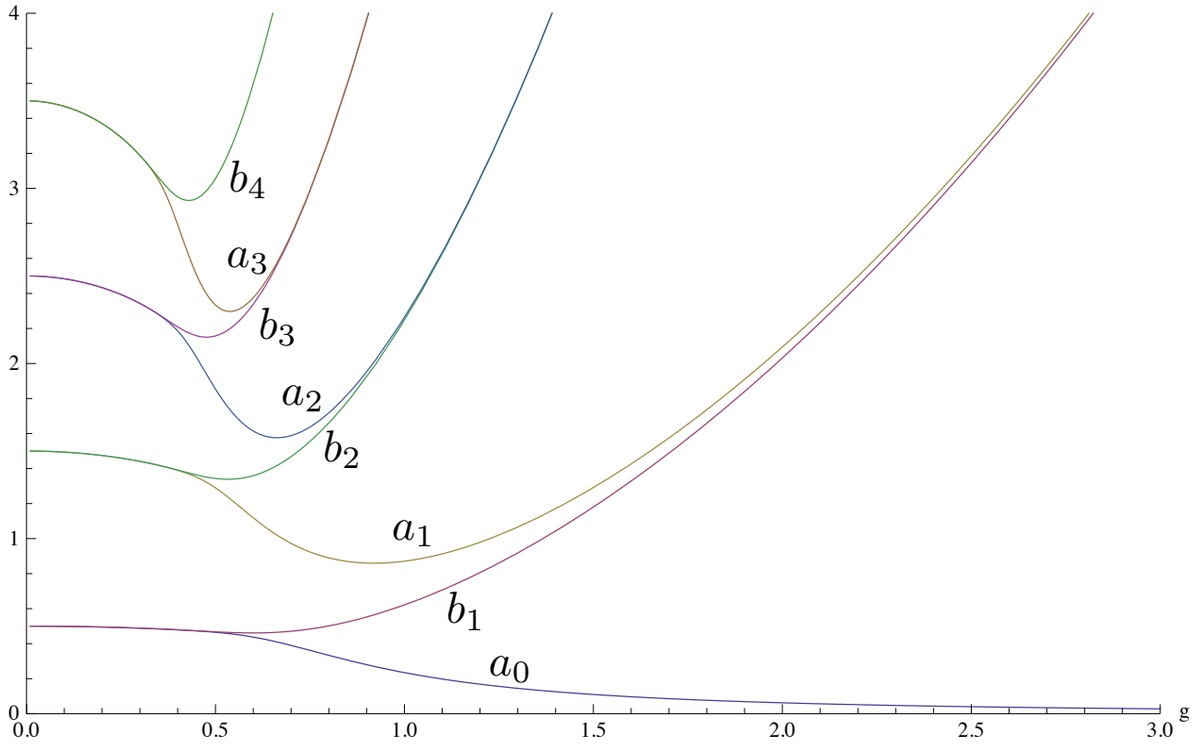
We claim (with Argyres in 4d) and (with Dunne in 2d) that our neutral bions and related other neutral topological molecules are semi-classical realization of 't Hooft's elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion. We showed this only at leading (but most important) order.





More than three decades ago, 't Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that in fact, we can, at least, in the semi-classical regime of QFT. (and perhaps even more.)

Mass gap in the small-S1 regime



Eigen-energies for CP(1) in reduced QM. In Born-Oppenheimer approx., the zero mode Hamiltonian at small g reduce to Mathieu ODE.

$$H^{\rm zero}_{\alpha_k} = -\frac{1}{2}\frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2}[1-\cos(2g\theta)]$$

$$m_g = \frac{C}{\sqrt{\lambda}}\left(1-\frac{7\lambda}{32\pi}+O(\lambda^2)\right)e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad \text{for } \mathbb{CP}^{N-1}$$

The functional form of the small-S1 result for CP(N-1) is same as large-N result on R2. This is the first microscopic derivation of the factor exp[-S_I/N] from microscopic considerations. This effect, at least in the small-S1 regime, solves the large-N vs. instanton puzzle. Clearly, this is an effect which survives large-N limit.

Resurgence Theory and Transseries

Ecalle (1980s) formalized asymptotic expansion with exponentially small terms (called trans-series) & generalized Borel resummation for them by incorporating the Stokes phenomenon.

Basic idea: Start with a formal power series, e.g. an asymptotic (divergent) expansion of Gevrey-1 type: $\Sigma\, a_n\, g^{2n}$ where $a_n \leq A\, n!\, c^n$ (generic in QFT).

Borel transformation maps this formal series to a convolutive subalgebra of germs at the origin in complex Borel plane.

Analytic continuation of the germ to a holomorphic function except a set of singularities (pole or branch points) in the complex Borel-plane.

Directional Laplace transforms to find sectorial sums by invoking Stokes phenomenon.

Resurgence theory of Ecalle

Main result: Borel-Ecalle resummation of a transseries exists and is unique, if the Borel transforms of all perturbative series are all "endlessly continuable" =Set of all singularities on all Riemann sheets on Borel plane do not form any natural boundaries.

Such transseries are called "resurgent functions": Example of transseries:

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

Formal: perturbative + (non-perturbative) x (perturbative)

Resurgence theory of Ecalle in QFTs

Pham, Delabaere,....(1990s): Using the theory of resurgent functions, they proved that the semi-classical (perturbative+ non-perturbative) transseries expansion in Quantum mechanics with double-well and periodic potentials are summable to finite, exact results.

In CP(N-1), by invoking "continuity", we can reduce QFTs in long distance limit to the quantum mechanical systems studied by Pham et.al. In particular, Pham et.al. result implies that in the small S1 regime, spectrum, mass gap etc. of the theory are resurgent functions. What I showed you was one of the first step of cancellations inherent to resurgence theory applied to QFT.

Resurgence theory in path integrals

Pham et. al. results are in Hamiltonian formalism. We wonder whether we can generalize this to path integrals of QM, because, path integral formulation generalize more easily to QFT.

Key step is in the analytic continuation of paths in field space (cf. Pham, and recent papers by Witten), to make sense of steepest descent and Stokes phenomenon in path integrals. (We actually use this implicitly, but need to make it more systematic.)

cf. a recent talk by Kontsevich "Resurgence from the path integral perspective", Perimeter Institute, August, 2012.

Graded Resurgence triangle

The structure of CP(N-1) and many QFTs is encoded into the following construct:

$f_{(0,0)}$ → pert. theory around pert. vacuum

kink x (pert. fluctuations) ← $e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)}$ $e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)}$ → bions x (pert. fluctuations)

$e^{-\frac{2A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,2)}$ $e^{-\frac{2A}{\lambda}} f_{(2,0)}$ $e^{-\frac{2A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,-2)}$

$e^{-\frac{3A}{\lambda}+3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,3)}$ $e^{-\frac{3A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(3,1)}$ $e^{-\frac{3A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-1)}$ $e^{-\frac{3A}{\lambda}-3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-3)}$

$e^{-\frac{4A}{\lambda}+4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,4)}$ $e^{-\frac{4A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,2)}$ $e^{-\frac{4A}{\lambda}} f_{(4,0)}$ $e^{-\frac{4A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-2)}$ $e^{-\frac{4A}{\lambda}-4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-4)}$

$\iddots$ $\vdots$ $\ddots$

No two column can mix with each other in the sense of cancellation of ambiguities.

N.P. confluence equations

In order QFT to have a meaningful semi-classical continuum definition, a set of perturbative--non-perturbative confluence equations must hold. Examples are

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \dots\Big)$$

Meaning, order by order hierarchical confluence equations:

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm\,, \qquad (\text{up to } e^{-4S_0})$$

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm + \mathrm{Im}\mathbb{B}_{[2,0]\pm}\mathrm{Re}[\mathcal{B}_{ii}] + \mathrm{Re}\mathbb{B}_{[4,0]}\mathrm{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$

$$0 = \dots$$

This has a very deep implication in QFT:

$$\text{Disc}\,\mathbb{B}_{[0,0]} = -2\pi i \lambda^{-r_2} P_{[2,0]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}), \tag{1}$$

Using dispersion relation, we obtain

$$\begin{aligned} a_{[0,0],q} &= \sum_{q'=0}^{\infty} a_{[2,0],q'} \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\ &= \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2}} \left[a_{[2,0],0} + \frac{2A}{(q+r_2-1)} a_{[2,0],1} + \frac{(2A)^2}{(q+r_2-1)(q+r_2-2)} a_{[2,0],2} + \dots \right] \\ &+ O\left(\left(\frac{1}{4A}\right)^q\right) \end{aligned} \tag{2}$$

Late terms in the perturbative expansion around the perturbative vacuum is given by early terms in the perturbative expansion around neutral bion sector.

1/q corrections to the late terms are dictated by the order by order pert. expansion around the neutral bion sector.

Exponentially suppressed corrections to the late terms is dictated by [Bion-Bion] etc. terms.

In other words, perturbation theory, if properly decoded, has all non-perturbative data!

Why extended supersymmetric theories are simpler?

$$
\begin{array}{ccccccc}
 & & & [0,0] & & & \\
 & & [1,1] & & [1,-1] & & \\
 & [2,2] & & \varnothing & & [2,-2] & \\
[3,3] & & \varnothing & & \varnothing & & [3,-3] \\
[4,4] & & \varnothing & & \varnothing & & \varnothing & & [4,-4] \\
 & & & \vdots & & & & & \ddots
\end{array}
$$

No neutral bion configurations, the confluence equation simplify into

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^{\pm}}\Big), \qquad 0 = \mathrm{Im}\Big(\mathbb{B}_{[1,1],\theta=0^{\pm}}\Big)$$

Extended supersymmetric theories must be Borel summable!

Same conclusion, Russo (2012) by other means.

Conclusions

Continuity and resurgence theory can be used in combination to provide a non-perturbative continuum definition of interesting asymptotically free theories, and more general QFTs.

The construction will have practical utility and region of overlap with lattice field theory. One can check predictions of the formalism numerically.