Conformal Geometry in the Bulk

A Boundary Calculus for Conformally Compact Manifolds

Andrew Waldron

Department of Mathematics
University of California

wally@math.ucdavis.edu

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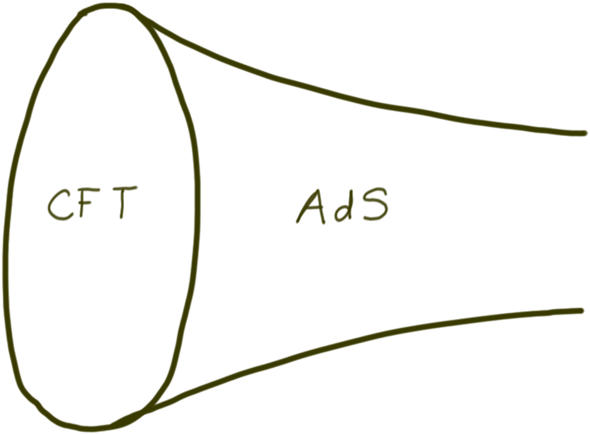
arXiv:1104.2991
with Rod Gover

arXiv:1104.4994, 1007.1724, 1003.3855, 0911.2477, 0903.1394, 0812.3364, 0810.2867
with Roberto Bonezzi, Olindo Corradini, Maxim Grigoriev, Emanuele Latini and Abrar Shaukat

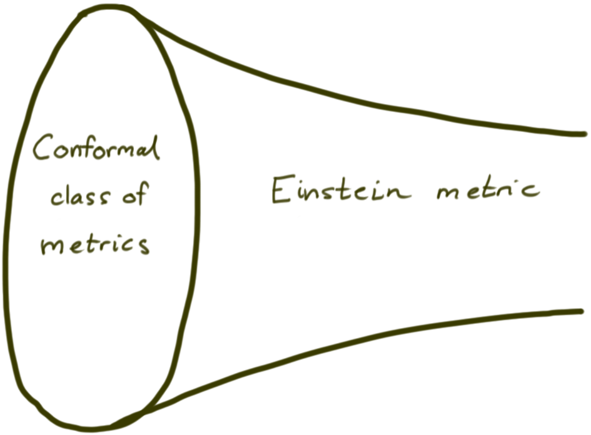
Main Idea

AdS/CFT $\Rightarrow$



Main Idea

or



Main Idea

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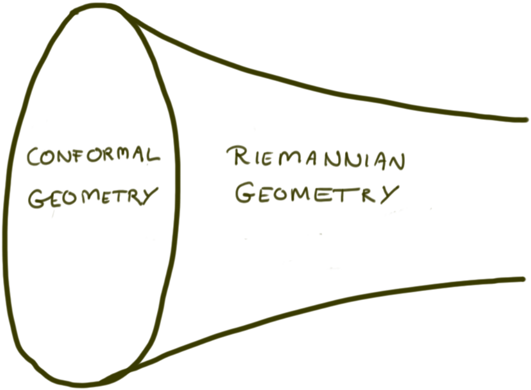
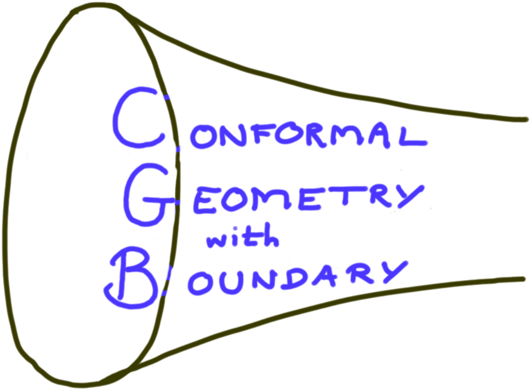


Main Idea

better

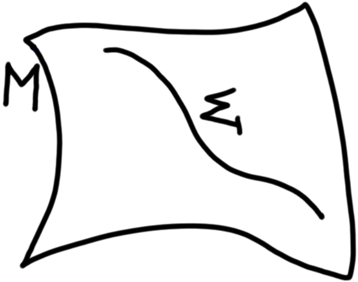


$\Rightarrow$ Solution Generating Algebra

(or "Almost Riemannian Geometry"$^{\diamond}$)

$\diamond$ *A.R. Gover, Almost conformally Einstein manifolds and obstructions, in Diff. Geom. Appl. 247, Matfyzpress, Prague, 2005. arXiv:math/0412393*

Describing the Boundary



$$M = \text{bulk}$$

$$(M, [g]) = \text{conformal manifold}$$

$$g_{\mu\nu} \sim \Omega^2(x) g_{\mu\nu}$$

$$\Sigma = \text{boundary}$$

inherits conformal class of metrics

$$(\Sigma, [g_\Sigma])$$

To say *where* the boundary is introduce an *almost* everywhere positive function $\sigma(x)$.

Σ is the zero locus of σ.

The Scale

Along Σ, the function σ encodes boundary data, in the bulk it is a spacetime varying Planck Mass/ Newton Constant.

- The *scale* $\sigma(x)$ is the gauge field for local choices of units

$$\sigma(x) \sim \Omega(x)\sigma(x)$$

- The double equivalence class $[g_{\mu\nu}, \sigma] = [\Omega^2 g_{\mu\nu}, \Omega\sigma]$ determines a canonical metric g^0 AWAY FROM Σ

$$[g_{\mu\nu}, \sigma] = [g^0_{\mu\nu}, 1]$$

in units $\kappa = 1$.

- Example: AdS

$$ds_0^2 = \frac{dx^2 + h(x)}{x^2}\,, \qquad \sigma_0 = 1\,,$$

but along Σ should use

$$ds^2 = dx^2 + h(x)\,, \qquad \sigma = x\,,$$

well defined at the boundary $\Sigma = \{x = 0\}$.

The Normal Tractor

Along Σ, $\nabla\sigma$ encodes the normal vector n and $\nabla . n$ the mean curvature H

Introduce *scale* and *normal* tractors

$$I = \begin{pmatrix} \sigma \\ \nabla\sigma \\ -\frac{1}{d}(\Delta\sigma + \sigma \mathrm{J}) \end{pmatrix}, \qquad N = \begin{pmatrix} 0 \\ \widehat{n} \\ -H \end{pmatrix},$$

$$d := \dim M\,, \quad R_{\mu\nu\rho\sigma} := W_{\mu\nu\rho\sigma} + (g_{\mu\rho}\mathrm{P}_{\nu\sigma} \pm 3 \text{ more})\,, \quad \mathrm{J} := \mathrm{P}^{\mu}_{\mu}$$

Theorem (Gover)

$$I^2 = 1 \quad \Rightarrow \quad I|_\Sigma = N$$

Away from Σ the scale tractor controls bulk geometry, along Σ it carries boundary information.

Tractors

Probably not surprising that to describe conformal geometry, one should use 6-vectors rather than 4-vectors! Spacetime remains 4-dimensional.

Weight w *tractors* are defined by their gauge transformation w.r.t. conformal transformations

$$T^M := \begin{pmatrix} T^+ \\ T^m \\ T^- \end{pmatrix} \mapsto \Omega^w \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \delta^m_n & 0 \\ -\frac{1}{2\Omega}\Upsilon^2 & -\frac{1}{\Omega}\Upsilon_n & \frac{1}{\Omega} \end{pmatrix} \begin{pmatrix} T^+ \\ T^n \\ T^- \end{pmatrix} =: \Omega^w U^M{}_N T^N ,$$

Here $\Upsilon_\mu := \Omega^{-1}\partial_\mu \Omega$ and $U^M{}_N \in SO(d,2)$.

Tractors give a tensor calculus for conformal geometry$^\diamondsuit$.

Example: $T^2 := 2T^+T^- + T^m T_m$ is a conformal invariant

$^\diamondsuit$ *T.N.Bailey, M.G.Eastwood, A.R.Gover, Thomas's structure bundle for conformal, projective and related structures, Rocky Mountain J.Math. 24 (1994) 1191.*

Parallel Scale Tractors

The tractor bundle is physically *very* interesting because parallel tractors correspond precisely to Einstein metrics.

The *tractor covariant derivative* on weight zero tractors

$$\nabla_\mu T^M := \begin{pmatrix} \partial_\mu T^+ - T_\mu \\ \nabla_\mu T^m + \mathrm{P}_\mu^m T^+ + e_\mu{}^m T^- \\ \partial_\mu T^- - \mathrm{P}_\mu^m T_m \end{pmatrix} \mapsto U^M{}_N \nabla_\mu T^N \,.$$

Theorem (Sasaki; Bailey, Eastwood, Gover; Nurowski)

$(M,[g])$ *conformally Einstein* $\Leftrightarrow$ $(M,[g])$ *admits a parallel scale tractor.*

Proof.

Call $I_M = (\rho, n_m, \sigma)$ and study

$$\nabla_\mu I^M = 0 \Leftrightarrow \begin{cases} \partial_\mu \sigma - n_\mu = 0 \\ \nabla_\mu n_\nu + \mathrm{P}_{\mu\nu}\sigma + g_{\mu\nu}\rho = 0 \\ \partial_\mu \rho - \mathrm{P}_{\mu\nu} n^\nu = 0 \end{cases}$$

□

Physics Dictionary[◇]

No physics depends on local choices of unit systems $\Rightarrow$ express any theory in tractors. Unification a lá 3 $\to$ 4-vectors!

- Einstein–Hilbert action $S[g,\sigma] = \int \frac{\sqrt{-g}}{\sigma^d} I^2$.
- I^M parallel $\Rightarrow I^2 = constant$; this is the cosmological constant!
- Replace derivatives by Thomas D-operator; unifies Laplacian and gradient!

$$D^M := \begin{pmatrix} w(d+2w-2) \\ (d+2w-2)\nabla \\ -\Delta - wJ \end{pmatrix}, \qquad D^M D_M = 0\,.$$

- Weights of tractors = masses; Breitenlohner–Freedman bounds for free!
- Wave equations for tractor tensors $\boxed{I \cdot D\, T = 0}$
- Example $T = \varphi$, weight w scalar, $\sigma = 1$,

$$I \cdot D\varphi = -\Big[\Delta - \frac{2J}{d} w(d+w-1)\Big]\varphi$$

Mass—Weyl-weight relationship $m^2 = -\frac{2J}{d}\Big[\Big(w + \frac{d-1}{2}\Big)^2 - \frac{(d-1)^2}{4}\Big]$.

◇ *Gover, Shaukat, AW*

Tractor Maxwell Theory

- Maxwell Tractor, V^M, weight w with gauge invariance

$$\delta V^M = D^M \xi = (d+2w) \begin{pmatrix} (w+1)\xi \\ \nabla \xi \\ \star \end{pmatrix}$$

- $D_M V^M$ is gauge inert so impose

$$D \cdot V = 0\,, \qquad \text{determines } V^-.$$

- Get gauge transformations of Stückelberg-massive Proca system

$$\delta V^+ = (d+2w)(w+1)\xi\,, \qquad V^m = (d+2w)\nabla^m \xi\,.$$

- Tractor Maxwell Field Strength

$$F^{MN} = D^M V^N - D^N V^M\,, \qquad \text{gauge invariant.}$$

- Equations of motion by coupling to scale

$$J^N = I_M F^{MN} = 0\,, \qquad \text{Proca Equation}$$

- For $w = -1$, V^+ decouples, get standard massless Maxwell.
- For $w = 1 - \frac{d}{2}$, scale tractor decouples!
- In four dimensions $1 - \frac{d}{2} = -1$ so this says Maxwell is Weyl invariant.
- In $d \neq 4$, get Weyl invariant Deser–Nepomechie theory

$$\Delta A_\mu - \frac{4}{d}\nabla^\nu \nabla_\mu A_\nu + \frac{d-4}{4}\Big(2\,\mathrm{P}_{\mu\nu}A^\nu - \frac{d+2}{2}A_\mu\Big) = 0\,.$$

Unify Massive, Massless and Partially Massless Theories

Example

Tractor Gravitons h_{MN} with

$$\delta h_{MN} = D_M \xi_N + D_N \xi_M \,, \qquad D^M h_{MN} - \frac{1}{2} D_N h^M_M = 0 \,,$$

Tractor Christoffels $2\Gamma_{MNR} = D_M h_{NR} + D_N h_{MR} - D_R h_{MN}$

$$G_{MN} = I^R \Gamma_{MNR} = 0 \,, \qquad \text{massive gravitons}$$

- Examine gauge transformations

$$\begin{cases} \delta h^{++} = (d+2w)(w+1)\xi^+ \\ \delta h^{m+} = (d+2w)\big[w\xi^m + \nabla^m \xi^+\big] \\ \delta h^{mn} = (d+2w)\big[\nabla^m \xi^n + \nabla^n \xi^m + \frac{2\mathrm{J}}{d}\eta^{mn}\xi^+\big] \,. \end{cases}$$

- $w = 0$, massless gravitons
- $w = -1$, partially massless gravitons

$$\delta h^{mn} = (d-2)\Big[\nabla^m \nabla^n + \frac{2\mathrm{J}}{d}\eta^{mn}\Big]\xi^+ \,.$$

Boundary Problems

Problem

Given a boundary tractor T_Σ, find a tractor T on M such that

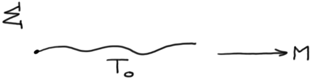
$$T|_\Sigma = T_\Sigma \qquad \textit{and} \qquad I \cdot D\ T = 0\,.$$

In a given Weyl frame, this is a Laplace type problem, so *could* just choose coordinates and study the resulting PDE.

Method

- Boundary data

- Extend T_Σ *arbitrarily* to T_0 with $T_0|_\Sigma = T_\Sigma$



- Iteratively find $T^{(1)}, T^{(2)}, \ldots$ approaching solution T, with $T^{(I)}|_\Sigma = T_\Sigma$

- Check solution T is *independent* of original choice of extension T_0.

Boundary Calculus

Observe that because $T|_\Sigma = T_\Sigma = T_0|_\Sigma$

$$T = T_0 + \sigma S\,, \qquad \text{for some } S.$$

This suggests to search for an expansion in the scale!

$$T^{(l)} = T_0 + \sigma T_1 + \sigma^2 T_2 + \cdots + \sigma^l T_l\,.$$

Need algebra of $I \cdot D$ and σ, remarkably

$$\boxed{[I \cdot D, \sigma] = (d + 2w) I^2}$$

Or calling $x := \sigma$, $h := d + 2w$, $y = -\frac{1}{I^2} I \cdot D$ we have the $\mathfrak{sl}(2)$ *solution generating* algebra

$$\boxed{[x, y] = h\,, \qquad [h, x] = 2x\,, \qquad [h, y] = -2y}$$

The Solution

Example

Given weighted tractor $hT = h_0 T$ and $T = T_0 + xT_1 + x^2 T_2 + \cdots$ then

$$yT = yT_0 - (h_0 - 2)T_1 + x\big(yT_1 - 2(h_0 - 3)T_2\big) + \cdots ,$$

so

$$T = T_0 + \frac{1}{h_0 - 2}\, x\, yT_0 + \frac{1}{2(h_0 - 2)(h_0 - 3)}\, x^2 y^2 T_0 + \cdots .$$

All order solution given by *solution generating operator*

$$\boxed{T = \; :K(z): \; T_0}$$

with

$$K(z) = z^{\frac{h_0 - 1}{2}} \Gamma(2 - h_0) J_{1-h_0}(2\sqrt{z}) = 1 + \frac{1}{h_0 - 2} z + \frac{1}{2(h_0 - 2)(h_0 - 3)} z^2 + \cdots$$

and $z = xy$ with normal ordering

$$: z^k : \; = x^k y^k .$$

Tangential Operators

We call a bulk operator $\mathcal{O}$ *tangential* if

$$\mathcal{O}\sigma = \sigma\mathcal{O}' \,, \qquad \text{for some } \mathcal{O}'$$

Example

- The tangential derivative $\nabla^T := \nabla - n\nabla_n$
- The solution generating operator $: K :$ obeys

$$: K : x = 0$$

  for same reason that $y : K : \; = 0$.

- The *holographic GJMS operator*

$$y^k \,, \qquad k \in 2\mathbb{N}$$

  acting on tractors with weight $h_0 = k + 1$, because $[x, y^k] = y^{k-1}k(h - k + 1)\,.$

When bulk operators are tangential they define boundary operators since $\mathcal{O}T|_\Sigma$ is independent of how T_Σ is extended to T.

Obstructions and Anomalies

When $h_0 = 2, 3, \ldots$, so $w + \frac{d}{2} = 1, \frac{3}{2}, 2, \ldots$, the recurrence

$$T_k = \frac{1}{k(h_0 - k - 1)} y T_{k-1} \qquad \textit{fails} \text{ for } T_{h_0 - 1} \,.$$

N.B., usually $T_k \sim y^k T_0$, so the operator y^k is the obstruction.

- For conformally Einstein bulk and $h_0 = 2, 4, 6, \ldots$, y^k vanishes.
- For $h_0 = 3, 5, 7, \ldots$ the tangential operator y^k is a holographic formula for the GJMS operator

$$P_{2k} = \Delta^k + \text{curvatures}\,.$$

  Conformally invariant boundary operators corresponding to conformal anomalies.

- Acting on log densities[1] y^{d-1} yields Branson's Q-curvature

$$Q = \frac{1}{((d-2)!!)^2}\, y^{d-1} U|_\Sigma \,.$$

  Holographic anomalies of Henningson–Skenderis.

[1] Under conformal transformations $U \mapsto U + \log \Omega$.

Log solutions

Using $[y, x^k] = -x^{k-1}k(h+k-1)$ we learn

$$y : K(z) : \; = \; : \Big(zK''(z) + K'(z)(2-h) + K(z) \Big) : y \,,$$

Operator problem now an ODE—Bessel type-equation solvable by Frobenius method:

- Second solution $= z^{h_0 - 1}\big($first solution $h_0 \to 2 - h_0\big)$
- $h_0 \in \mathbb{N}$, Log solution $= \big($degree $h_0 - 2$ polynomial$\big)$ $+ \; z^{h_0-1} \log z \times \big($second solution$\big) +$ "finite terms"
- Log solution requires second scale τ, at definite weight only $\log(\sigma/\tau)$ can appear. $\tau|_\Sigma \neq 0$
- $\log z$ is completely formal because $z = xy$, but $\log \tau$ can play rôle of "$\log y$".
- Algebra of $\log x$,

$$[y, \log x] = -\frac{1}{x}(h-1)$$

- Must also require solution generating operator to be tangential.

The Solution

Remarkably, can solve to all orders at log weights

$$T = \mathcal{O}\, T_0\,, \qquad \text{solves } I \cdot D\, T = 0$$

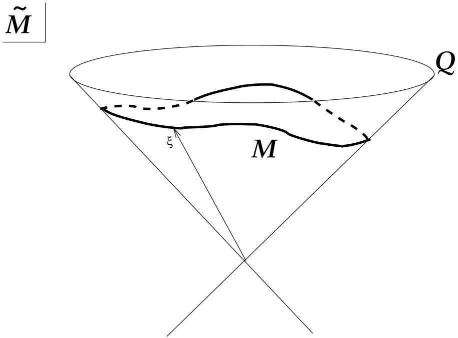
$$\mathcal{O} = \;:F_{h_0-2}(z):\; - \;\frac{:z^{h_0} B(z):}{(h_0-1)!(h_0-2)!}$$
$$- \;\frac{x^{h_0-1}\log x \;\; :K_{h_0}(z):\; y^{h_0-1} \;-\; x^{h_0-1}\;\; :K_{h_0}(z):\; \Big[\log\tau \;\; y^{h_0-1}\Big]_W}{(h_0-1)!(h_0-2)!}$$

- $F_{h_0-2} = 1 + \cdots$ is the standard solution up to orders before obstruction—"infinities".
- log terms multiply second solution $K_{h_0} = 1 + \cdots$
- careful Weyl ordering of operators y and $\log\tau$ ensures tangentiality $\mathcal{O}x = 0$.
- $B = 1 + \cdots$ are non-log finite terms. Explicit formulae for all terms.

Solution of wave equation boundary problem for arbitrary tensors in *any* curved bulk.

Ambient Tractors

Flat model for conformal manifold



- Ambient space $\widetilde{M} = \mathbb{R}^{d+1,1}$
- Lightcone $Q = \{X^M X_M = 0\}$
- Conformal manifold $M = \{\text{lightlike rays}\}$

Momentum Cone

Tractors are equivalence classes of weighted ambient tensors T (Gover, Peterson, Čap)

$$T \sim T + X^2 S\,, \qquad X^M \nabla_M T = wT\,.$$

Tractor operators respect equivalence classes

$$\mathcal{O} X^2 = X^2 \mathcal{O}'$$

Fundamental operators $\leftrightarrow$ momentum space representation of the ambient conformal group (Gover, AW)

Canonical Tractor	X^M
Weight	$w = \nabla_X$
Double D-operator	$D_{MN} = X_N \nabla_M - X_M \nabla_N$
Thomas D-operator	$D_M = \nabla_M (d + 2\nabla_X - 2) - X_M \Delta$

Curved Cone

Metric on curved ambient space $(\widetilde{M}, g_{MN})$:

$$g_{MN} = \nabla_M X_N$$

where X is now hypersurface orthogonal homothetic vector field.

Consequences:

$$\mathcal{L}_X g_{MN} = 2g_{MN}\,, \quad \nabla_{[M} X_{N]} = 0\,, \quad X_M = \nabla_M \frac{1}{2} X^2\,, \quad g_{MN} = \frac{1}{2} \nabla_M \partial_N X^2\,.$$

So X^2 is homothetic potential and defines a curved cone.

Define tractors as before $\Rightarrow$ arbitrary curved space.

Remarkably have an $\mathfrak{sl}(2) \cong \mathfrak{sp}(2)$ algebra from operators (GJMS)

$$\mathbb{Q} = \begin{pmatrix} X^2 & \nabla_X + \frac{d+2}{2} \\ \nabla_X + \frac{d+2}{2} & \Delta \end{pmatrix}\,, \qquad [Q_{ij}, Q_{kl}] = \epsilon_{kj} Q_{ik} + \text{ (3 more)}$$

Two Times Physics

Itzhak Bars:

$$H \to Q_{ij}\,, \qquad \mathbb{R}^{d-1,1} \to \mathbb{R}^{d,2}$$

because *Howe dual pair*

$$\mathfrak{sp}(2) \otimes \mathfrak{so}(d,2) \subset \mathfrak{sp}\big(2(d+2)\big)$$

Particle action

$$S = \int dt \Big[P_M \dot{X}^M - \lambda^{ij} Q_{ij} \Big]\,, \quad \mathbb{Q} = \begin{pmatrix} X^2 & X.P \\ X.P & P^2 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} \text{relativistic particle} \\ \text{AdS particle} \\ H\text{-atom} \\ \text{Harmonic Oscillator} \\ \quad \vdots \end{array} \right.$$

Bars proposed

$$\text{Gravity} \leftrightarrow \left\{ \begin{array}{c} \text{triplets of Hamiltonians in } 2(d+2) \text{ dimensional phase space} \\ \text{obeying } \mathfrak{sp}(2) \text{ algebra} \end{array} \right\}$$

Confirm this proposal using tractors! (Bonezzi, Latini, AW)

Gravity

$$\begin{cases} [Q_{ij}, Q_{kl}] = \epsilon_{kj} Q_{il} + \epsilon_{ki} Q_{jl} + \epsilon_{lj} Q_{ik} + \epsilon_{li} Q_{jk} \\ \mathbb{Q}\Psi = 0 \end{cases}$$

- Symplectic Gauge Invariance

$$\mathbb{Q} \mapsto \mathbb{Q} + [\mathbb{Q}, \epsilon]\,, \qquad \Psi \mapsto \Psi + \epsilon\Psi$$

- Expand $\mathbb{Q}, \epsilon$ in powers of operator ∇ $\rightarrow$ infinitely many fields
- Solve $\mathfrak{sp}(2)$ conditions

$$\mathbb{Q} = \begin{pmatrix} X^M G_{MN} X^M & X^M(\nabla_M + A_M) + \frac{d+2}{2} \\ X^M(\nabla_M + A_M) + \frac{d+2}{2} & (\nabla^M + A^M) G_{MN} (\nabla^N + A^N) \end{pmatrix},$$

$$\epsilon = \alpha + \xi^M(\nabla_M + A_M)\,,$$

with $G_{MN} = \nabla_M X_N$, $X^M F_{MN}(A) = 0$.

Many Actions

- Lagrange multipliers for Hamiltonian constraints

$$S(G_{MN},A_M,\Psi,\Omega,\Theta,\Lambda)=\int\sqrt{G}\Big(\Omega\widetilde{\nabla}^2+\Theta\big[X.\widetilde{\nabla}+\frac{d+2}{2}\big]+\Lambda X^2\Big)\Psi$$

- Θ fixes weight $\nabla_X\Psi=(w-\frac{d}{2}-1)\Psi$
- Λ says $\Psi=\delta(X^2)\phi$ so $\phi\sim\phi+X^2\chi$
- $\Rightarrow S=\int\sqrt{G}\delta(X^2)T(G,A,\Omega,\phi)$
- $T=\phi\big(\nabla+A\big)^2\Omega$ must be a tractor: in Maxwell gauge $X.A=w$

$$T=\phi\Big(\frac{1}{w}A^M D_M-\frac{1}{d-2}(D^M A_M)+A^2\Big)\Omega$$

- T tractor $\Rightarrow$ d-dimensional action $S=\int\sqrt{-g}\,T$
- Residual $SO(1,1)$ gauge invariance

$$\delta\Omega=\alpha\Omega\,,\qquad\delta\phi=-\alpha\phi\,,\qquad\delta A^M=\frac{1}{d-2}D^m\alpha$$

- Singlet $\Omega\phi =: \varphi^2$ is gauge invariant.
- Integrate out A_M leaves only φ and metric

$$S = \int \sqrt{-g}\varphi\Big[\Delta - \frac{d-2}{2}\mathsf{J}\Big]\varphi$$

CONFORMALLY IMPROVED SCALAR

- In terms of scale

$$\varphi = \sigma^{1-\frac{d}{2}}$$

- Tractor Einstein–Hilbert action

$$S = \int \frac{\sqrt{-g}}{\sigma^d} I^2$$

Weyl invariant

- Choose $\sigma = 1$,

$$S = \int \sqrt{-g}R\,.$$

Conclusions and Outlook

- Transversal tensors.
- Ubiquity of $I \cdot D$, harmonic Weyl tensor $\rightarrow I.\not{D} W_{MNRS} = 0$ for Weyl tractor.
- Global solution?
- Correlator calculus.
- Two times and dualities.