Scale without Conformal Invariance

Jean-François Fortin

Department of Physics, University of California, San Diego
La Jolla, CA

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University of California, Davis

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with Benjamín Grinstein and Andreas Stergiou

Scale and conformal invariance in two dimensions

Does scale invariance imply conformal invariance ?

- Polchinski following Zamolodchikov Polchinski (1988) & Zamolodchikov (1986)
  - Unitarity
  - Finiteness of EM tensor correlation functions
  - $\Rightarrow$ Scale invariance implies conformal invariance from conservation of EM tensor

"Counter" examples

- Non-linear σ model Hull, Townsend (1986)
  - Non-existence of EM tensor two-point correlation functions
- Theory of elasticity Cardy, Riva (2005)
  - Non-reflection-positive

Scale invariance implies conformal invariance

Scale and conformal invariance in $d > 2$ dimensions

Does scale invariance imply conformal invariance ?

- No proof *à la* Polchinski
  - Conservation of EM tensor $\Rightarrow$ Not enough information

  $\Rightarrow$ Scale invariance does not necessarily imply conformal invariance

No interesting counterexamples

- AdS/CFT Kerr-AdS black holes in $d = 5, 7$ dimensions Awad, Johnson (1999)
  - Conformal invariance broken to scale invariance by black hole rotation
- Maxwell theory in $d \neq 4$ dimensions Jackiw, Pi (2011) & El-Showk, Nakayama, Rychkov (2011)
  - Free field theory
  - Scale invariance broken by interactions

Study of non-conformal scale-invariant QFTs

Scale invariance does not necessarily imply conformal invariance but no proper counterexamples $\Rightarrow$ Possible proof !

- Without proper counterexamples
  - Physical implications of non-conformal scale-invariant QFTs (correlation functions in non-conformal scale-invariant QFTs versus CFTs ?)
- With proper counterexamples
  - Scale invariance conditions weaker than conformal invariance conditions (plentiful examples ?)

Uncharted territory !

Outline

1. Historical review
2. Scale versus conformal invariance
   - Preliminaries
   - Scale invariance and new improved energy-momentum tensor
   - RG flows along scale-invariant trajectories
   - Scale invariance and recurrent behaviors
   - Scale invariance, gradient flows and *a*-theorem
   - Why dilatation generators generate dilatations
3. Scale-invariant trajectories
   - Polchinski–Dorigoni–Rychkov argument
   - Systematic approach
   - Scheme-dependence of β-functions
   - Examples
4. Discussion and conclusion
   - Features and future work

Preliminaries ($d > 2$)

- Dilatation current Polchinski (1988)
  - $\mathcal{D}^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)$
  - $T_\nu{}^\mu(x)$ any symmetric EM tensor following from spacetime nature of scale transformations
  - $V^\mu(x)$ local operator (virial current) contributing to scale dimensions of fields
  - Freedom in choice of $T_\nu{}^\mu(x)$ compensated by freedom in choice of $V^\mu(x)$
- Scale invariance $\Rightarrow$ $T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$

- Conformal current Polchinski (1988)
  - $\mathcal{C}^{\mu}_{\nu}(x) = v^{\nu}(x) T_{\nu}{}^{\mu}(x) - \partial \cdot v(x) V'^{\mu}(x) + \partial_{\nu}\partial \cdot v(x) L^{\nu\mu}(x)$
  - $T_{\nu}{}^{\mu}(x)$ any symmetric EM tensor following from spacetime nature of conformal transformations
  - $V'^{\mu}(x)$ local operator corresponding to ambiguity in choice of dilatation current
  - $L^{\nu\mu}(x)$ local symmetric operator correcting position dependence of scale factor
  - $\partial \cdot v(x)$ scale factor (general linear function of x^{μ})
  - Freedom in choice of $T_{\nu}{}^{\mu}(x)$ compensated by freedom in choice of $V'^{\mu}(x)$ and $L^{\nu\mu}(x)$

- Conformal invariance $\Rightarrow$ $T_{\mu}{}^{\mu}(x) = \partial_{\mu} V'^{\mu}(x) = \partial_{\mu}\partial_{\nu} L^{\nu\mu}(x)$

- Conformal invariance $\Rightarrow$ Existence of symmetric traceless energy-momentum tensor

Scale without conformal invariance

Non-conformal scale-invariant QFTs Polchinski (1988)

- Scale invariance $\Rightarrow$ $T_{\mu}{}^{\mu}(x) = \partial_{\mu} V^{\mu}(x)$
- Conformal invariance $\Rightarrow$ $T_{\mu}(x) = \partial_{\mu}\partial_{\nu} L^{\nu\mu}(x)$
- Scale without conformal invariance

  $\Rightarrow$ $T_{\mu}{}^{\mu}(x) = \partial_{\mu} V^{\mu}(x)$ where $V^{\mu}(x) \neq J^{\mu}(x) + \partial_{\nu} L^{\nu\mu}(x)$ with $\partial_{\mu} J^{\mu}(x) = 0$
- Constraints on possible virial current candidates
  - Gauge-invariant spatial integral
  - Fixed $d-1$ scale dimension in d spacetime dimensions
- No suitable virial current $\Rightarrow$ Scale invariance implies conformal invariance (examples: ϕ^p in $d = n - \epsilon$ for $(p, n) = (6, 3), (4, 4)$ and $(3, 6)$)

Virial current candidates

Most general classically scale-invariant renormalizable theory in $d = 4$ spacetime dimensions Jack, Osborn (1985)

$$\begin{aligned}\mathcal{L} = &- \mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F^A_{\mu\nu} F^{A\mu\nu} + \tfrac{1}{2} Z^{\frac{1}{2}}_{ab} Z^{\frac{1}{2}}_{ac} D_\mu \phi_b D^\mu \phi_c \\ &+ \tfrac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} \bar{\psi}_j i \bar{\sigma}^\mu D_\mu \psi_k - \tfrac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} D_\mu \bar{\psi}_j i \bar{\sigma}^\mu \psi_k \\ &- \tfrac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d \\ &- \tfrac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \tfrac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j\end{aligned}$$

- $A^A_\mu(x)$ gauge fields
- $\phi_a(x)$ real scalar fields
- $\psi^\alpha_i(x)$ Weyl fermions
- Dimensional regularization ($d = 4 - \epsilon$)

Virial current candidates and new improved EM tensor

- Virial current $V^\mu(x) = Q_{ab}\phi_a D^\mu \phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}^\mu \psi_j$
  - $Q_{ba} = -Q_{ab}$
  - $P^*_{ji} = -P_{ij}$
- New improved energy-momentum tensor $\Theta_\nu{}^\mu(x)$ Callan, Coleman, Jackiw (1970)
  - Finite
  - Not renormalized
  - Anomalous trace Robertson (1991)

$$\begin{aligned}\Theta_\mu{}^\mu(x) = & \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + \gamma_{aa'} D^2 \phi_a \phi_{a'} \\ & - \gamma^*_{i'i} \bar{\psi}_i i\bar{\sigma}^\mu D_\mu \psi_{i'} + \gamma_{ii'} D_\mu \bar{\psi}_i i\bar{\sigma}^\mu \psi_{i'} \\ & - \frac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} \\ & \qquad - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\ & - \frac{1}{2}(\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'})\phi_a\psi_i\psi_j + \text{h.c.}\end{aligned}$$

- β-functions from vertex corrections and wavefunction renormalizations ($d = 4$ spacetime dimensions)
  - RG time $t = \ln(\mu_0/\mu)$

$$\beta_A = -\frac{dg_A}{dt} = \gamma_A g_A \quad \text{(no sum)}$$

$$\begin{aligned}\beta_{abcd} &= -\frac{d\lambda_{abcd}}{dt} \\ &= -(\lambda\gamma^\lambda)_{abcd} + \gamma_{a'a}\lambda_{a'bcd} + \gamma_{b'b}\lambda_{ab'cd} + \gamma_{c'c}\lambda_{abc'd} + \gamma_{d'd}\lambda_{abcd'}\end{aligned}$$

$$\beta_{a|ij} = -\frac{dy_{a|ij}}{dt} = -(y\gamma^y)_{a|ij} + \gamma_{a'a}y_{a'|ij} + \gamma_{i'i}y_{a|i'j} + \gamma_{j'j}y_{a|ij'}$$

- Divergence of dilatation current

$$\begin{aligned}\partial_\mu \mathcal{D}^\mu(x) =& \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} \\ & - (\gamma^*_{i'i} + P^*_{i'i}) \bar{\psi}_i i \bar{\sigma}^\mu D_\mu \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_\mu \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'} \\ & - \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a} \lambda_{a'bcd} - \gamma_{b'b} \lambda_{ab'cd} \\ & \qquad - \gamma_{c'c} \lambda_{abc'd} - \gamma_{d'd} \lambda_{abcd'}) \phi_a \phi_b \phi_c \phi_d \\ & - \frac{1}{2} (\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'}) \phi_a \psi_i \psi_j + \text{h.c.}\end{aligned}$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$\begin{aligned}\beta_A &= 0 \\ \beta_{abcd} &= -Q_{a'a} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'} \\ \beta_{a|ij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'}\end{aligned}$$

- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$\beta_A = \beta_{abcd} = \beta_{a|ij} = 0$$

Interlude: Current conservation

- Divergence of current $J^\mu(x)$ without use of EOMs Collins (1984)

$$\partial_\mu J^\mu(x) = \Delta_{\mathrm{EOM}} + \Delta_{\mathrm{Classical}} + \Delta_{\mathrm{Anomaly}}$$

- Green's function of elementary fields with current $J^\mu(x)$ and Ward identity

  ✔ $\Delta_{\mathrm{EOM}} \Rightarrow$ Expected contact terms from Ward identity
  ✗ $\Delta_{\mathrm{Classical}} \Rightarrow$ Usual non-anomalous classical violation
  ✗ $\Delta_{\mathrm{Anomaly}} \Rightarrow$ Possible anomalous violation in divergent Green's function

- Example: $\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} + \bar{\psi}_i (i\gamma^\mu D_\mu \delta_{ij} - M_{ij}) \psi_j$
  - Vector current $J^{\mu a}_V(x) = \bar{\psi}\gamma^\mu t^a \psi$ with $\Delta_{\mathrm{EOM}} \neq 0$, $\Delta_{\mathrm{Classical}} = i\bar{\psi}[M, t^a]\psi$ and $\Delta_{\mathrm{Anomaly}} = 0$
  - Axial current $J^{\mu a}_A(x) = \bar{\psi}\frac{1}{2}[\gamma^\mu, \gamma^5] t^a \psi$ with $\Delta_{\mathrm{EOM}} \neq 0$, $\Delta_{\mathrm{Classical}} = i\bar{\psi}\gamma^5\{M, t^a\}\psi$ and $\Delta_{\mathrm{Anomaly}} = \frac{1}{2}\bar{\psi}\{\gamma^\mu, \gamma^5\} t^a D_\mu \psi - \frac{1}{2} D_\mu \bar{\psi}\{\gamma^\mu, \gamma^5\} t^a \psi$

Virial current and unitarity bounds

- New improved energy-momentum tensor ⇒ Finite and not renormalized Callan, Coleman, Jackiw (1970)
- Operators related to EOMs ⇒ Finite and not renormalized Politzer (1980) & Robertson (1991)
- Virial current ⇒ Finite and not renormalized
  - Unconserved current with scale dimension exactly 3
- Unitarity bounds for conformal versus scale-invariant QFTs Grinstein, Intriligator, Rothstein (2008)
- Non-trivial virial current ⇒ Non-conformal scale-invariant QFTs

RG flows along scale-invariant trajectories

Scale-invariant solution $(g_A, \lambda_{abcd}, y_{a|ij}) \Rightarrow$ RG trajectory

$$\bar{g}_A(t) = g_A$$
$$\bar{\lambda}_{abcd}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'}$$
$$\bar{y}_{a|ij}(t) = \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'}$$

$$\widehat{Z}_{aa'}(t) = \left(e^{Qt}\right)_{aa'}$$
$$\widehat{Z}_{ii'}(t) = \left(e^{Pt}\right)_{ii'}$$

- $(\bar{g}_A(t,g,\lambda,y), \bar{\lambda}_{abcd}(t,g,\lambda,y), \bar{y}_{a|ij}(t,g,\lambda,y))$ also scale-invariant solution
- Q_{ab} and P_{ij} constant along RG trajectory
- $\widehat{Z}_{ab}(t)$ orthogonal and $\widehat{Z}_{ij}(t)$ unitary $\Rightarrow$ Always non-vanishing β-functions along scale-invariant trajectory

Scale invariance and recurrent behaviors

RG flows along scale-invariant trajectories $\Rightarrow$ Recurrent behaviors !
Lorenz (1963,1964), Wilson (1971) & Kogut, Wilson (1974)

- Virial current $\Rightarrow$ Transformation in symmetry group of kinetic terms $(SO(N_S) \times U(N_F))$
    - Q_{ab} antisymmetric and P_{ij} antihermitian $\Rightarrow$ Purely imaginary eigenvalues
    - $\widehat{Z}_{ab}(t)$ and $\widehat{Z}_{ij}(t)$ in $SO(N_S) \times U(N_F)$

$\Rightarrow$ Periodic (limit cycle) or quasi-periodic (ergodicity) scale-invariant trajectories

Recurrent behaviors

Intuition from $\mathcal{D}^{\mu}(x) = x^{\nu}\Theta_{\nu}{}^{\mu}(x) - V^{\mu}(x)$

- RG flow $\Rightarrow$ Generated by scale transformation $(x^{\nu}\Theta_{\nu}{}^{\mu}(x))$
- RG flow $\Rightarrow$ Related to virial current through conservation of dilatation current
- Virial current $\Rightarrow$ Generates internal transformation of the fields
  - Internal transformation in compact group $SO(N_S) \times U(N_F)$

  $\Rightarrow$ Rotate back to or close to identity
- RG flow return back to or close to identity $\Rightarrow$ Recurrent behavior

Scale-invariant trajectories ?

RG flows ∼ Field redefinitions ⇒ Scale-invariant trajectories or fixed points ?

- RG-time-dependent field redefinitions ⇒ Generates RG flows
  Wegner (1974) & Latorre, Morris (2001)
  - RG-time-dependent field redefinitions ⇒ All exact RG flows (Wilson, Wegner, Polchinski, etc.)

β-function operators ∼ Redundant operators ⇒ Scale-invariant trajectories or fixed points ?

- Wavefunction renormalization operators ⇒ Redundant operators
  - Redundant β-function operators necessary for scale invariance

Non-conformal scale-invariant QFTs ⇒ Non-trivial RG flows (recurrent behaviors)

Scale invariance, gradient flows and a-theorem

- Gradient flow

$$\beta_i(g) = -\frac{dg_i}{dt} = G_{ij}(g)\frac{\partial c(g)}{\partial g_j}$$

  - G_{ij} positive-definite metric
  - Potential $c(g)$ function of couplings

- Potential $c(g)$ monotonically decreasing along RG trajectory

$$\frac{dc(g(t))}{dt} = -G^{ij}(g)\beta_i\beta_j \leq 0$$

  - Recurrent behaviors (scale-invariant trajectories) $\not\Leftrightarrow$ Gradient flows (scale implies conformal invariance) Wallace, Zia (1975)

- a-theorem Barnes, Intriligator, Wecht, Wright (2004)
  - RG flow $\Rightarrow$ Irreversible process (integrating out DOFs)
  - $c(g) \sim$ measure of number of massless DOFs
  - a-theorem $\Rightarrow$ weak ($c_{IR} < c_{UV}$), stronger ($\frac{dc}{dt} \leq 0$), ~~strongest~~ (RG flows as gradient flows)

Why dilatation generators generate dilatations

Dilatation generators do not generate dilatations in non-scale-invariant QFTs Coleman, Jackiw (1971)

- Quantum anomalies at low orders
  - Anomalous dimensions
  - ⇒ Possible to absorb into redefinition of scale dimensions of fields
  - ✔ Preserve scale invariance
- Quantum anomalies at high orders
  - β-functions
  - ⇒ Not possible to absorb
  - ✗ Break scale invariance

Why dilatation generators generate dilatations in non-conformal scale-invariant QFTs ?

- β-functions on scale-invariant trajectories
  - Both vertex correction and wavefunction renormalization contributions
  - Very specific form for vertex correction contribution
  - Equivalent in form to wavefunction renormalization contribution (redundant operators)

  $\Rightarrow$ Also possible to absorb into redefinition of scale dimensions of fields

  ✓ Preserve scale invariance !

Ward identity for scale invariance

Callan-Symanzik equation for effective action Callan (1970) & Symanzik (1970)

$$\left[M\frac{\partial}{\partial M}+\beta_i\frac{\partial}{\partial g_i}+\gamma_j^{\ i}\int d^4x\,\varphi_i(x)\frac{\delta}{\delta\varphi_j(x)}\right]\Gamma[\varphi(x),g,M]=0$$

- In non-scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✗ β-functions
- In CFTs
  - ✔ Anomalous dimensions
  - ✔ Vanishing β-functions

$$\left[M\frac{\partial}{\partial M}+(\gamma_j^{\ i}+Q_j^{\ i})\int d^4x\,\varphi_i(x)\frac{\delta}{\delta\varphi_j(x)}\right]\Gamma[\varphi(x),g,M]=0$$

- In non-conformal scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✔ β-functions (redundant operators)

Poincaré algebra augmented with dilatation charge

- β-functions on scale-invariant trajectories
  - Quantum-mechanical generation of scale dimensions
  - Appropriate scale dimensions required by virial current
  - $\Rightarrow$ Conserved dilatation current $\mathcal{D}^\mu(x)$
- Poincaré algebra with dilatation charge $D = \int d^3x\, \mathcal{D}^0(x)$

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[D, P_\mu] &= -iP_\mu
\end{aligned}$$

- Algebra action on fields $\mathcal{O}_I(x)$

$$\begin{aligned}
[M_{\mu\nu}, \mathcal{O}_I(x)] &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\mathcal{O}_I(x) \\
[P_\mu, \mathcal{O}_I(x)] &= -i\partial_\mu\mathcal{O}_I(x) \\
[D, \mathcal{O}_I(x)] &= -i(x\cdot\partial + \Delta)\mathcal{O}_I(x)
\end{aligned}$$

- New classical scale dimensions of fields due to virial current

$$[D, \phi_a(x)] = -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x)$$
$$[D, \psi_i(x)] = -i(x \cdot \partial + \tfrac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x)$$

  - How do non-conformal scale-invariant QFTs know about new scale dimensions ?

  $\Rightarrow$ Generated by β-functions !

- Quantum-mechanical scale dimensions of fields

$$\Delta_{ab} = \delta_{ab} + \gamma_{ab} + Q_{ab}$$
$$\Delta_{ij} = \tfrac{3}{2}\delta_{ij} + \gamma_{ij} + P_{ij}$$

Scale-invariant trajectories ??

β-functions $\sim$ Anomalous dimensions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Shift β-functions away $\Rightarrow$ Scheme change
  - ✗ Non-conformal scale-invariant QFTs with traceless EM tensor
- Shift β-functions away $\Rightarrow$ Global shift
  - ✗ Conformal fixed points become conformal trajectories

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows

Non-conformal scale-invariant correlation functions

- Scalar fields $\mathcal{O}_I(x)$ with scale dimensions Δ_I

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\rangle = \frac{g_{IJ}}{(x_1 - x_2)^{\Delta_I+\Delta_J}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \sum_{\substack{\delta_1+\delta_2+\delta_3= \\ \Delta_I+\Delta_J+\Delta_K}} \frac{c_{IJK}^{\delta_1\delta_2\delta_3}}{(x_1 - x_2)^{\delta_1}(x_2 - x_3)^{\delta_2}(x_3 - x_1)^{\delta_3}}$$

  - Non-vanishing two-point functions with $\Delta_I \neq \Delta_J$ contrary to CFTs

- Two-point correlation functions of fundamental real scalar fields

$$\langle \phi_a(x)\phi_b(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\, G^\phi\, (x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$

  - G^ϕ constant real symmetric matrix

Polchinski–Dorigoni–Rychkov argument at one loop

Non-conformal scale-invariant β-functions

$$\beta_{abcd} = \mathcal{Q}_{abcd}$$
$$\beta_{a|ij} = \mathcal{P}_{a|ij}$$

$$\mathcal{Q}_{abcd} = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}$$
$$\mathcal{P}_{a|ij} = -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'}$$

- Real scalar fields only Polchinski (1988)
  - $\mathcal{Q}_{abcd}\beta_{abcd}^{(\text{one-loop})} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

  $\Rightarrow$ Scale invariance implies conformal invariance
- Real scalar fields and Weyl fermions Dorigoni, Rychkov (2009)
  - $\mathcal{P}_{a|ij}^{*}\beta_{a|ij}^{(\text{one-loop})} = 0 \Rightarrow \mathcal{P}_{a|ij} = 0$
  - $\mathcal{Q}_{abcd}\beta_{abcd}^{(\text{one-loop})} = 0$ using $\mathcal{P}_{a|ij} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

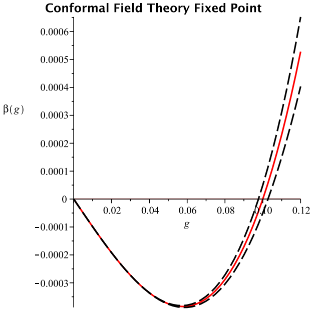
  $\Rightarrow$ Scale invariance implies conformal invariance

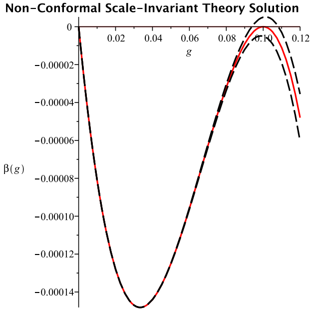
Polchinski–Dorigoni–Rychkov argument at two loops

- Real scalar fields only JFF, Grinstein, Stergiou (2011)
  - $\mathcal{Q}_{abcd}\beta_{abcd}^{(\text{two-loop})}=0\Rightarrow\mathcal{Q}_{abcd}=0$
  - $\Rightarrow$ Scale invariance implies conformal invariance
- One real scalar field only and Weyl fermions *ibid*
  - $\mathcal{P}_{a|ij}^{*}\beta_{a|ij}^{(\text{two-loop})}=0\Rightarrow\mathcal{P}_{a|ij}=0$
  - $\mathcal{Q}_{abcd}\equiv0$
  - $\Rightarrow$ Scale invariance implies conformal invariance (also at all loops)
- Real scalar fields and Weyl fermions *ibid*
  - $\mathcal{P}_{a|ij}^{*}\beta_{a|ij}^{(\text{two-loop})}\neq0$
  - $\Rightarrow$ Scale invariance does NOT imply conformal invariance
  - Obstruction due to $y^3\lambda$ and $y\lambda^2$ terms (also obstruction to gradient flow interpretation Wallace, Zia (1975))

Schematically,

- Conformal field theory fixed point
  - Stable with respect to higher-order corrections
- Non-conformal scale-invariant theory solution
  - Would-be conformal fixed point at lowest order unstable with respect to higher-order corrections





Systematic approach

Scale-invariant trajectories at weak coupling

$$g_A = \sum_{n\geq 1} g_A^{(n)} \epsilon^{n-\frac{1}{2}} \qquad \lambda_{abcd} = \sum_{n\geq 1} \lambda_{abcd}^{(n)} \epsilon^{n} \qquad y_{a|ij} = \sum_{n\geq 1} y_{a|ij}^{(n)} \epsilon^{n-\frac{1}{2}}$$

$$Q_{ab} = \sum_{n\geq 2} Q_{ab}^{(n)} \epsilon^{n} \qquad P_{ij} = \sum_{n\geq 2} P_{ij}^{(n)} \epsilon^{n}$$

- ϵ small parameter
  - Obvious choice in $d = 4 - \epsilon$
  - One-loop gauge coupling β-function coefficient in $d = 4$ Banks, Zaks (1982)
- Form of expansions determined by β-functions
  - For coupling constants $\Rightarrow$ Lowest-order terms in β-functions (would-be conformal fixed points)
  - For virial current $\Rightarrow$ Higher-order terms in β-functions due to Polchinski–Dorigoni–Rychkov argument

Interlude: Scheme-(in)dependence of β-functions

Change of scheme $\bar{\lambda} = \lambda + a^{(1)}\lambda^2 + a^{(2)}\lambda^3 + \cdots$

$$\beta = b^{(1)}\lambda^2 + b^{(2)}\lambda^3 + b^{(3)}\lambda^4 + \cdots$$
$$\bar{\beta} = \bar{b}^{(1)}\bar{\lambda}^2 + \bar{b}^{(2)}\bar{\lambda}^3 + \bar{b}^{(3)}\bar{\lambda}^4 + \cdots$$

$$\bar{b}^{(1)} = b^{(1)}$$
$$\bar{b}^{(2)} = b^{(2)}$$
$$\bar{b}^{(3)} = b^{(3)} + f(b^{(1)}, b^{(2)}, a^{(1)}, a^{(2)})$$

- One coupling constant case
  - Scheme-independence $\Rightarrow$ Only two lowest-order terms
  - Scheme-dependence $\Rightarrow$ All higher-order terms
  - Scheme-dependent terms can all be set to vanish
  - High-precision numerical analysis possible (but useless)

Change of scheme

$$(\bar{g}_A, \bar{\lambda}_{abcd}, \bar{y}_{a|ij}) = (g_A, \lambda_{abcd}, y_{a|ij}) + (a_A^{(1)}, a_{abcd}^{(1)}, a_{a|ij}^{(1)}) + \cdots$$

$$(\beta_A, \beta_{abcd}, \beta_{a|ij}) = (b_A^{(1)}, b_{abcd}^{(1)}, b_{a|ij}^{(1)}) + (b_A^{(2)}, b_{abcd}^{(2)}, b_{a|ij}^{(2)}) + \cdots$$

$$(\bar{\beta}_A, \bar{\beta}_{abcd}, \bar{\beta}_{a|ij}) = (\bar{b}_A^{(1)}, \bar{b}_{abcd}^{(1)}, \bar{b}_{a|ij}^{(1)}) + (\bar{b}_A^{(2)}, \bar{b}_{abcd}^{(2)}, \bar{b}_{a|ij}^{(2)}) + \cdots$$

$$(\bar{b}_A^{(1)}, \bar{b}_{abcd}^{(1)}, \bar{b}_{a|ij}^{(1)}) = (b_A^{(1)}, b_{abcd}^{(1)}, b_{a|ij}^{(1)})$$

$$(\bar{b}_A^{(2)}, \bar{b}_{abcd}^{(2)}, \bar{b}_{a|ij}^{(2)}) \neq (b_A^{(2)}, b_{abcd}^{(2)}, b_{a|ij}^{(2)})$$

- General case
  - Scheme-independence ⇒ Only lowest-order terms
  - Scheme-dependence ⇒ All higher-order terms
  - Scheme-dependent terms cannot all be set to vanish
  - High-precision numerical analysis not possible (but useful)

Examples

Physical $d = 4$ case

- No proper example yet $\Rightarrow$ Maybe none ?
  - Technically difficult to generate β-functions
- $SU(2)$ gauge theory with two real scalars (singlet) and two active flavors of Weyl fermions (fundamental)
  - Unbounded-from-below scalar potential
  - Only found numerically $\Rightarrow$ Trustworthy ?

Features and future work

Physics of non-conformal scale-invariant QFTs

- Less constrained than CFTs
- β-functions $\sim$ Anomalous dimensions
- Rare RG flows (recurrent behaviors)
  - RG flows $\neq$ Gradient flows
  - Strongest version of a-theorem violated

Future work

- Phenomenological applications
  - Cyclic unparticle physics JFF, Grinstein, Stergiou (2011)
- Generic examples (with β-functions at higher order) ?