Space-Time, Quantum Mechanics

+

Scattering Amplitudes

with

F. Cachazo
C. Cheung
J. Kaplan
J. Bourjaily
J. Trnka
S. Caron-Huot

also

E. Witten
L. Dolan
P. Goddard
M. Spradlin
A. Volovich
S. Gonchenov

J. Maldacena
F. Alday
D. Gaiotto
P. Vieira
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Z. Bern
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D. Kosower
G. Korchemsky
E. Sokatchev
J. Henn
J. Drummond

R. Penrose
A. Hodges
L. Mason
D. Skinner
M. Bullimore

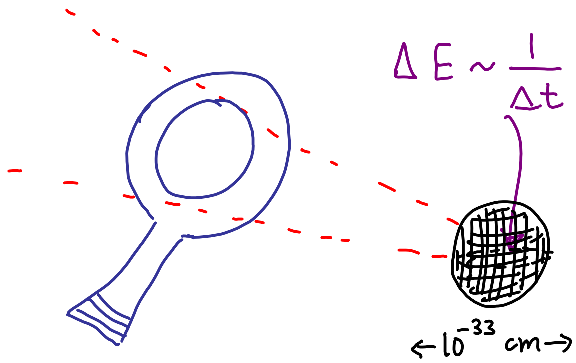
Motivations

Gravity + QM

$\Downarrow$

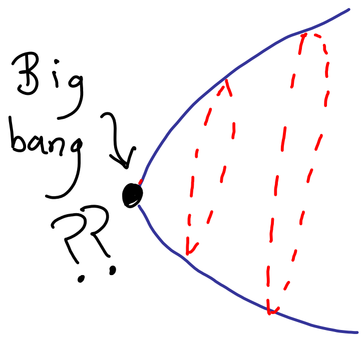
"Space-time is Doomed"

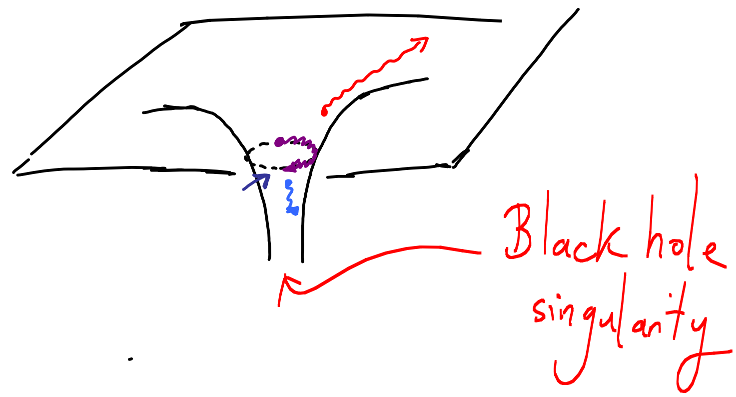
$\Delta E \sim \frac{1}{\Delta t} \longrightarrow$ eventually make Black Hole!



No Operational meaning to distance $< 10^{-33}$ cm, times $< 10^{-43}$ s, ...

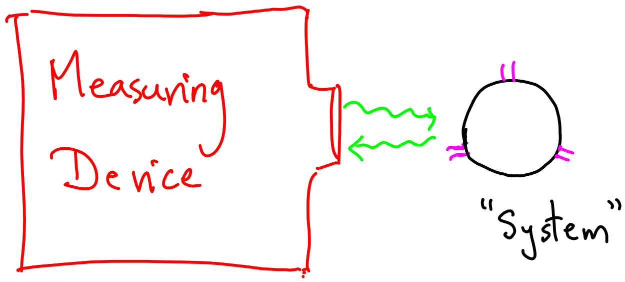
End of Space-Time





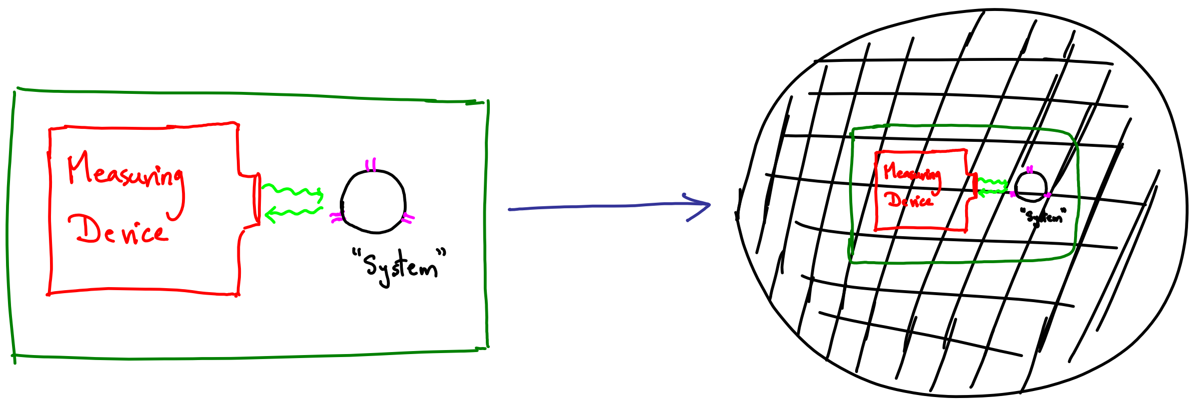
Our theories just <u>break</u> <u>down</u> when gravity is strong and quantum gravity effects are dominant.

Exact Quantum Predictions

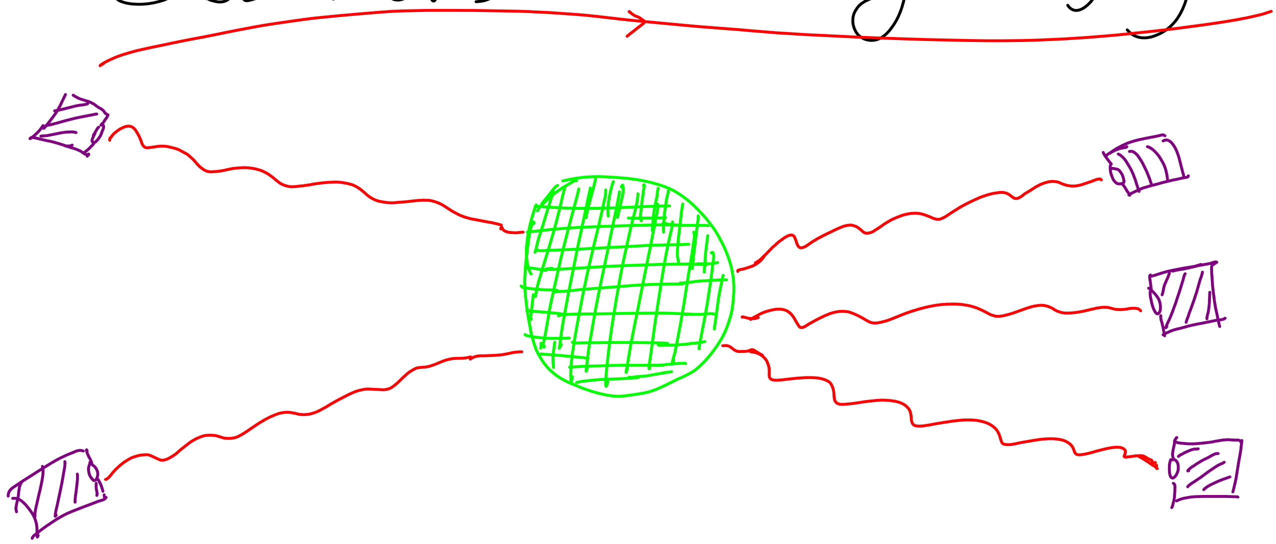


Infinitely many measurements with an Infinitely large measuring apparatus!

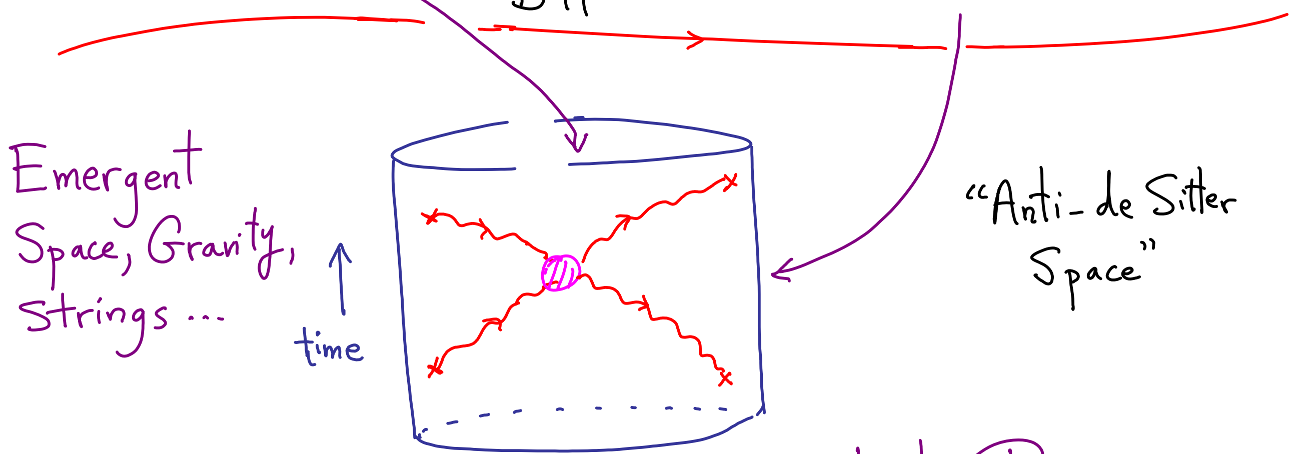
No Local Observables!



Observables on "Boundary at Infinity"

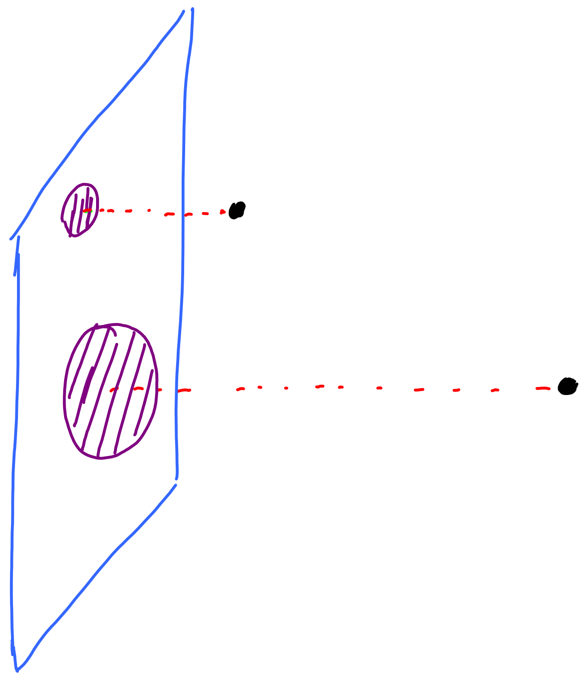


$$(\text{Quantum Gravity})_{D+1} = (\text{Quantum Field Theory})_D \; (!)$$

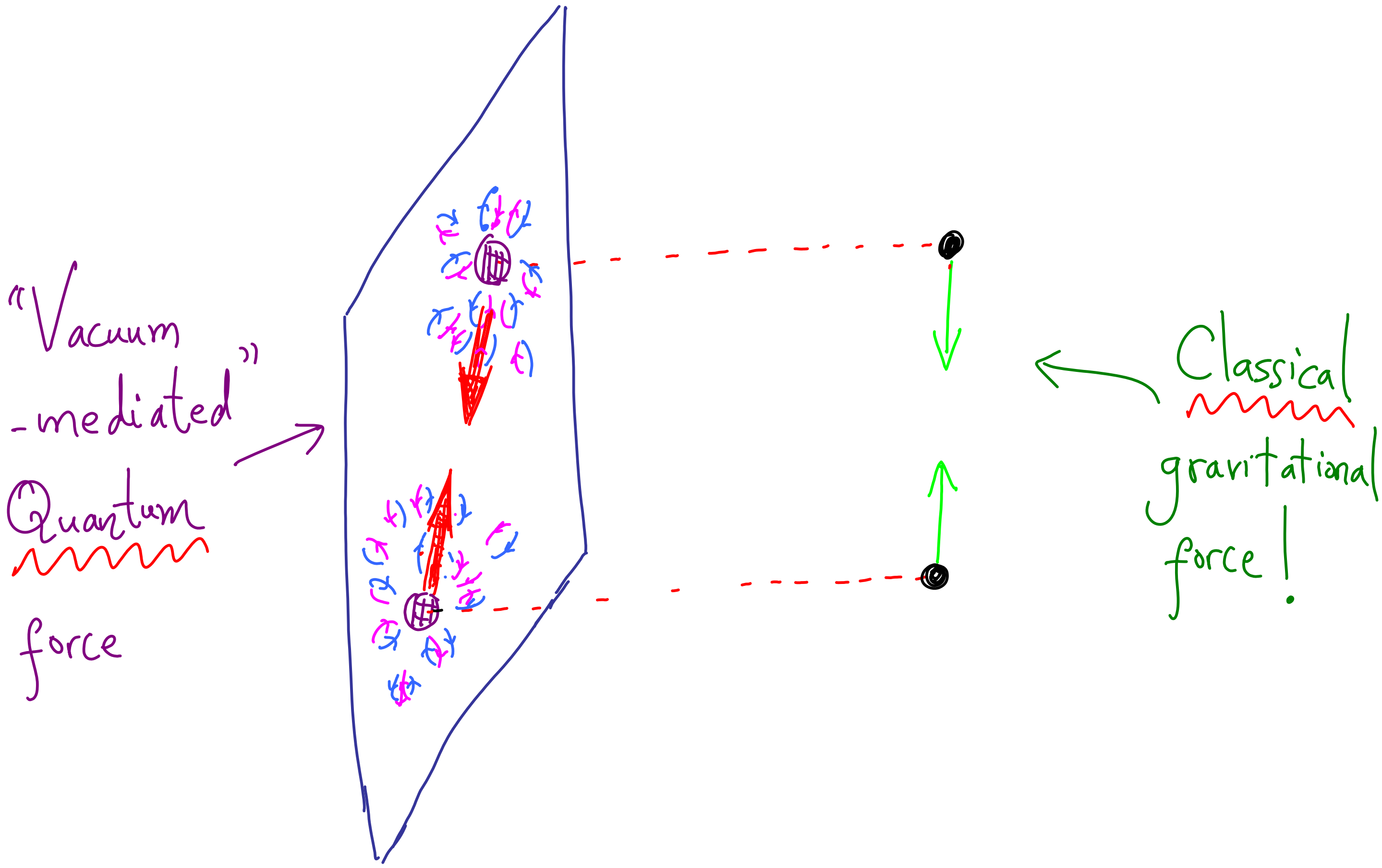


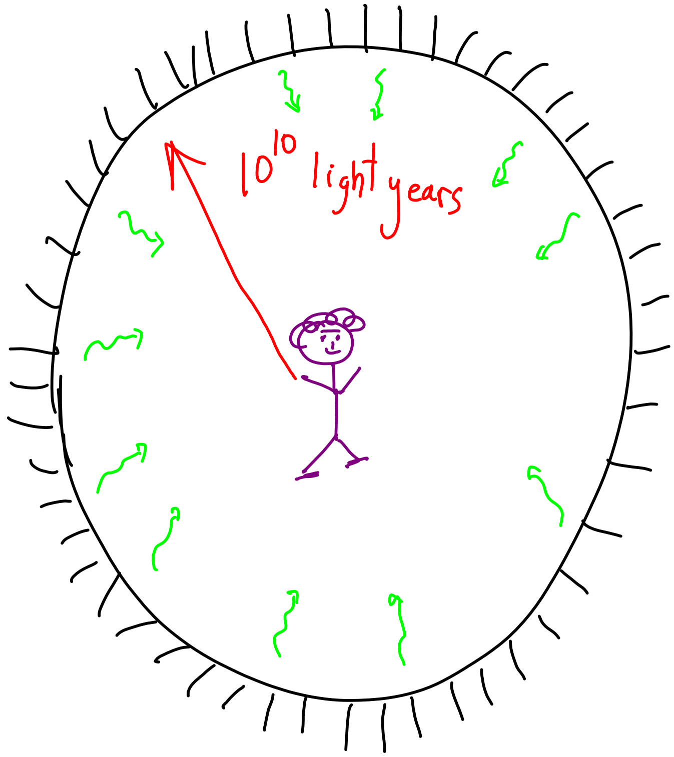
String Theory = Particle Physics

[Weakly interacting] [strongly interacting]



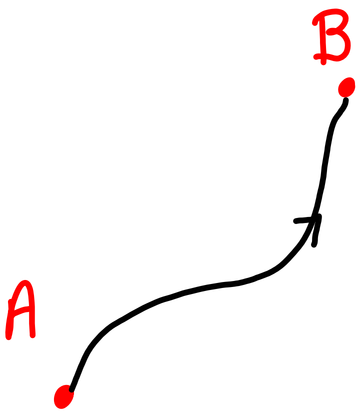
Position ⟷ Scale size


"Vacuum
-mediated"
Quantum
force
Classical
gravitational
force!



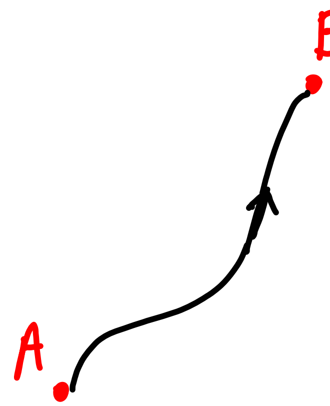
What are the correct observables?!?

Emergent Space-time?



$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

Manifestly Determinstic

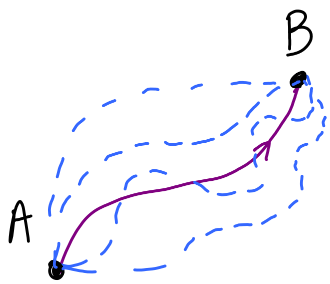


$x(t)$ minimizes action

$$S = \int dt \left[\tfrac{1}{2} m\dot{x}^2 - V(x)\right]$$

<u>Not</u> manifestly determinstic

Quantum Mechanics

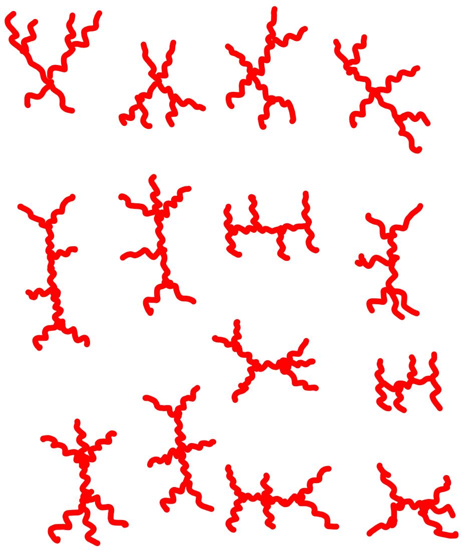


All paths are taken.

$$\text{Amp} = \sum_{\text{paths}} e^{iS/\hbar}$$

$\hbar \to 0$ limit of QM = Least action principle, <u>not</u> $F = ma$!

Feynman Follies



$+ \cdots$

220 Diagrams

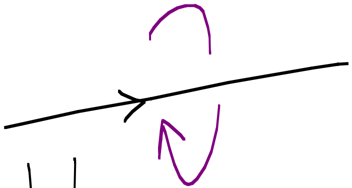
10's of thousands of terms ...

$$\mathrm{Amp}(1^+2^-3^+4^-5^+6^+) = \frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} \, !$$

3 terms for $(+-+-+-)$ ....

Q: What makes Feynman Diagrams so complicated, obscuring simplicity?

A: Insistence on Manifest spacetime locality + unitarity!



photon

2 helicities ± 1.

Locality $\Longrightarrow$ Field $A_\mu(x) = \epsilon_\mu(p)\, e^{ip\cdot x}$

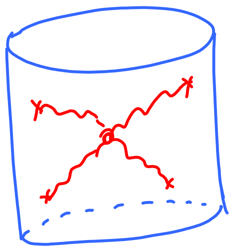
4 components!

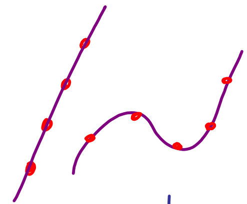
$\epsilon \cdot p = 0$, also $\epsilon_\mu(p) \sim \epsilon_\mu(p) + \alpha\, p_\mu$

$$A_\mu(x) \sim A_\mu(x) + \partial_\mu \alpha$$

All the trouble! $\longrightarrow$ "Gauge Redundancy"

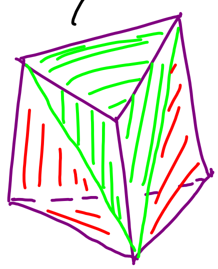
Sitting Under our Noses for 60 yrs





Twistor Theory

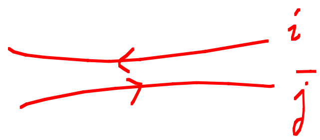
Scattering without Spacetime – emergent locality + unitarity



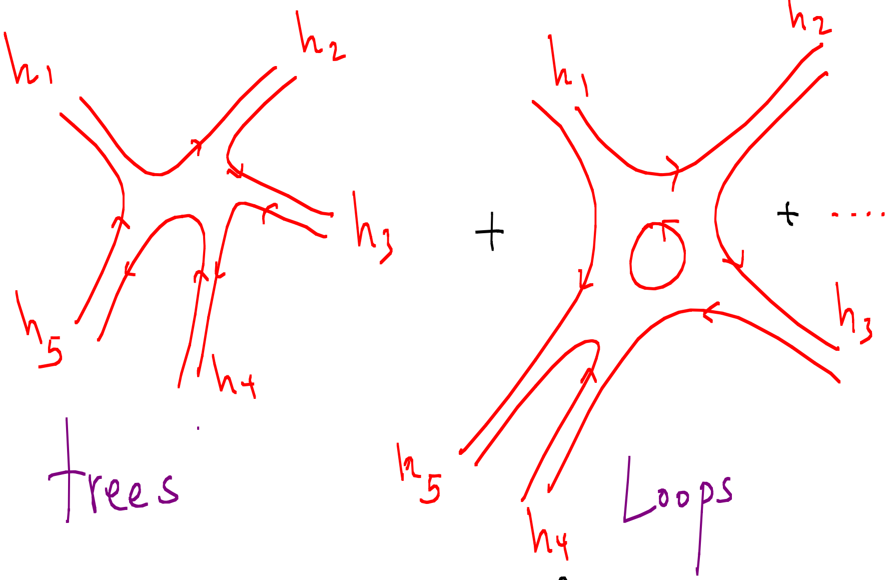
Alegraic Geometry

Cast of Characters

· Gluon scattering amplitudes

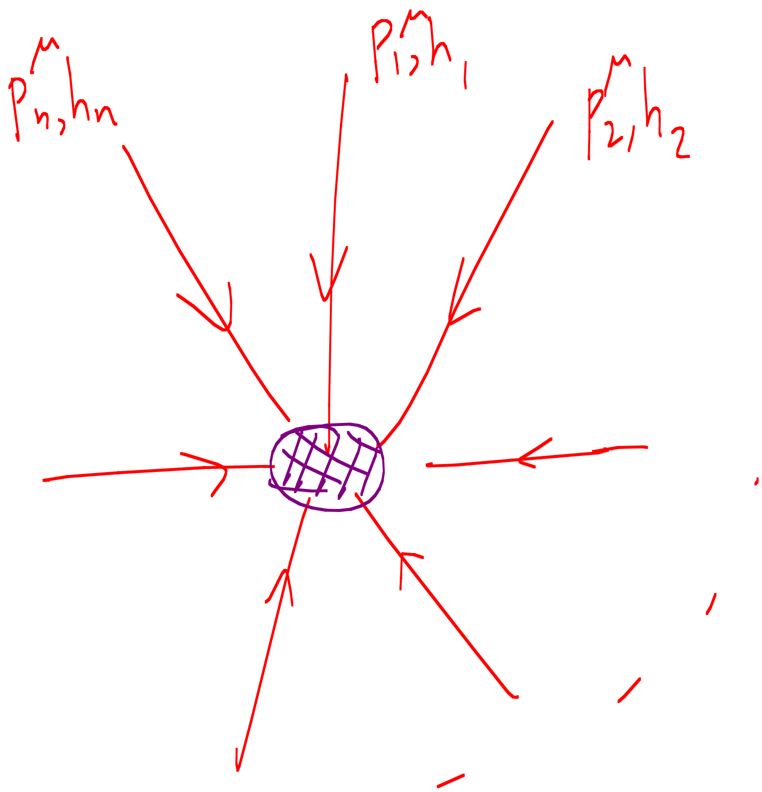


N_c colors



Restrict to planar limit

Planar loop integrand is well-defined



$$\left[p_1^\mu + \cdots + p_n^\mu = 0 \right]$$

$a = 1, \ldots, n$

$M_n[p_a, h_a; l_i]$ l_i : loop momenta

$p^\mu = (p^0, \vec{p})$, massless $p^2 = 0$.

$$P_{A\dot{A}} = \begin{pmatrix} p^0+p^3 & p^1+ip^2 \\ p^1-ip^2 & p^0-p^3 \end{pmatrix}. \quad \det P_{A\dot{A}} = p^2 = 0$$

$$\Rightarrow \quad P_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}.$$

Lorentz symmetry $SL(2) \times SL(2)$,

$\lambda \to M\lambda$, $\tilde{\lambda} \to \tilde{M}\tilde{\lambda}$.

Invariants $\langle \lambda_1 \lambda_2 \rangle = \det(\lambda_1, \lambda_2)$; $[\tilde{\lambda}_1 \tilde{\lambda}_2]$

$$M_n(\lambda_a, \tilde{\lambda}_a, h_a), \quad M_n(t_a \lambda_a, t_a^{-1} \tilde{\lambda}_a, h_a)$$

$$= t_a^{-2h_a} M_n(\lambda_a, \tilde{\lambda}_a, h_a)$$

e.g.

$$M_6(1^+ 2^- 3^+ 4^- 5^+ 6^+) = \frac{\langle 24 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle}$$

$$M_6(t_2 \lambda_2) = t_2^{2} M_6(\lambda_2) \Leftrightarrow \text{helicity of 2 is } -1.$$

Finally – simplest gauge theory of all is "maximally supersymmetric, $\mathcal{N}=4$ Super Yang Mills":

"Harmonic Oscillator of the 21st Century"

Unifies helicities.

$$
\begin{array}{cc}
 & +1 \\
Q_{1,2,3,4} \downarrow & +\frac{1}{2} \\
 & 0 \\
\tilde{Q}_{1,2,3,4} \uparrow & -\frac{1}{2} \\
 & -1
\end{array}
\Bigg\}
$$

"Supermultiplet"

$$|\tilde{\eta}\rangle = e^{Q\tilde{\eta}}|+1\rangle$$

$$= |+1\rangle + \eta\,|+\tfrac{1}{2}\rangle + \cdots + \eta^4\,|-\rangle$$

$$\mathcal{M}_n(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a) = \sum_{k=0}^{n} \mathcal{M}_{n,k}(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a)$$

contains amps with k - helicity gluons.

Turns out $\mathcal{M}_{n,0} = \mathcal{M}_{n,1} = 0$

$\mathcal{M}_{n,2}$ = "MHV" amplitude

$\mathcal{M}_{n,3}$ = "NMHV" amplitude

$\vdots$

In summary: we are after a theory for

$$M_{n,k}\left[\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a, \text{loop var}\right]$$

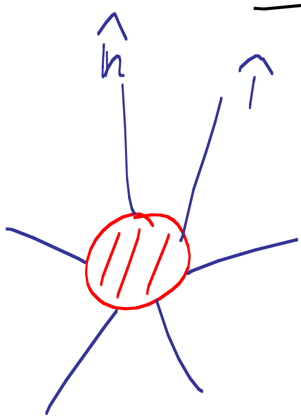
without Unitary evolution through Spacetime.

Emergent Space-time, Emergent QM

Tree Amplitudes:

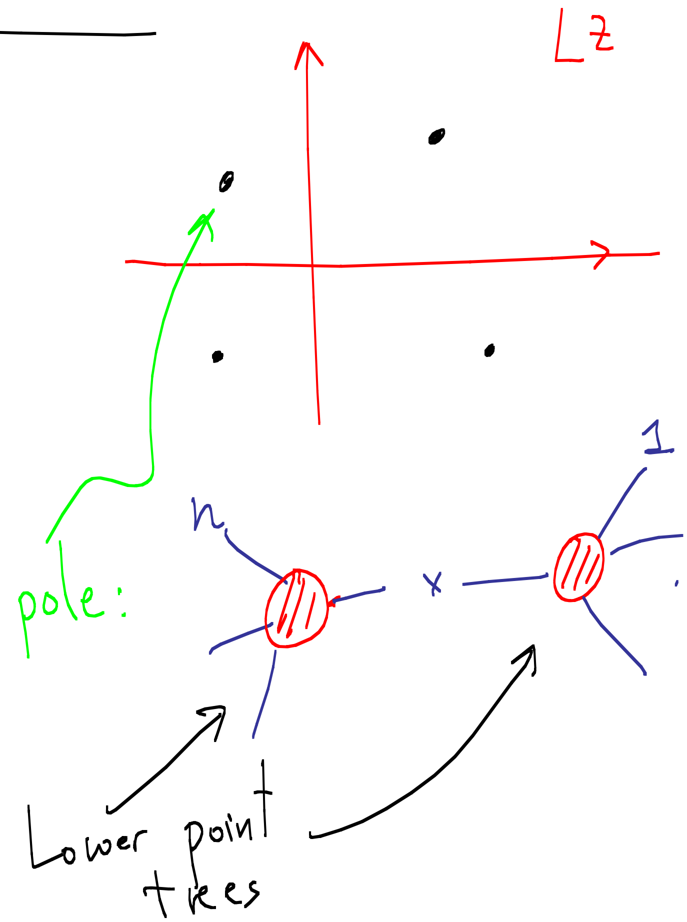
Gathering "Data"

"BCFW Recursion"

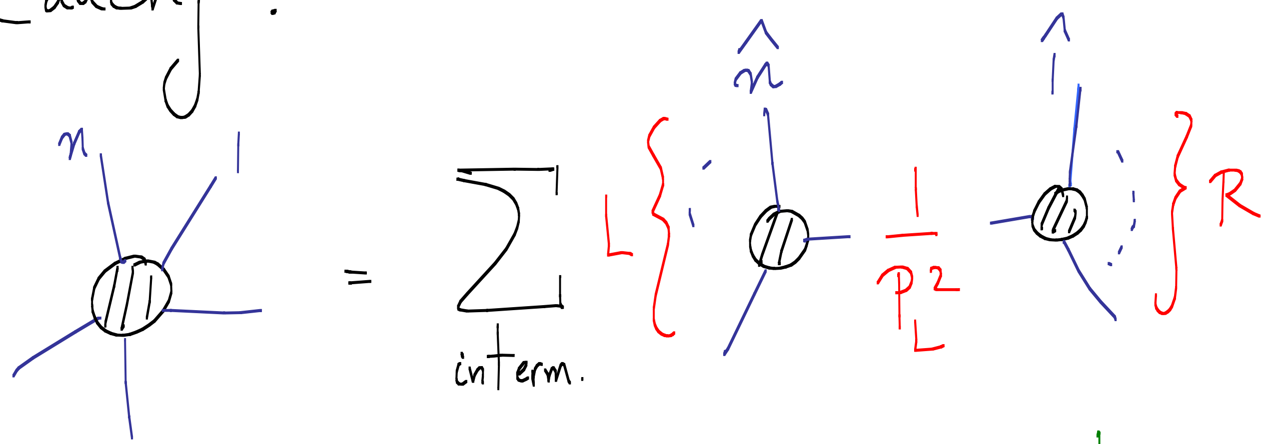


Deform $\lambda_n \to \lambda_n + z\lambda_1$

$\tilde{\lambda}_1 \to \tilde{\lambda}_1 - z\tilde{\lambda}_n$

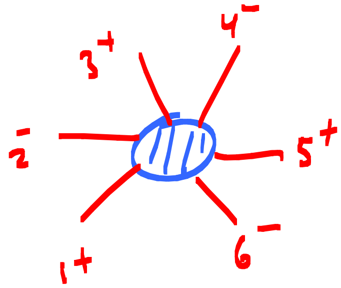


Cauchy:



On-Shell recursion relation!

BCFW 6pt



$$= \frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle} \frac{1}{(p_1+p_2+p_3)^2} \times \frac{1}{\langle 6|5+4|3]} \frac{1}{\langle 4|5+6|1]}$$

"Spurious" Poles: Don't occur in local theories!

$$+ \left\{ i \to i+2 \right\} + \left\{ i \to i+4 \right\}$$

Remarkable 6-term Id

$$\frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle} \frac{1}{(p_1+p_2+p_3)^2} \times \frac{1}{\langle 6|5+4|3]} \frac{1}{\langle 4|5+6|1]} + \{i \to i+2\} + \{i \to i+4\}$$

$$= \frac{\langle 3|(2+4)|6]^4}{[23][34]\langle 56\rangle\langle 61\rangle} \frac{1}{(p_5+p_6+p_1)^2} \times \frac{1}{\langle 1|6+5|4]} \frac{1}{\langle 5|6+1|2]} + \{i \to i+2\} + \{i \to i+4\}$$

Guarantees
- Parity
- Cyclicity
- No Spurious Poles

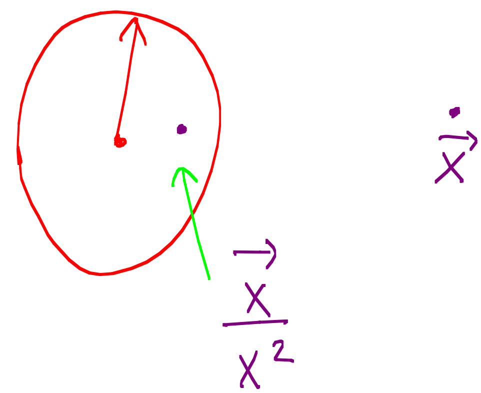
7-pt 12 terms
8-pt 20 terms
40 terms
⋮ ⋮

SOME POWERFUL MATHEMATICAL STRUCTURE IS AT WORK!

Infinitely Many Hidden Symmetries

Theories of massless particles enjoy conformal invariance — the remarkable symmetry under inversions

$$x^\mu \longrightarrow \frac{x^\mu}{x^2}$$

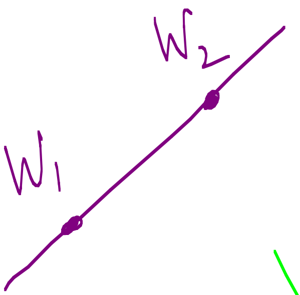


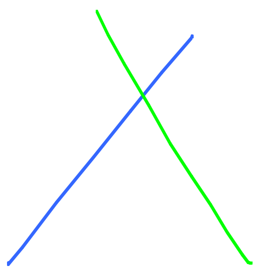
Twistor Space

• $W = \begin{pmatrix}\tilde{\mu}_{\dot{A}} \\ \tilde{\lambda}^{A'}\end{pmatrix}$, $W \to LW$

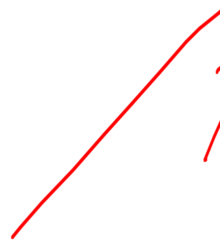
$\det L = 1$

are conf. transf.





Spacetime



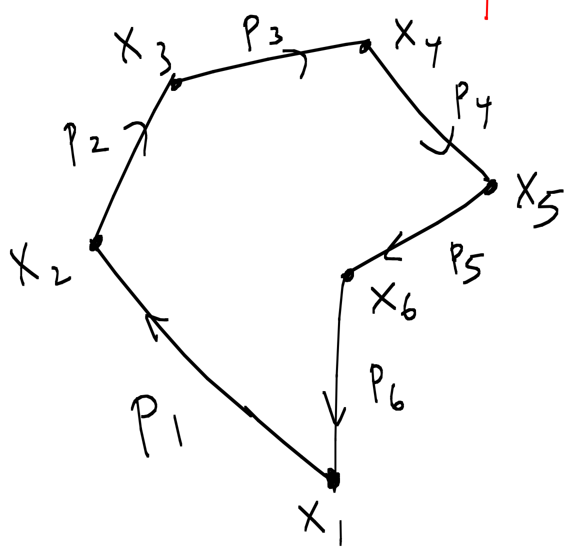
$\tilde{\mu}_A = X_{A\dot{A}}\tilde{\lambda}^{\dot{A}}$

null ray

• $X = \dfrac{\mu_1\lambda_2 - \mu_2\lambda_1}{\langle 12\rangle}$

X • Y • $\longleftrightarrow$ null $\quad (X-Y)^2 = 0$

Dual (Super) Conformal Symmetry



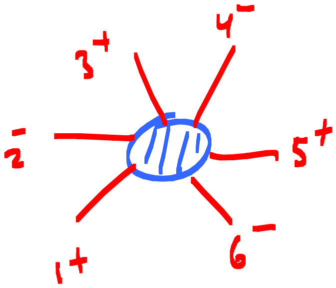
$$p_a = x_{a+1} - x_a$$

"Experimental" observation
– amplitudes invariant under

Conf. transf. on

<u>this</u> X space!

[Term by term for BCFW form of trees]

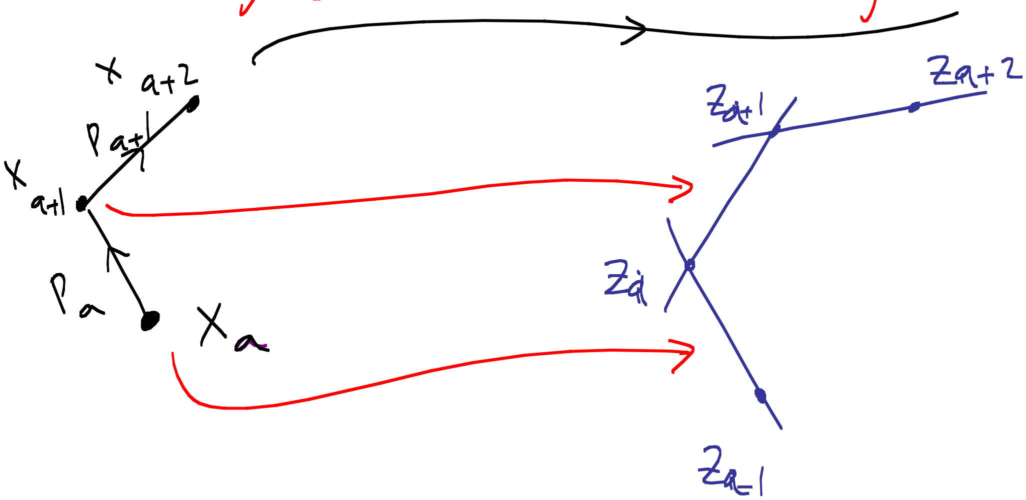


$$= \frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle} \frac{1}{(p_1+p_2+p_3)^2} \times \frac{1}{\langle 6|5+4|3]} \frac{1}{\langle 4|5+6|1]}$$

"Spurious" Poles

Are there because these BCFW terms know about both spacetimes!

"Momentum" Twistor Space



$$Z_a = \begin{pmatrix} \mu_a \\ \lambda_a \\ \eta_a \end{pmatrix}, \qquad \tilde{\lambda}_a = \frac{\langle a-1\, a\rangle \mu_{a+1} + \text{cyclic}}{\langle a-1\, a\rangle\langle a\, a+1\rangle}$$

$$\tilde{\eta}_a = \frac{\langle a-1\, a\rangle \eta_{a+1} + \text{cyc.}}{}$$

(Super) Conformal + Dual (Super) Conformal

$\Big\Downarrow$ generate

"Yangian Algebra"

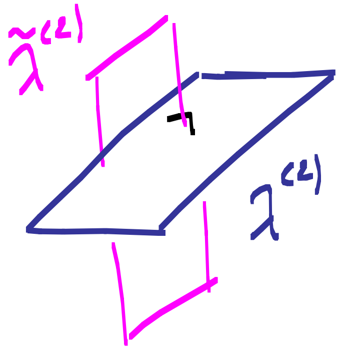
Infinite Dimensional Symmetry

Completely Invisible In Usual Formalism

A New Formulation

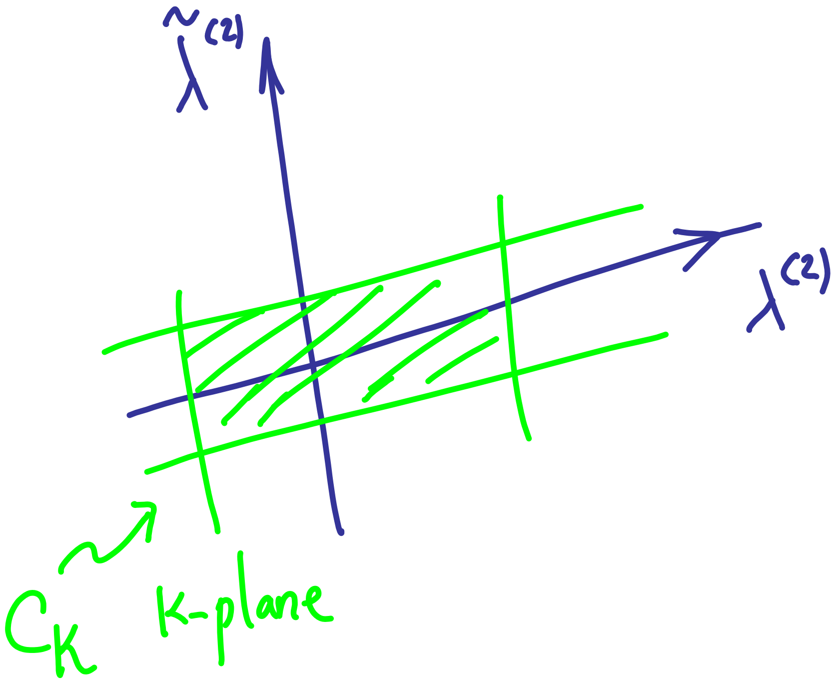
Start by thinking about momentum conservation afresh!

$$\lambda_A^{a}, \quad \tilde{\lambda}_{\dot{A}}^{\dot{a}}$$



mom. conservation:

$$\lambda \cdot \tilde{\lambda} = 0.$$



Note: parity invariant since

$\lambda \leftrightarrow \tilde{\lambda}$

k plane $\leftrightarrow$ n-k plane

Note: impossible for $k = 0, 1, n-1, n$.

Good!

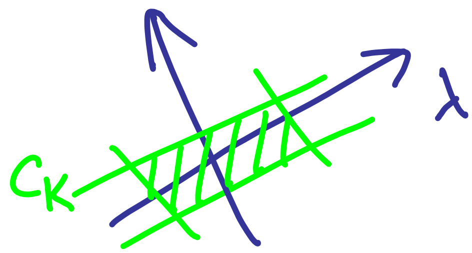
Eqns:

$$C = \begin{bmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_k \end{bmatrix} = C_{\alpha a}$$

Invariance under $GL(k)$ $\quad C_{\alpha a} \to L_\alpha^{\ \beta} C_{\beta a}$.

Space of k-planes in n-dim : Grassmannian $G(k,n)$

$$\dim G(k,n) = kn - k^2 = k(n-k)$$



$$\int d^{2\times k}\rho_\alpha \underbrace{\delta^2\left[C_{\alpha a}\rho_\alpha - \lambda_a\right]}_{C \text{ contains } \lambda} \underbrace{\delta^2\left[C_{\alpha a}\tilde{\lambda}_a\right]}_{C \text{ orthogonal to } \tilde{\lambda}} \underbrace{\delta^4\left[C_{\alpha a}\tilde{\eta}_a\right]}_{\text{SUSY partner}}$$

Motivation : preserve GL(k)

This object is very simple in Twistor Space:

$$\prod_{\alpha=1}^{k} \delta^{4|4}\left[C_{\alpha a} W_a \right]$$

Manifests (Super) conformal symmetry

$k = 0, 1, n-1, n$: no possible planes.

$k = 2$ unique : $C = 2$ plane.

General k : integrate over all k-planes!

$$\int \frac{d^{k\times n} C_{\alpha a}}{(1 2 \ldots k)(2 3 \ldots k+1)\ldots(n 1 \ldots k-1)}$$

simplest + most natural GL(k) invariant measure!

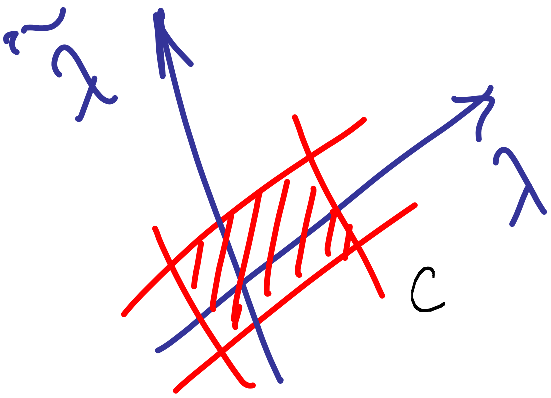
$(m_1 \ldots m_k)$: $k \times k$ minor of C made of columns $m_1, \ldots, m_k$.

$$\boxed{\mathcal{L}_{n,k} = \underbrace{\int \frac{d^{k(n-k)} C_{\alpha a}}{(12\cdots k)\cdots(n1\cdots k-1)}}_{\text{simplest measure}} \times \underbrace{\prod_{\alpha} \delta^{4|4}\left[C_{\alpha a} W_a\right]}_{\text{simplest dependence on kinematics}}}$$

$\updownarrow$

All-Loop Scattering in $\mathcal{N}=4$ SYM!

Manifest Dual Superconformal Invariance



C contains λ plane:

so really an integral over $(k-2)$ planes in n dimensions!

Natural linear transformation mapping $k \times k$ minors to $(k-2)\times(k-2)$ minors...

$$\mathcal{L}_{n,k} \rightarrow \int \frac{d^{p\times(n-p)} D_{\alpha a}}{(12\cdots p)\cdots(n1\cdots(p-1))} \times \prod_{\alpha=1}^{p} \delta^{4|4}\left[D_{\alpha a} \mathcal{Z}_a\right]$$

momentum-twistor variables

Identical Structure!

Dual superconformal symmetry manifest

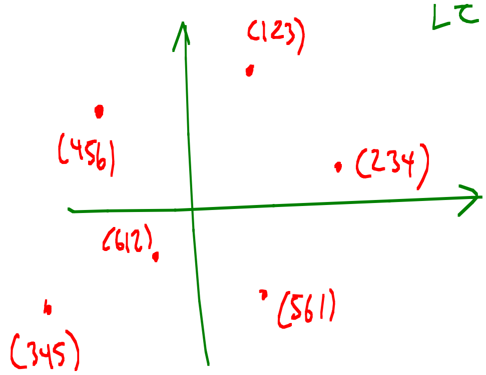
The Grassmannian Formulation makes no mention of locality or Unitarity — but makes all symmetries — The Yangian — manifest.

Quick Example

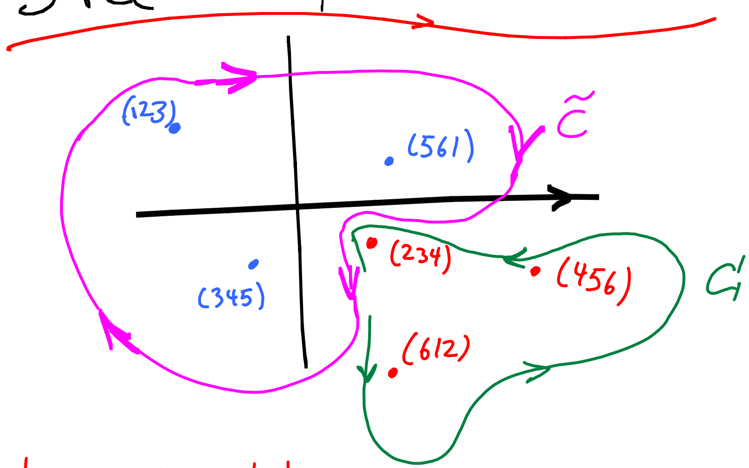
First non-trivial $k=3$, $n=6$, NMHV, $(k-2)(n-k-2) = 1$ variable!

$$\mathcal{Z}_{6,3} = \int \frac{d\tau}{(123)(\tau)\cdots(612)(\tau)}$$

each minor linear in τ



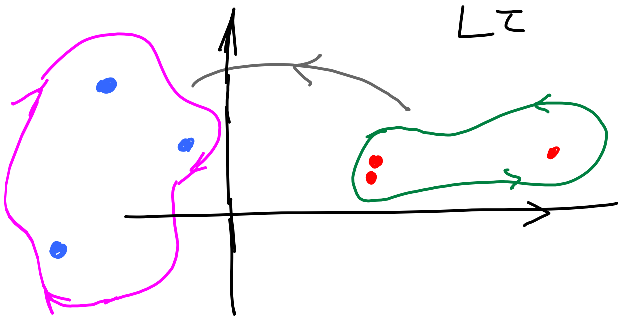
Tree Amplitude



[Unique choices respecting cyclic symmetry]

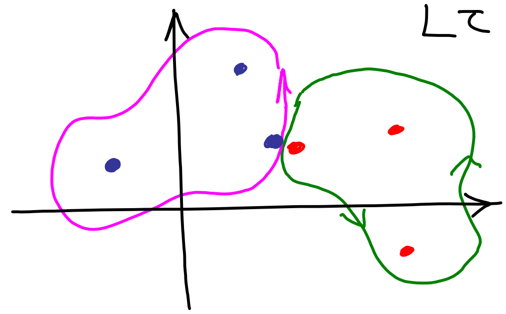
- residues : BCFW terms
- residues : P[BCFW] terms
- Cauchy : BCFW = P[BCFW] = Remarkable 6-term identity!

Spurious Poles

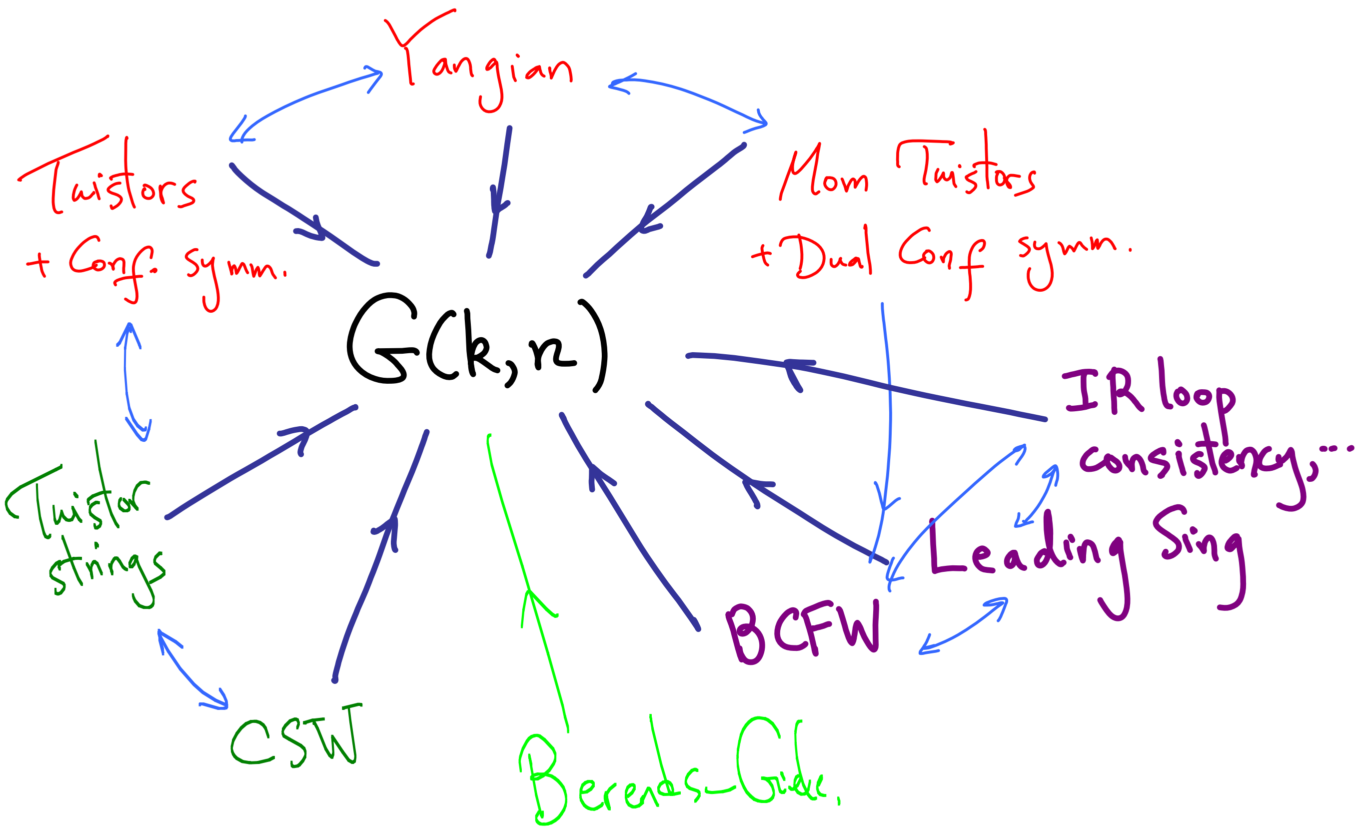


Contour can be deformed away from singularity

Physical Poles

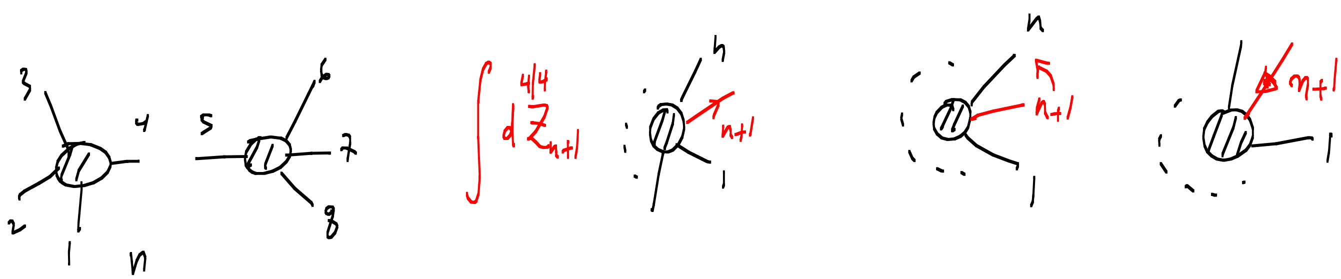


Can't deform contour to avoid singularity

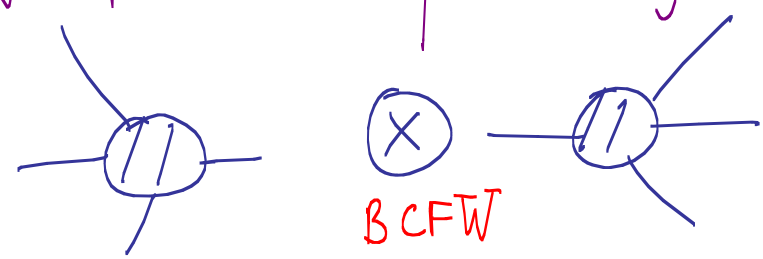


Yangian
Twistors
+ Conf. symm.
Mom Twistors
+ Dual Conf symm.
G(k,n)
IR loop
consistency, ...
Twistor
strings
Leading Sing
BCFW
CSW
Berends-Giele

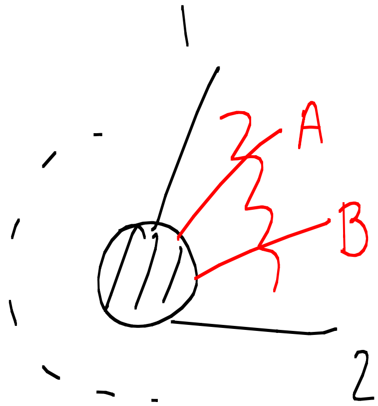
Basic Operations on Yangian Invariants



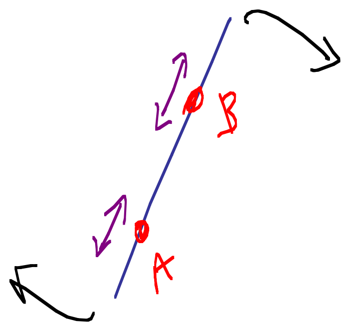
BCFW terms composed from these



Origin of Loops



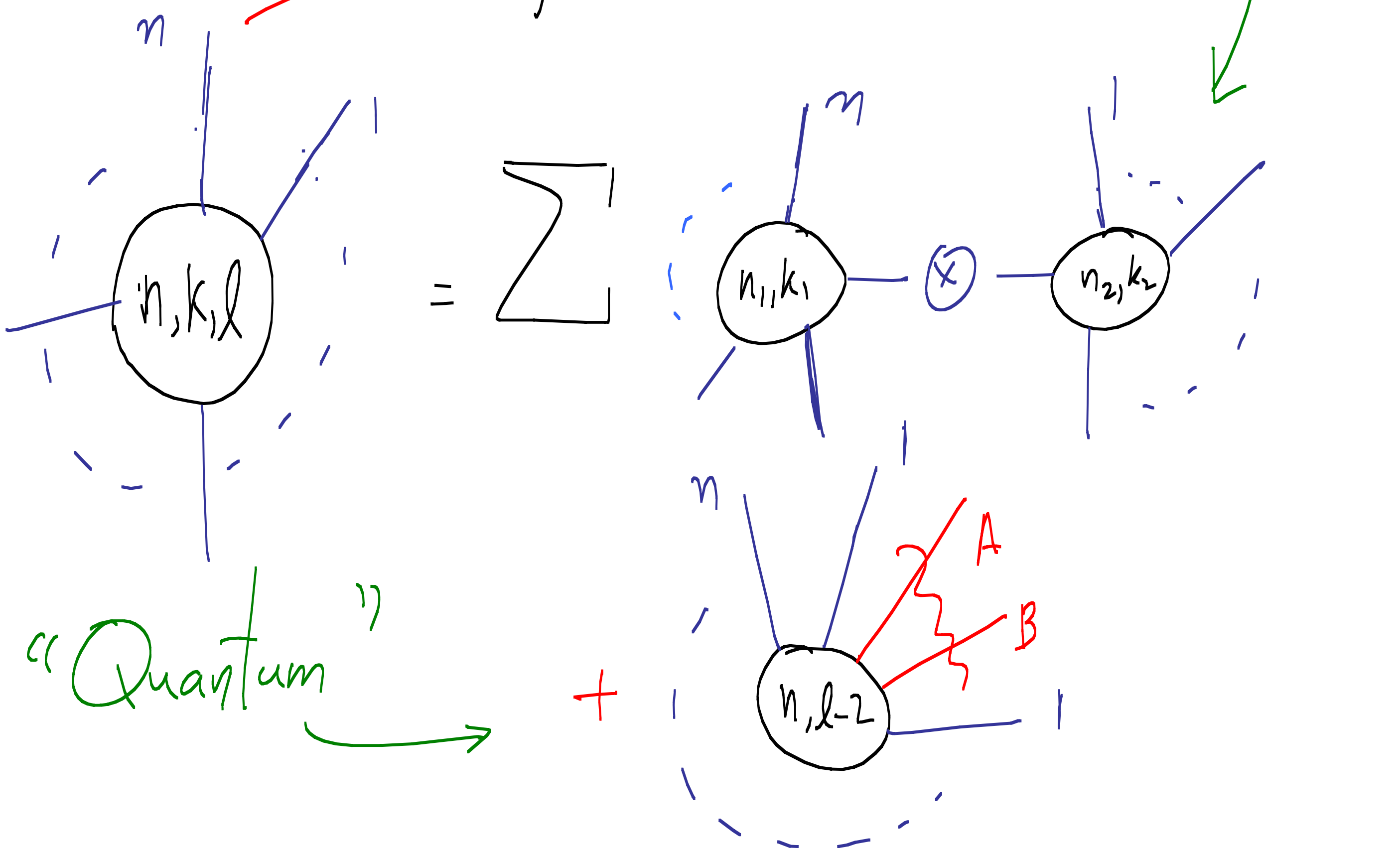
$$\int d^{4|4}Z_A \, d^{4|4}Z_B$$



"Entangled removal" of a pair of particles

= Loop integral

All-loop Recursion

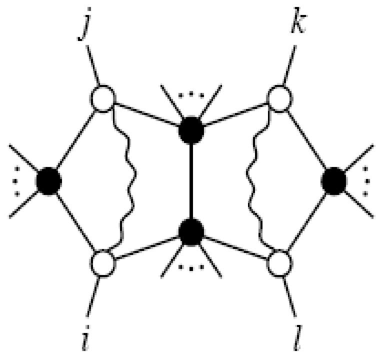


Complete definition of theory built from the Grassmannian, making Yangian symmetry manifest. No reference to spacetime, Lagrangians, Path Integrals, Gauge redundancies, ...

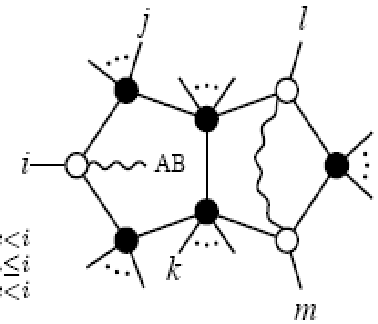
Previously Impossible Calculations Now Routine.

Reveal even further simplicity.

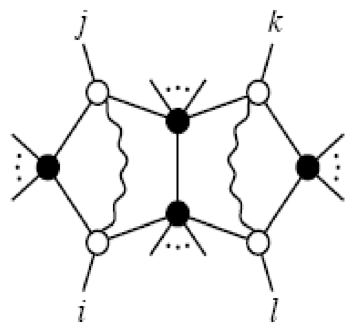
$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$



$$\mathcal{A}^{2-\text{loop}}_{\text{NMHV}} = \sum_{\substack{i<j<l<m\leq k<i\\ i<j<k<l<m\leq i\\ i\leq l<m\leq j<k<i}}$$

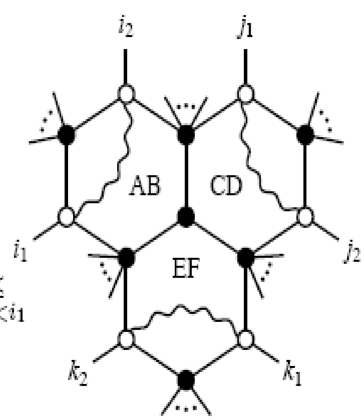


$$\times\,[i, j, j+1, k, k+1] \quad + \frac{1}{2}\sum_{i<j<k<l<i}$$

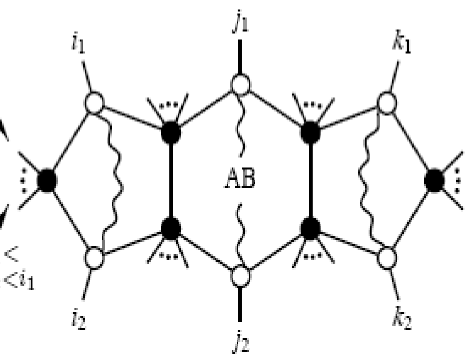


$$\times \left\{ \begin{array}{c} \mathcal{A}^{\text{tree}}_{\text{NMHV}}(j,\ldots,k;\ l,\ldots,i) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(i,\ldots,j) \\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(k,\ldots,l) \end{array} \right\}$$

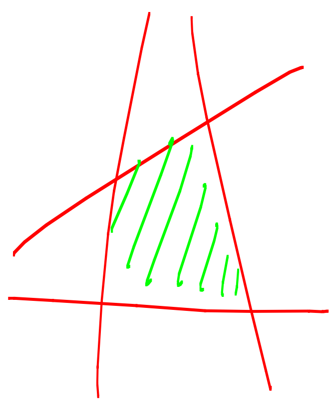
$$\mathcal{A}^{3-\text{loop}}_{\text{MHV}} = \frac{1}{3}\sum_{\substack{i_1\leq i_2<j_1\leq\\ \leq j_2<k_1\leq k_2<i_1}}$$



$$+\frac{1}{2}\sum_{\substack{i_1\leq j_1<k_1<\\ <k_2\leq j_2<i_2<i_1}}$$



· In a specific sense, amplitudes are to be thought of as "the volume" of some polytope:



Different "triangulations" make different properties (Yangian, locality, Unitarity...) manifest.

Our solution should be thought of as providing one class of triangulations

– but we need to more deeply understanding what the object is that is being triangulated!

• This incredible structure has been hiding our noses for ~50 yrs! The basic structure of gauge theories have been begging for a deeper explanation...

Stay tuned — there is much to be understood, and challenges to overcome to make more direct contact with more general theories (+ the real world!).