Axial anomalies in hydrodynamics:

Dam T. Son (INT, University of Washington)

Ref.: DTS, Piotr Surówka, arXiv:0906.5044

Plan of the talk

- Hydrodynamics as a low-energy effective theory
- Relativistic hydrodynamics
- Triangle anomaly: a new hydrodynamic effect

Place of hydrodynamics in theoretical physics

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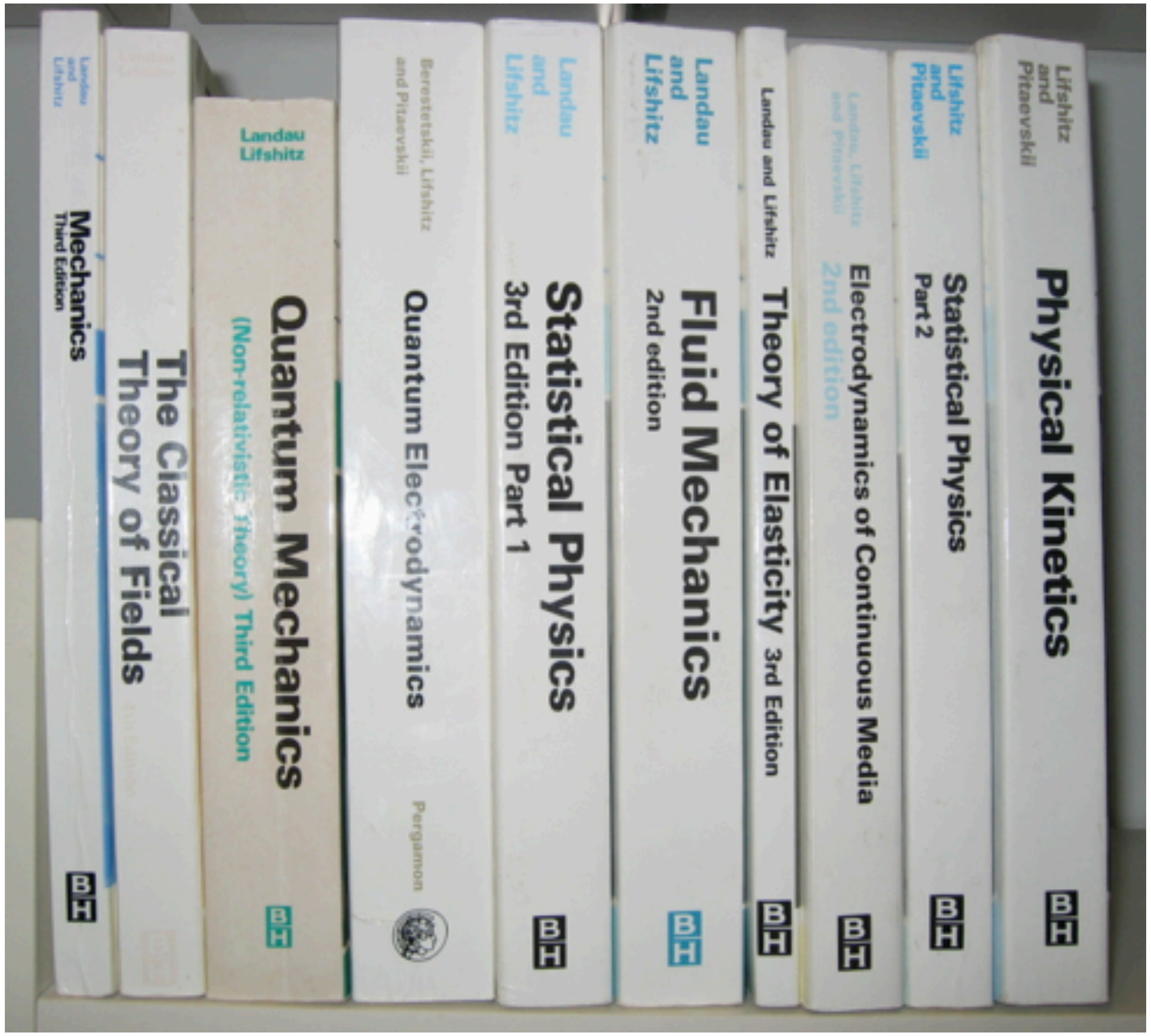
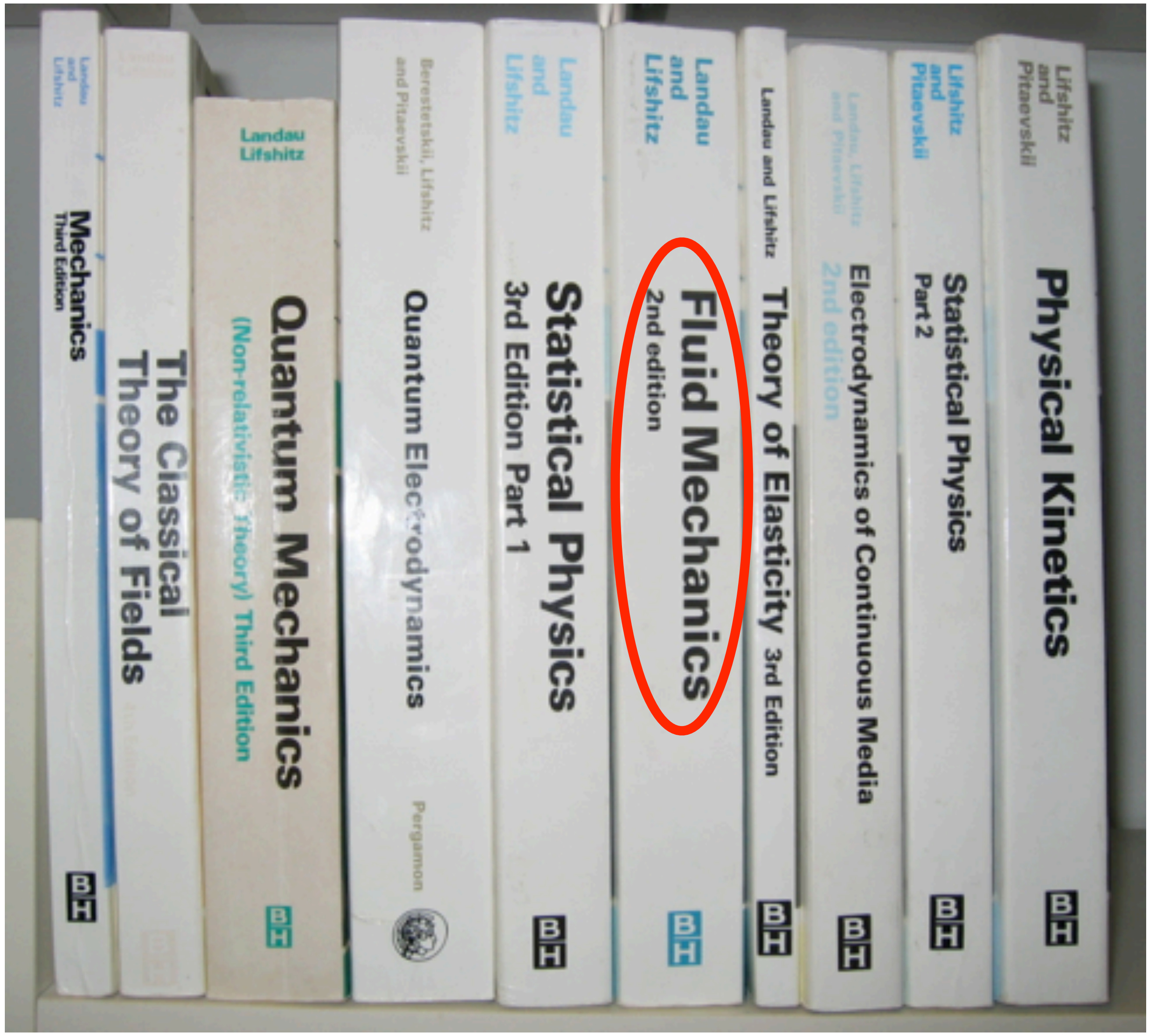


Place of hydrodynamics in theoretical physics



A low-energy effective theory

Consider a thermal system: $T \neq 0$

Dynamics at large distances $\ell \gg \lambda_{\rm mfp}$

governed by hydrodynamics

Degrees of freedom in hydrodynamics

D.o.f. that relax arbitrarily slowly in the long-wavelength limit:

- Conserved densities
- Goldstone modes (superfluids)
- Massless U(1) gauge field (magnetohydrodynamics)

Relativistic hydrodynamics

Conservation laws: $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\mu j^\mu = 0$$

(if ∃ conserved charge)

Constitutive equations: local thermal equilibrium

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

$$j^\mu = n u^\mu$$

Total: 5 equations, 5 unknowns

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$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$$

$$j^\mu = n u^\mu + \nu^\mu$$

Total: 5 equations, 5 unknowns

Dissipative terms

$$\tau^{ij} = -\eta(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\vec{\nabla}\cdot\vec{u}) - \zeta\delta^{ij}\vec{\nabla}\cdot\vec{u} \qquad \nu^i = -\sigma T \partial^i \left(\frac{\mu}{T}\right)$$

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shear viscosity

bulk viscosity

conductivity
(diffusion)

Parity-odd effects?

- QFT: may have *chiral fermions*
    - example: QCD with massless quarks
- Parity invariance does not forbid

$$j^{5\mu} = n^5 u^\mu + \xi(T,\mu)\omega^\mu$$

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta \qquad \text{vorticity}$$

- The same order in derivatives as dissipative terms (viscosity, diffusion)

Landau-Lifshitz frame

- We can also have correction to the stress-energy tensor

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \xi'(u^\mu \omega^\nu + \omega^\mu u^\nu)$$

  - Can be eliminated by redefinition of u^μ

$$u^\mu \to u^\mu - \frac{\xi'}{\epsilon + P}\omega^\mu$$

Only a linear combination $\xi - \frac{n}{\epsilon + P}\xi'$ has physical meaning

Let us set $\xi' = 0$

New effect: chiral separation

- Rotating piece of quark matter
- Initially only vector charge density ≠ 0
- Rotation: lead to j^5: chiral charge density develops
- Can be thought of as chiral separation: left- and right-handed quarks move differently in rotation fluid
- Similar effect in nonrelativistic fluids?

Chiral separation by rotation

Chiral separation by rotation

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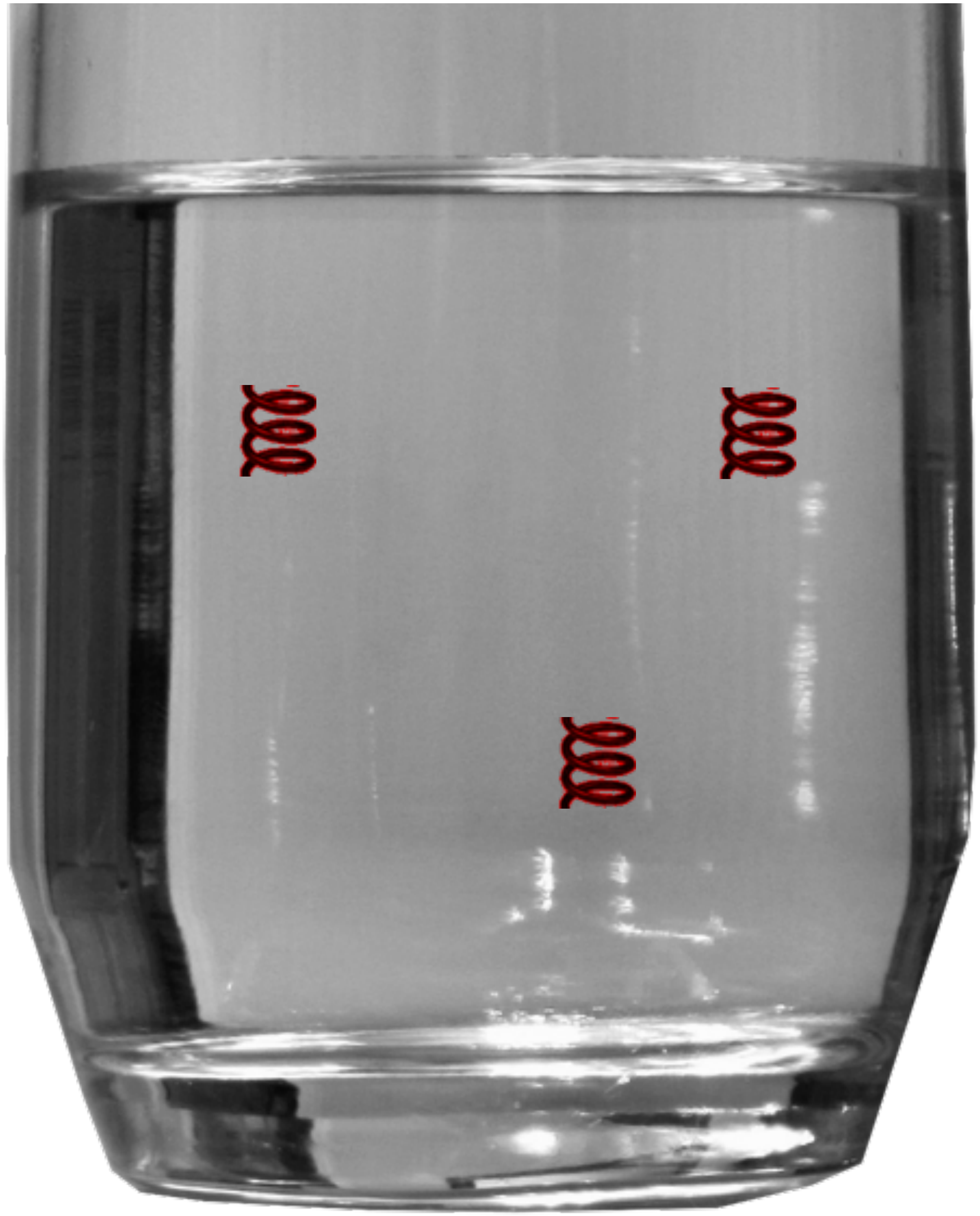

Chiral separation by rotation

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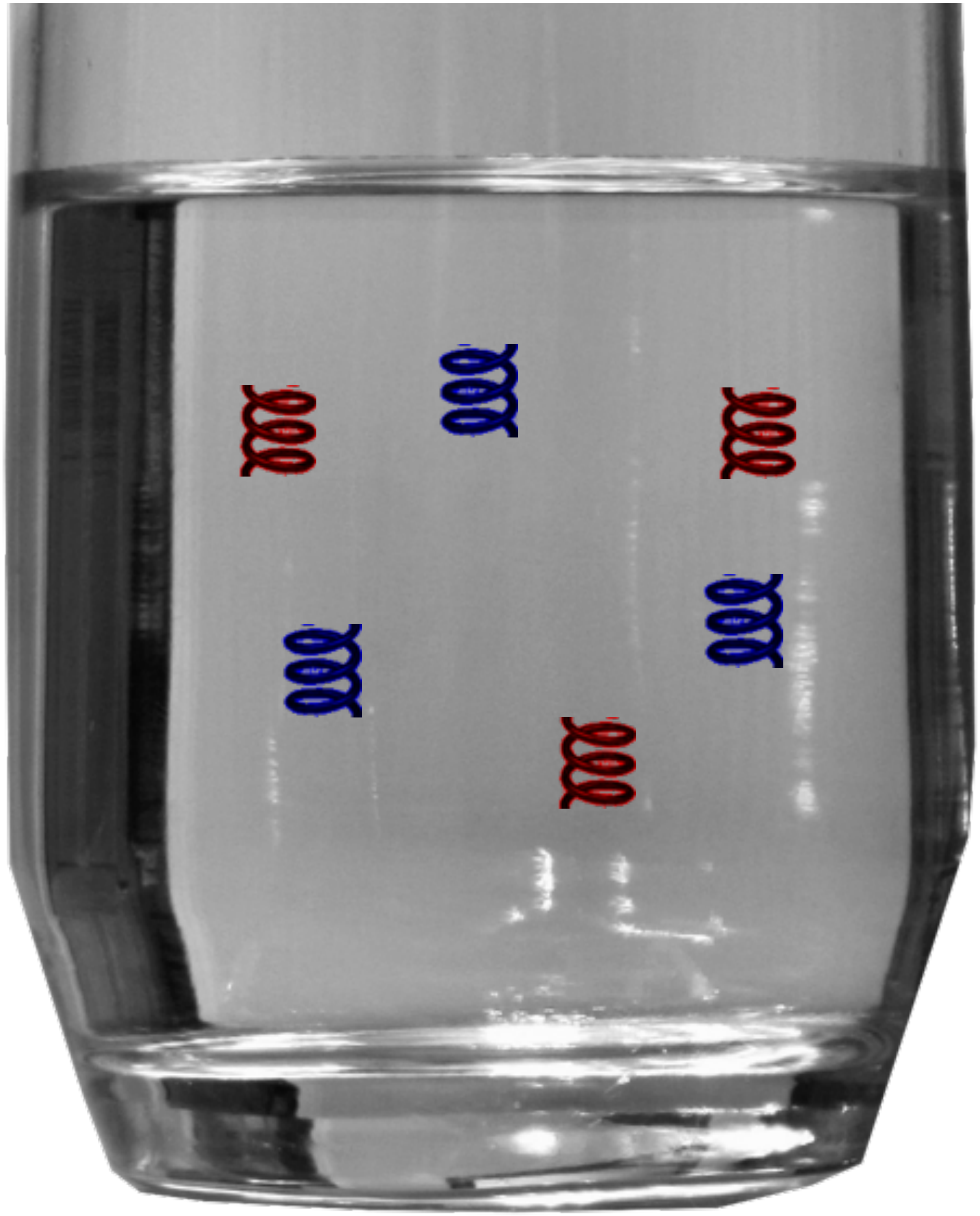

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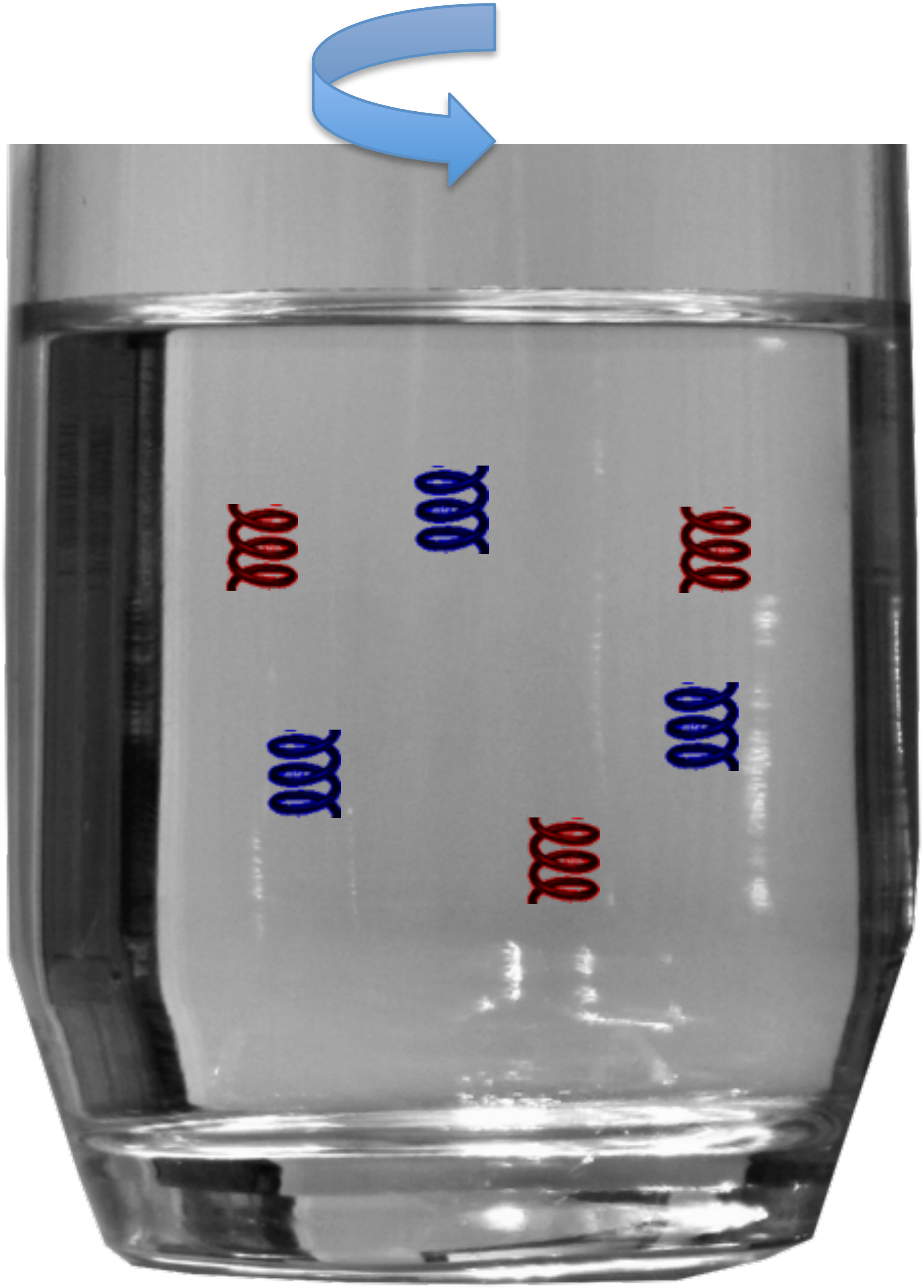

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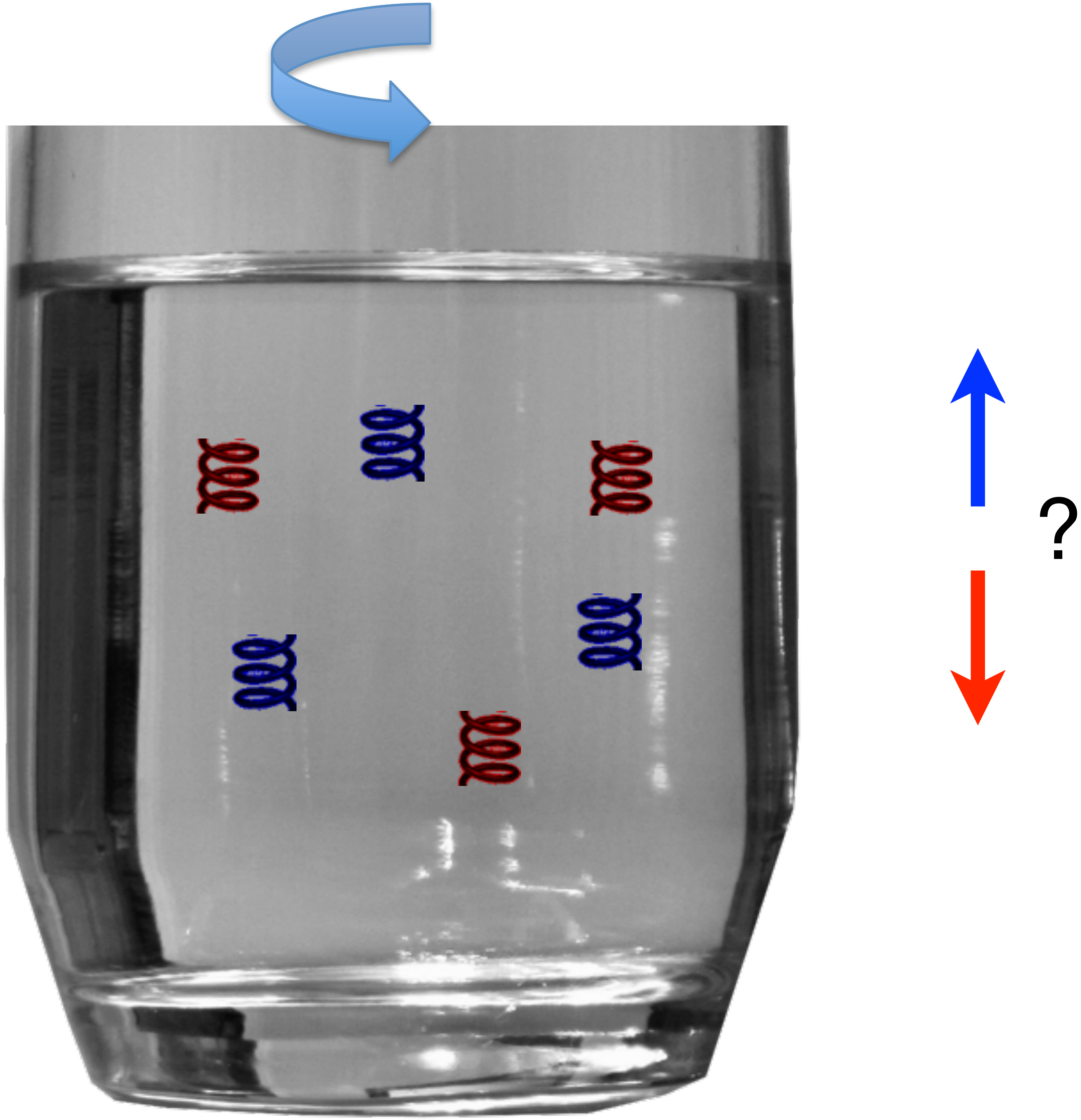


Can chiral separation occur in rigid rotation?

- If a chiral molecule rotates with respect to the liquid, it will moves
- In rigid rotation, molecules rotate with liquid
- $\Rightarrow$ no current in rigid rotation.
- Chiral separation occurs at higher orders in derivative expansion Andreev DTS Spivak

$$j_i^{\text{chiral}} \sim (\partial_i v_j + \partial_j v_i)\omega_j + \cdots$$

but NOT $\xcancel{\mathbf{j}^{\text{chiral}} \sim \omega = \nabla \times \mathbf{v}}$

Relativistic theories are different

- There can be current ~ vorticity
- It is related to triangle anomalies

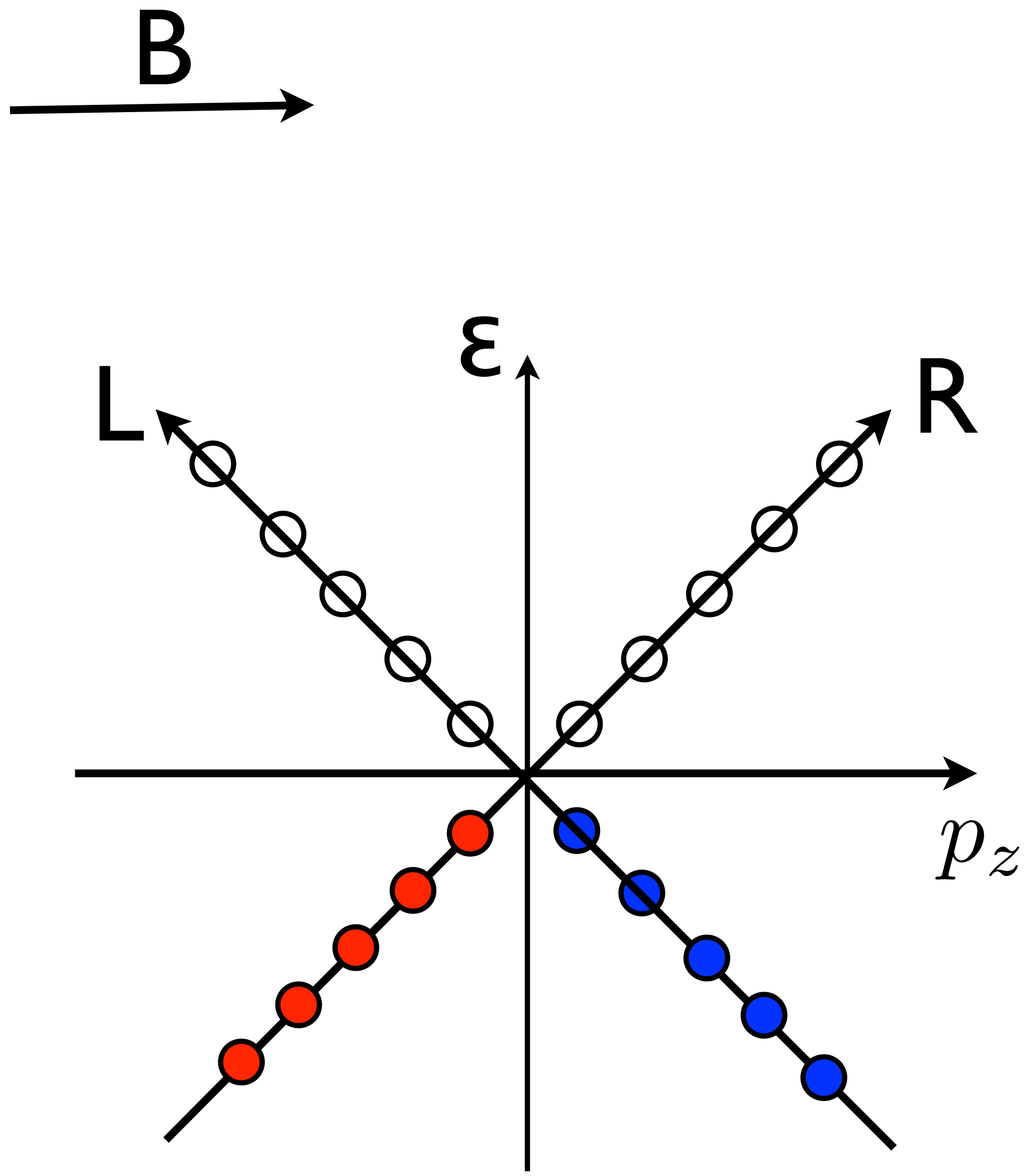
$$\partial_\mu j^{5\mu} = \# E \cdot B$$

  but the effect is there even in the absence of external field

- The kinetic coefficient ξ is determined completely by anomalies and equation of state

Anomalies

Massless fermions: lowest Landau level is chiral



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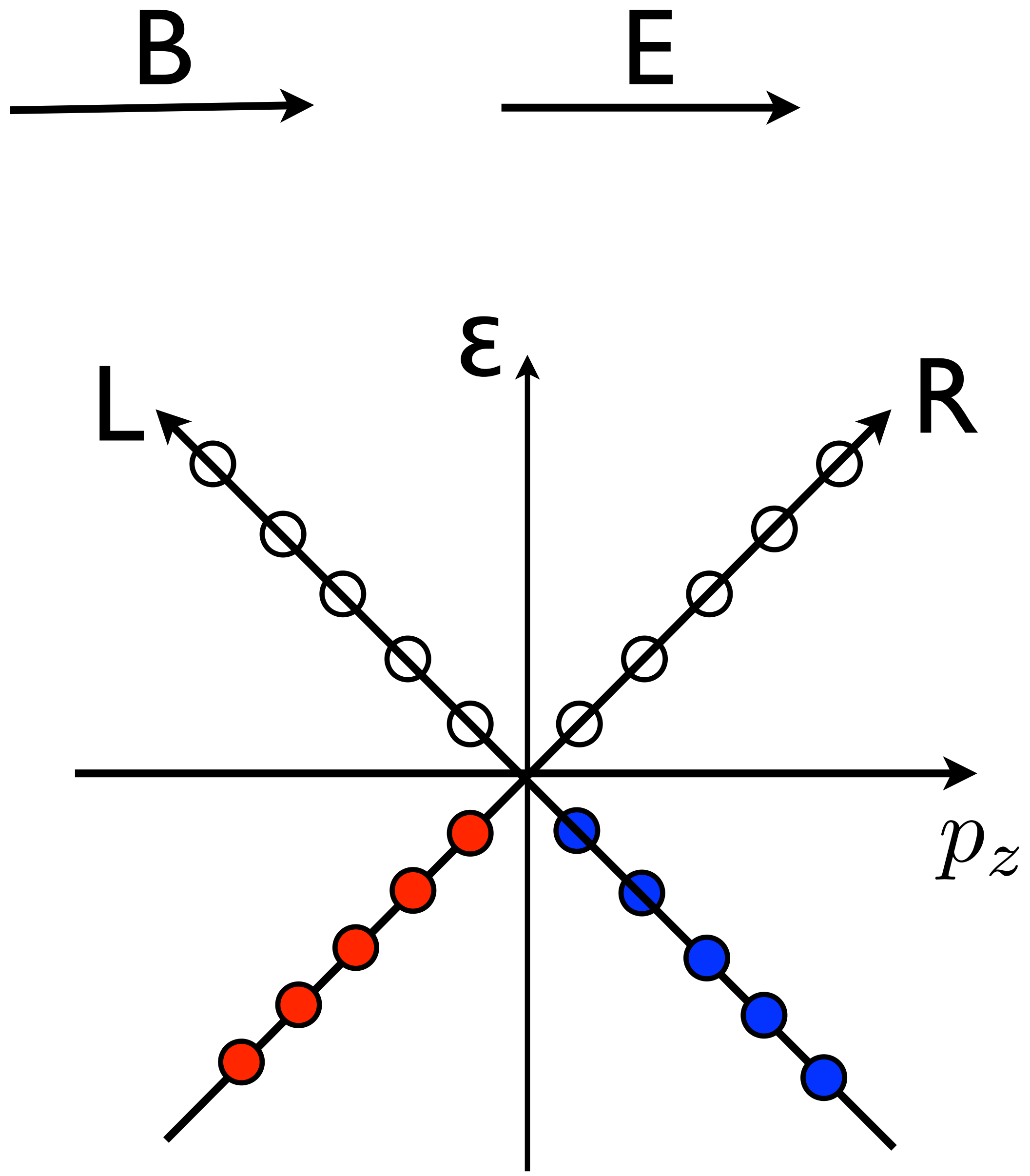

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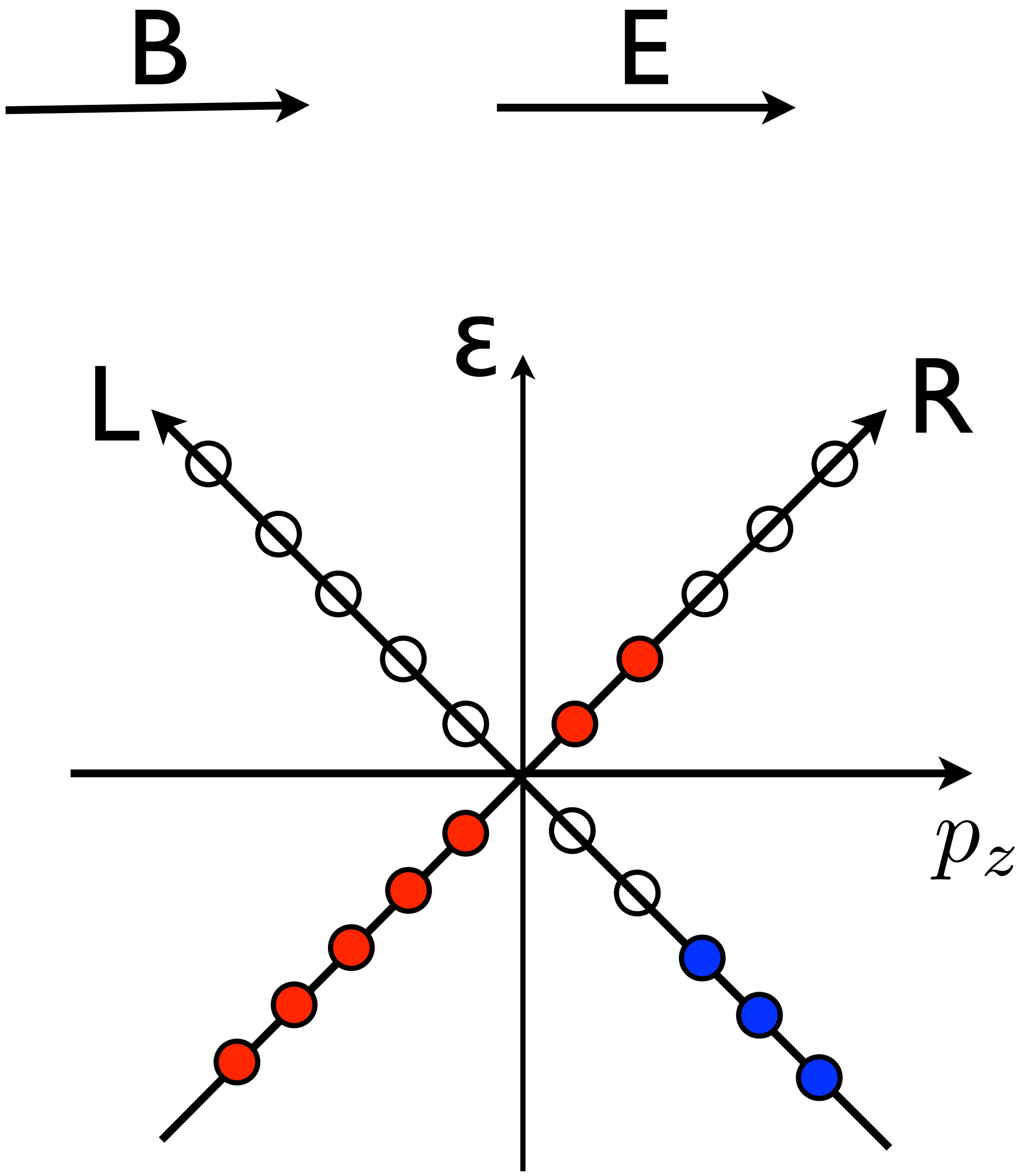

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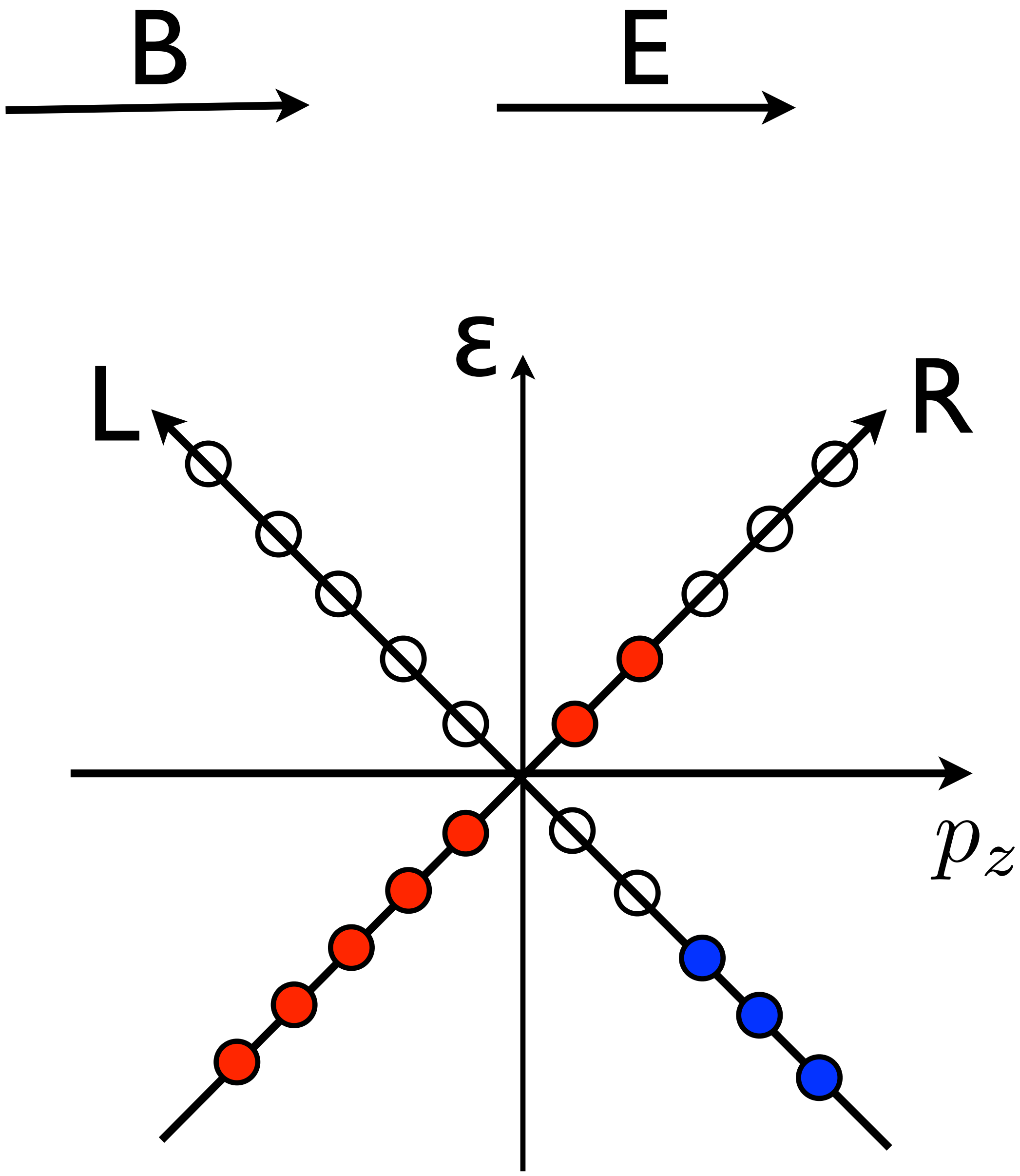

$$\frac{d}{dt}(N_R - N_L) \sim E \cdot B$$

Forbidden by Landau?

- Terms with epsilon tensor do not appear in the standard Landau-Lifshitz treatment of hydrodynamics
- Was it deliberate?

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- Was it deliberate?

Possible reason: 2nd law of thermodynamics

Dissipative terms

Standard textbook manipulations (single U(1) charge)

$$\partial_\mu[(\epsilon+P)u^\mu u^\nu]+\partial^\nu P+\partial_\mu \tau^{\mu\nu}=0$$

$$\partial_\mu(nu^\mu)+\partial_\mu \nu^\mu=0$$

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$$\partial_\mu[(Ts+\mu n)u^\mu u^\nu]+\partial^\nu P+\partial_\mu\tau^{\mu\nu}=0$$

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$$\partial_\mu(su^\mu)=\frac{\mu}{T}\partial_\mu\nu^\mu+\frac{1}{T}\quad u_\nu\,\partial_\mu\tau^{\mu\nu}$$

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$$\partial_\mu\left(su^\mu-\frac{\mu}{T}\nu^\mu\right)=\qquad\frac{\mu}{T}\partial_\mu\nu^\mu+\frac{1}{T}\quad u_\nu\,\partial_\mu\tau^{\mu\nu}$$

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↑

entropy current s^μ

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↑

entropy current s^μ

Positivity of entropy production constrains the dissipation terms: only three kinetic coefficients η, ζ, and σ (right hand side positive-definite)

Is there a place for a new kinetic coefficient?

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) = -\frac{1}{T} \tau^{\mu\nu} \partial_\mu u_\nu - \nu^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

Consider a theory with a single conserved chiral charge

Can we add to the current: $\nu^\mu = \cdots + \xi \omega^\mu$?

Problem: Extra term in current would lead to

$$\partial_\mu s^\mu = \cdots - \xi \omega^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$ not manifestly zero

This can have either sign, and can overwhelm other terms

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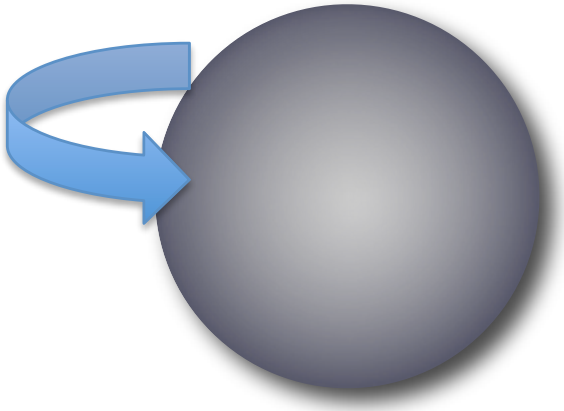
Forbidden by 2nd law of thermodynamics?

Holography

The first indication that standard hydrodynamic equations are not complete comes from considering



rotating 3-sphere of N=4 SYM plasma ↔ rotating BH

If the sphere is large: hydrodynamics should work
  no shear flow: corrections ~ 1/R^2
Instead: corrections ~ 1/R Bhattacharyya, Lahiri, Loganayagam, Minwalla

Holography (II)

Erdmenger et al. arXiv:0809.2488

Banerjee et al. arXiv:0809.2596

considered N=4 super Yang Mills at strong coupling
finite T and μ

should be described by a hydrodynamic theory

discovered that there is a current ~ vorticity

Found the kinetic coefficient ξ(T,μ)

$$\xi = \frac{N^2}{4\sqrt{3}\pi^2}\mu^2 \left(\sqrt{1+\frac{2}{3}\frac{\mu^2}{\pi^2T^2}}+1\right)\left(3\sqrt{1+\frac{2}{3}\frac{\mu^2}{\pi^2T^2}}-1\right)^{-1}$$

Fluid-gravity correspondence

- Long-distance dynamics of black-brane horizons (in AdS) are described by hydrodynamic equations
    - finite-T field theory ↔ AdS black holes

described by hydrodynamics

- Charged black branes in Einstein-Maxwell theory: hydrodynamics with conserved charges
- Anomalies: Chern-Simons term in 5D action of gauge fields

A holographic fluid

$$S = \frac{1}{8\pi G}\int d^5x\,\sqrt{-g}\left(R - 12 - \frac{1}{4}F_{AB}^2 + \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB} F_{CD}\right)$$

encodes anomalies

Black brane solution (Eddington coordinates)

$$ds^2 = 2dvdr - r^2 f(r,m,q)dv^2 + r^2 d\vec{x}^2 \qquad f(m,q,r) = 1 - \frac{m^4}{r^4} + \frac{q^2}{r^6}$$

$$A_0(r) = \#\frac{q}{r^2}$$

Boosted black brane: also a solution

$$ds^2 = -2u_\mu dx^\mu dr + r^2(P_{\mu\nu} - f u_\mu u_\nu)dx^\mu dx^\nu$$

$$A_\mu(r) = -u_\mu \# \frac{q}{r^2}$$

Promoting parameters into variables

$$u_\mu \to u_\mu(x) \qquad m \to m(x) \qquad q \to q(x)$$

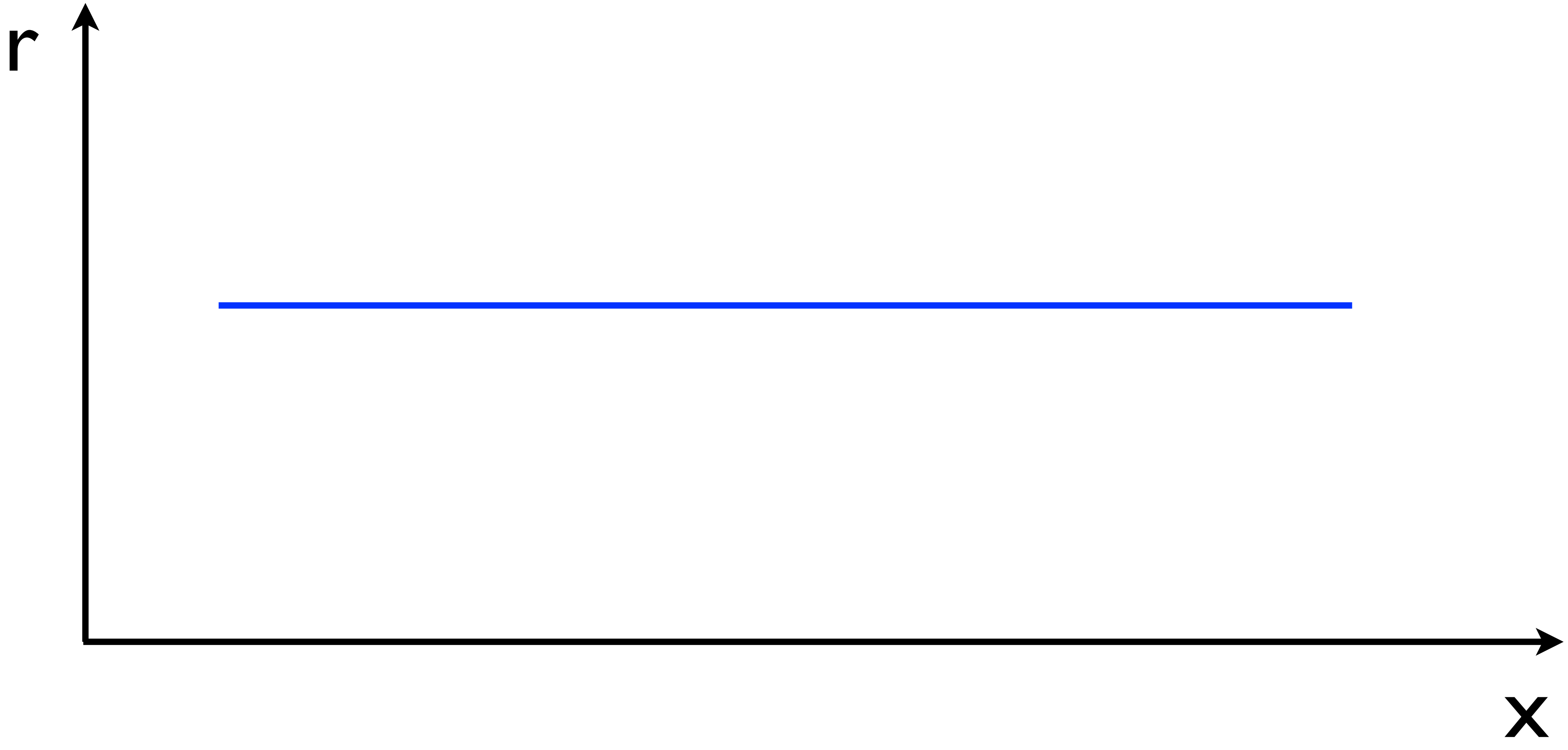
$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(m, q, u) + g^{1}_{\mu\nu}$$

proportional to ∇m, ∇q, ∇u

Solve for g^1 perturbatively in derivaties

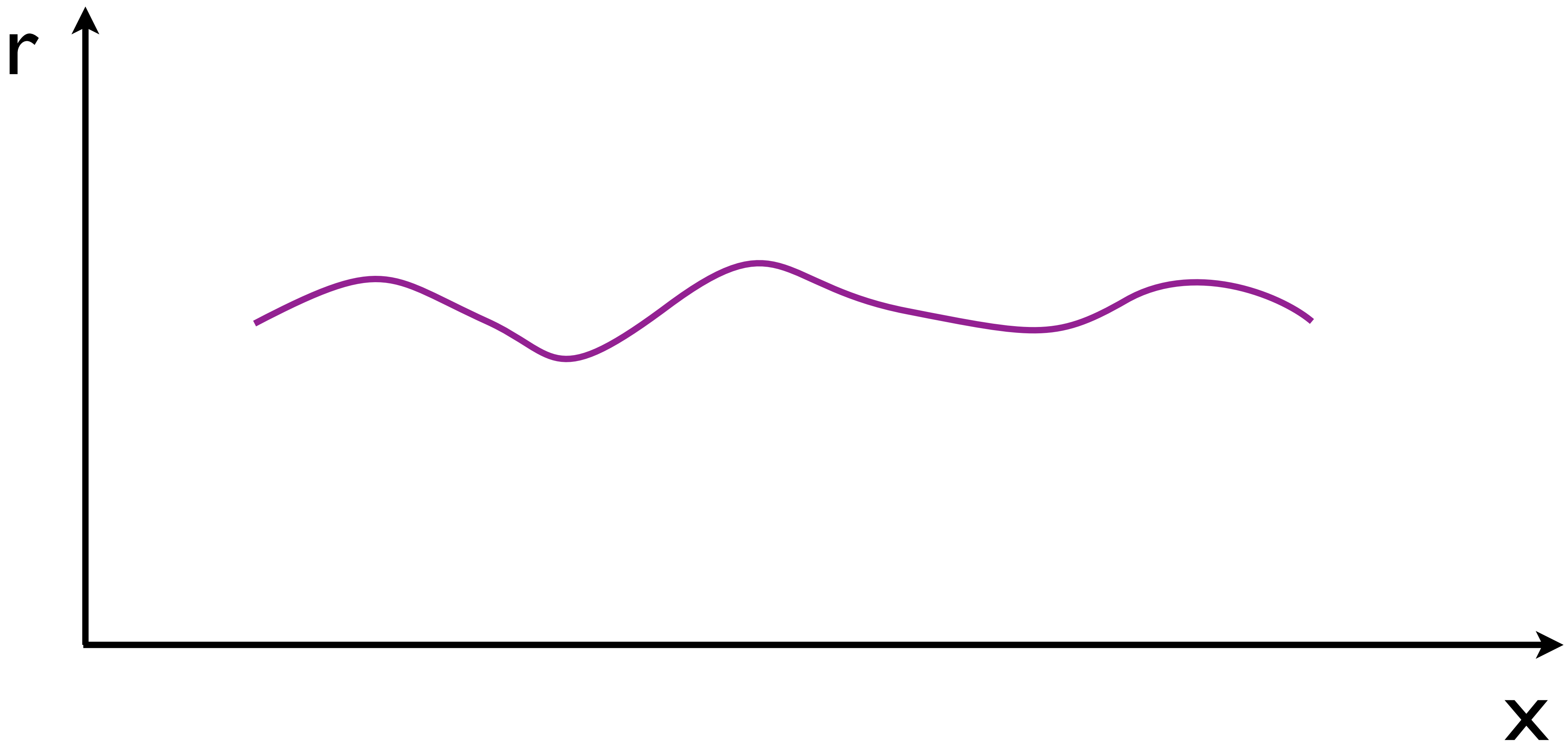
Condition: no singularity outside the horizon

In picture



BH horizon in equilibrium

In picture



BH horizon out of equilibrium

Learning from holography

- Chern-Simons term enters the equation of motion

$$\Box A^{\mu} \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta}$$

Learning from holography

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$$\Box \underset{\substack{\uparrow \\ i}}{A^{\mu}} \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\underset{\substack{\uparrow \\ 0}}{\nu}\underset{\substack{\uparrow \\ r}}{\lambda}} F_{\underset{\substack{\uparrow \\ j}}{\alpha}\underset{\substack{\uparrow \\ k}}{\beta}}$$

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$$A_i \sim u_i$$

Learning from holography

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$$A_i \sim u_i$$

- This lead to correction to the gauge field
  - $\delta A_i \sim \varepsilon_{ijk} \partial_j u_k$
- Current is read out from asymptotics of A near the boundary: $j \sim \omega$

Back to hydrodynamics

- How can the argument based on 2nd law of thermodynamics fail?
  - 2nd law not valid? unlikely...
  - Maybe we were not careful enough?

$$\partial_\mu s^\mu = \cdots - \xi \omega^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

Can this be a total derivative?

If yes, then all we need to to is to modify s^μ

$$s^\mu \to s^\mu + D(T, \mu)\omega^\mu$$

so our task is to find D so that

$$\partial_\mu [D(T,\mu)\omega^\mu] = \xi(T,\mu)\omega^\mu \partial_\mu \left(\frac{\mu}{T}\right)$$

for all solutions to hydrodynamic equations

This is possible for a special class of ξ(T,μ) (expressible in terms of a function of 1 variable: μ/T

but we are still not able to relate ξ to anomalies

Turning on external fields

- To see where anomalies enter, we turn on external background U(1) field A_μ
- Theory still makes sense if A_μ is non dynamical
- Now the energy-momentum and charge are not conserved

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda$$

$$\partial_\mu j^\mu = -\frac{C}{8}\epsilon^{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho}$$

- Power counting: A~1, F~O(p): right hand side has to be taken into account

Anomalous hydrodynamics

- These equations have to be supplemented by the constitutive relations:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu}$$ +viscosities

$$j^\mu = nu^\mu + \xi\omega^\mu + \xi_B B^\mu \qquad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$$

+diffusion+Ohmic current

- We demand that there exist an entropy current with positive derivative: $\partial_\mu s_\mu \geq 0$
- The most general entropy current is

$$s^\mu = su^\mu - \frac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu$$

Entropy production

- Positivity of entropy production completely fixes all functions ξ, ξ_B, D, D_B

$$\xi = C \left(\mu^2 - \frac{2}{3} \frac{n\mu^3}{\epsilon + P} \right)$$

anomaly coefficient

$$\xi_B = C \left(\mu - \frac{1}{2} \frac{n\mu^2}{\epsilon + P} \right)$$

$$j^\mu = \cdots + \xi \omega^\mu + \xi_B B^\mu$$

These expressions have been checked for N=4 SYM

A more convenient “frame”

$$j^\mu = n u^\mu + C\mu^2 \omega^\mu$$

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \frac{2}{3} C \mu^3 (u^\mu \omega^\nu + u^\nu \omega^\mu)$$

can be eliminated by redefinition of u

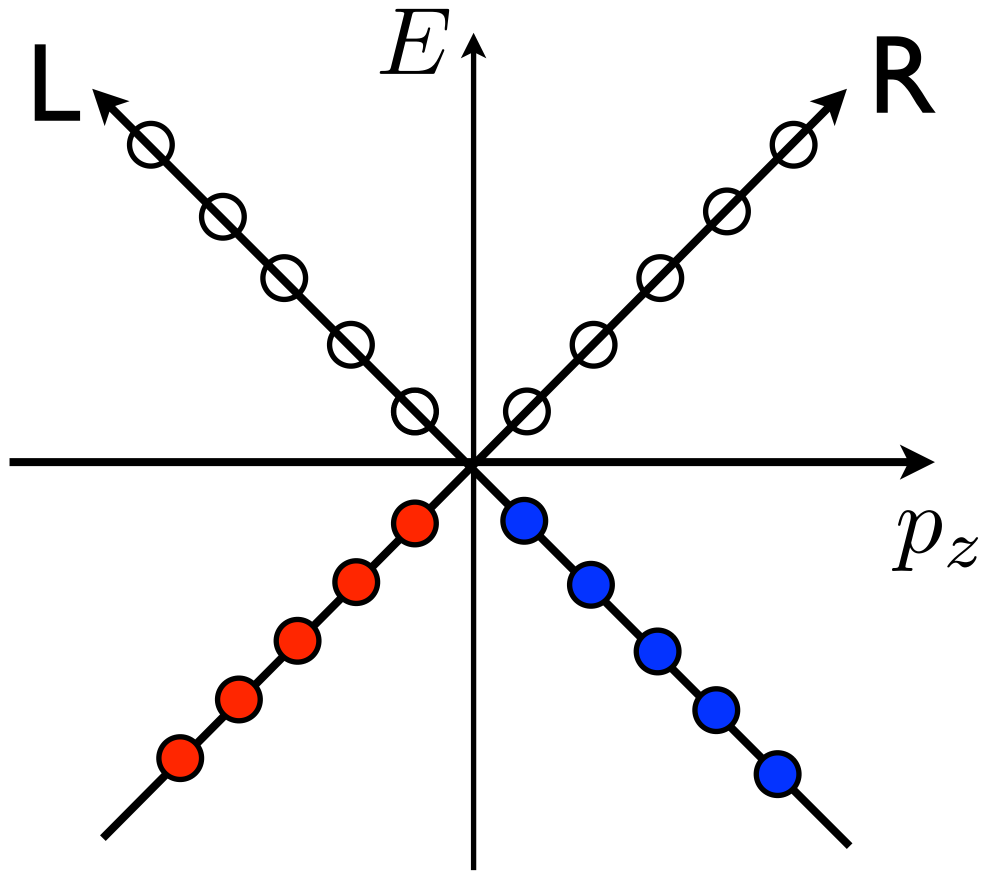
anomalous terms are “quantized”

Current induced by magnetic field

Spectrum of Dirac operator:

$$E^2 = 2nB + p_z^2$$

All states LR degenerate except for n=0



$$j_{\rm L} \sim -C\mu B$$
$$j_{\rm R} \sim C\mu B$$

$$j_5 = j_R - j_L \sim C\mu B$$

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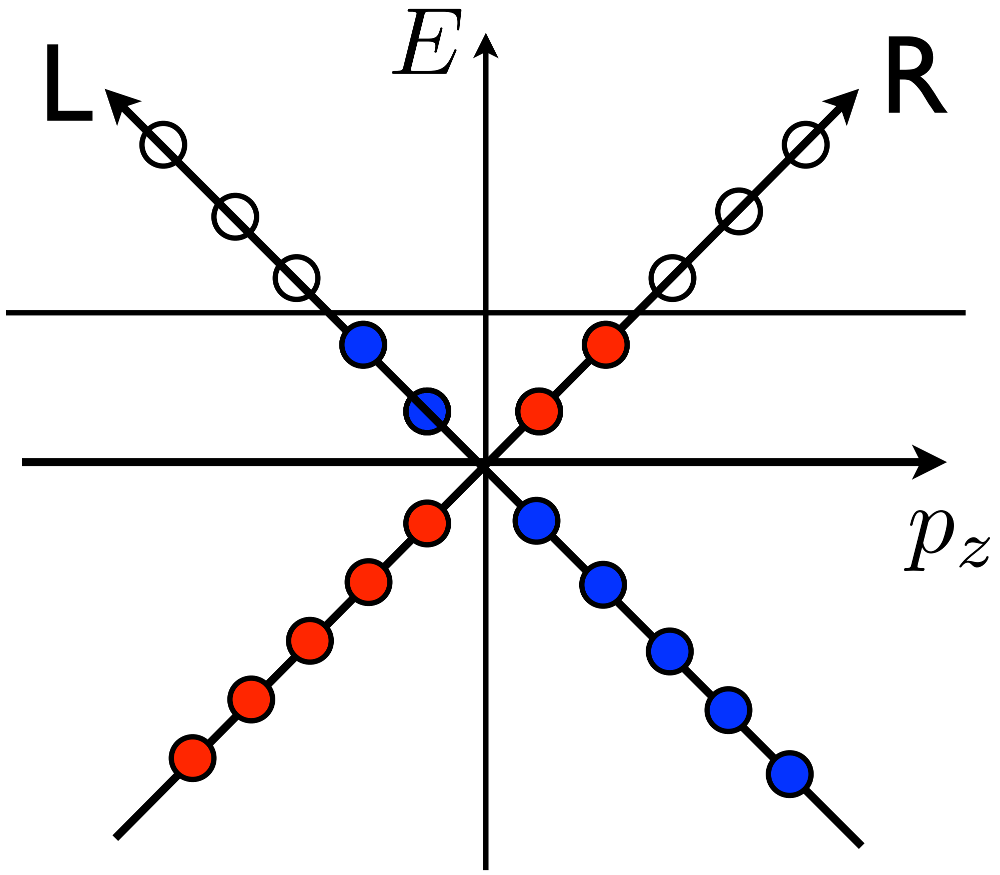


$$j_{\mathrm{L}} \sim -C\mu B$$
$$j_{\mathrm{R}} \sim C\mu B$$

$$j_5 = j_R - j_L \sim C\mu B$$

If there is only right-handed fermions:

$$j^\mu = nu^\mu + C\mu B^\mu$$

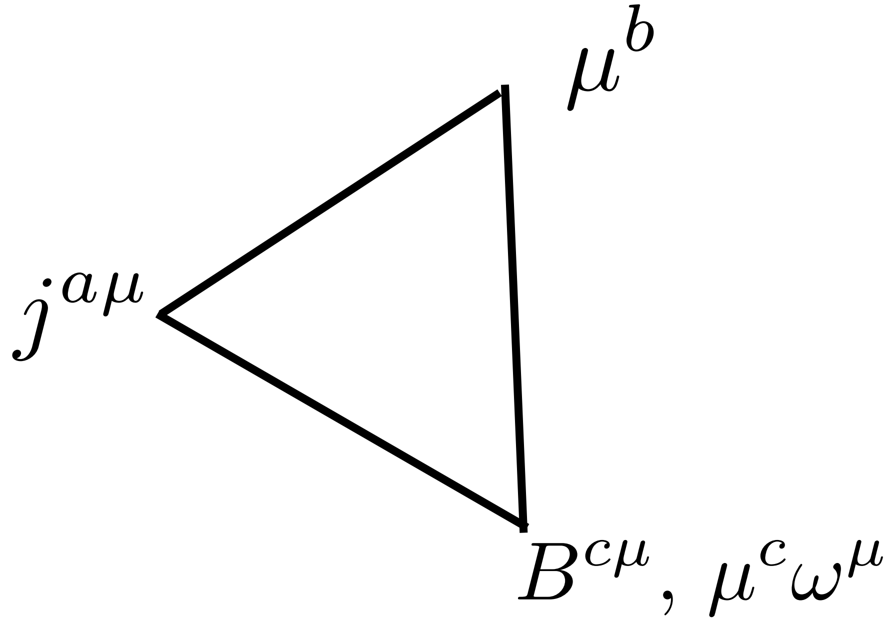
$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + \frac{C}{2}\mu^2(u^\mu B^\nu + u^\nu B^\mu)$$

going to the Landau-Lifshitz frame gives the correct ξ_B

No similar picture for vorticity induced current

Multiple charges

In the case when there are multiple conserved charges: anomalous contribution to each current



$$\sim C^{abc}$$

$$j^{a\mu} = \cdots + \# C^{abc}\mu^b\mu^c\omega^\mu + \# C^{abc}\mu^b B^{c\mu}$$

(these are gauge invariant, non-conserved currents)

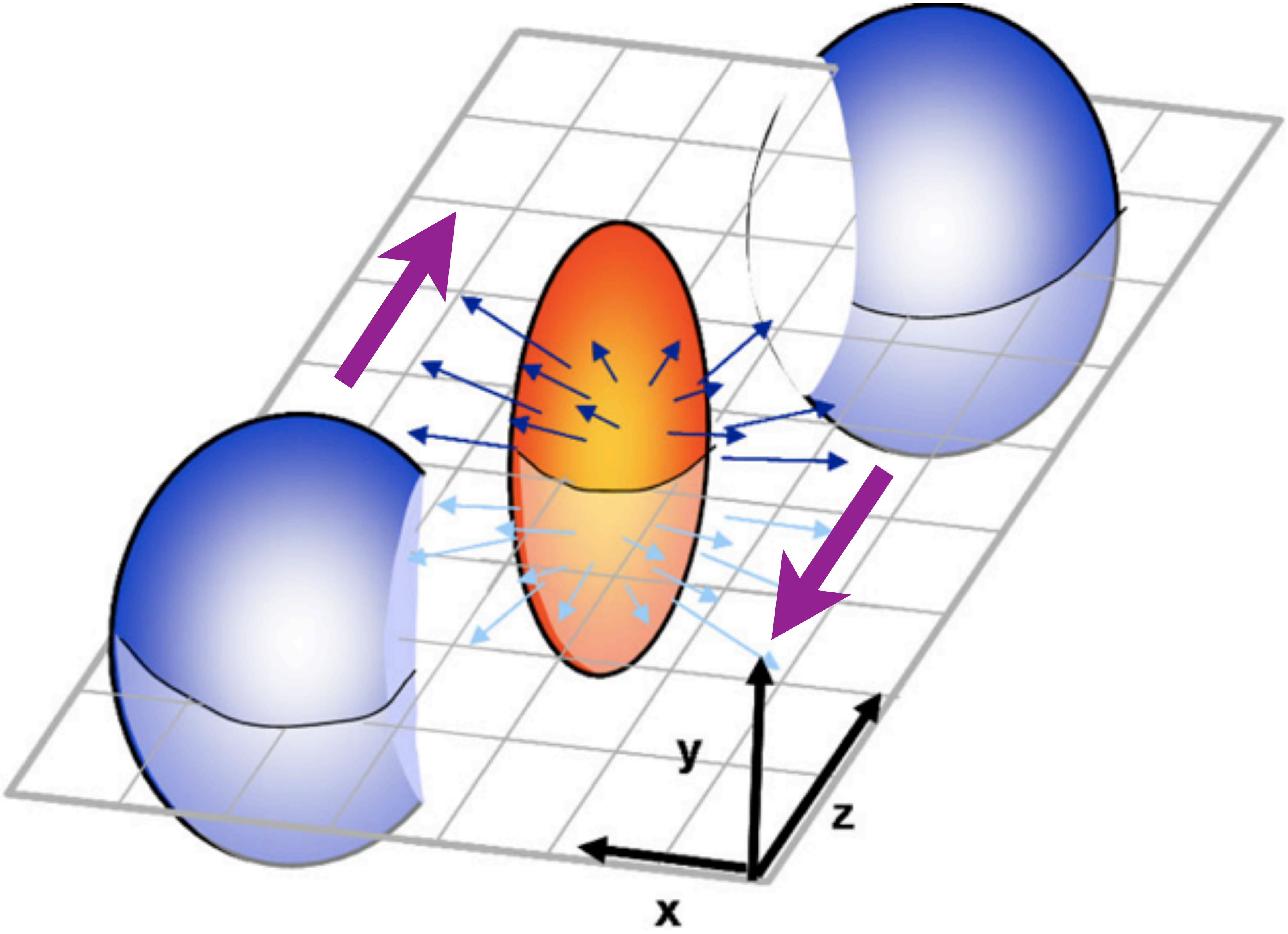
Multiple charges (II)

Example: theory with one massless Dirac fermion

$$j^\mu = \frac{1}{2\pi^2}(2\mu\mu_5\omega^\mu + \mu_5 B^\mu + \mu B_5^\mu)$$

$$j_5^\mu = \frac{1}{2\pi^2}((\mu^2 + \mu_5^2)\omega^\mu + \mu B^\mu + \mu_5 B_5^\mu)$$

Observable effect on heavy-ion collsions?

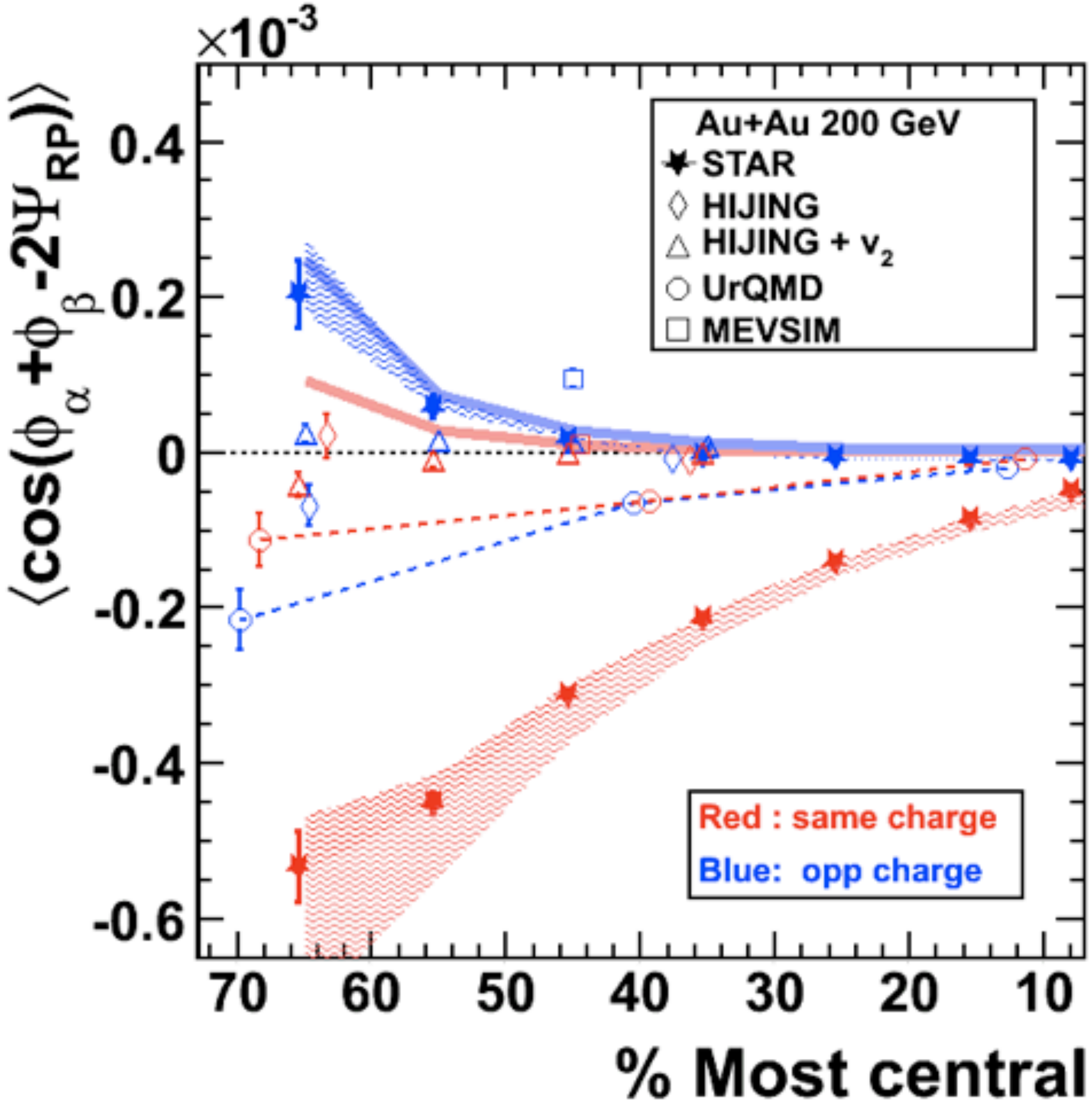


Chiral charges accumulate at the poles: what happens when they decay?

“Chiral magnetic effect”

- Large axial chemical potential μ_5 for some reason
- Leads to a vector current: charge separation
- π^+ and π^- would have anticorrelation in momenta
- Some experimental signal?
- Attempts to explain the signal by $j \sim \mu_5 B$ Kharzeev et al

STAR result



Abelev et al. PRL 2009 (arxiv:0909.1739)

From kinetic theory?

- The anomalous hydrodynamics current also exists in weakly coupled theories
- Should be derivable from kinetic theory, for example from Landau's Fermi liquid theories
- which kind of corrections to Landau's Fermi liquid theory?
  - should distinguish left- and right-handed quarks
- Berry's curvature on the Fermi surface?

Conclusions

- Anomalies affect hydrodynamic behavior of relativistic fluids
- First seen in holographic models, but can be found by reconciling anomalies and 2nd law
- Further studies of experimental significance needed
- Anomalies in Landau's Fermi liquid theory?