Chern-Simons theory: That obscure object of desire

University of California @ Davis

Davis, April 14, 2008

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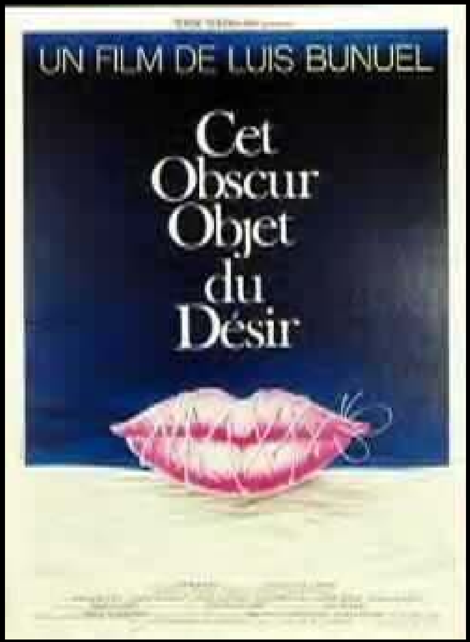

1977 film about a neurotic relationship between a middle aged man and a beautiful young woman who drives him crazy.

She seduces and promises but never yields to the guy's wishes.

The situation repeats itself endlessly, but with a new surprising twist every time.

It is frustrating and nerve-wracking, but it's also addictive.

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A typical Yang-Mills action is something like this:

$$I[A] = \frac{1}{4g} \int_{M^4} \sqrt{g}\, g^{\mu\alpha} g^{\nu\beta} \gamma_{ab} F^a{}_{\mu\nu} F^b{}_{\alpha\beta} d^4x$$

A typical Chern-Simons action is something like this:

$$I[A] = \kappa \int_{M^3} \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle$$

or this…

$$I[A] = \kappa \int_{M^{2n+1}} \left\langle A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1} \right\rangle$$

Chern-Simons lagrangians define gauge field theories in a different class:

- They are explicit functions of the connection (A), not local functions of the curvature (F).

- Yet, they yield gauge-invariant field equations.

- Related to homotopic/topological invariants on fiber bundles: characteristic classes.

- They require no metric; just a Lie algebra (not necessarily semisimple); no adjustable parameters, conformally invariant. More fundamental(?)

- They are very sensitive to the dimension.

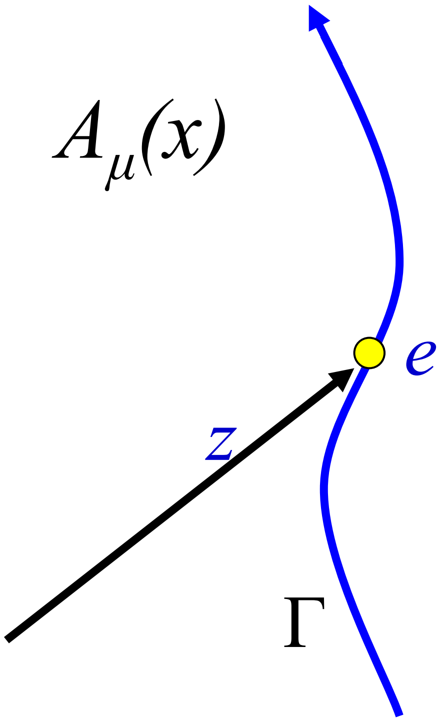
- They naturally couple to branes.

- Their quantization corresponds to sum over holonomies in an embedding space.

- CS theories are not exotic but a rather common occurrence in nature: Anomalies, quantum Hall effect, 11D supergravity (CJS), superconductivity, $2+1$ gravity, $2n+1$ gravity, all of classical mechanics,...

1. CS action in *0+1* dimensions

E-M coupling



$$I[A,z] = \int j^\mu A_\mu d^D x$$

$$j^\mu(x) = e\int_{-\infty}^{+\infty} \dot{z}^\mu(\tau)\delta(x - z(\tau))d\tau$$

$$I[A(z)] = \int eA_\mu(z)dz^\mu = e\int_\Gamma A(z)$$

Gauge invariance $A_\mu(x) \rightarrow A_\mu(x)+\partial_\mu \Omega(x)$, is ensured by current conservation, $\partial_\mu j^\mu = 0$, provided $\Omega(z(+\infty))=\Omega(z(-\infty))$.

Not quite gauge invariant, but quasi-invariant.

$$I[z] = \int j^{\mu} A_{\mu} d^4 x$$

This expression is invariant under

- Lorentz transformations, ($\Lambda^{\mu}{}_{\nu}$)
- Gauge transformations $A \rightarrow A + d\Omega(\mathrm{x})$
- Gen. coordinate transf. $z^{\mu} \rightarrow z'^{\mu}(z)$

This coupling is consistent with the minimal derivative substitution

$$p_{\mu} \rightarrow p_{\mu} - eA_{\mu}(z), \quad \partial_{\mu} \rightarrow \partial_{\mu} + ieA_{\mu}(z)$$

Good for quantization:

$$\partial_{\mu}\Psi \rightarrow \left(\partial_{\mu} + ieA_{\mu}(z)\right)\Psi$$

The simplest Chern-Simons action

Take

$$I[A]=\kappa\int_{M^{2n+1}}\left\langle A\wedge(dA)^{n}+c_1A^{3}\wedge(dA)^{n-1}+\cdots+c_nA^{2n+1}\right\rangle$$

for an abelian connection and set $n=0$:

$$I[A]=e\int_{\Gamma}A(z)$$

0+1 CS theory

Is this a sensible action?

$$1=0$$

? ? ?

not exactly…

Varying the action,

$$\delta I[A] = e\int_{\Gamma} \delta A(z) = 0$$

This only means $\delta A = d\Omega$ with $\Omega(-\infty) = \Omega(\infty)$, or $\partial\Gamma = 0$.

The classical configurations are arbitrary *U(1)* connections with PBC or living in a periodic *1d* spacetime

Alternatively, *I* can also be viewed as an action for the embedding coordinates z^{μ},

$$I[z] = e\int_{\Gamma} A(z)$$

Varying the action,

$$\delta I[z] = e\int_{\Gamma} \delta z^{\mu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})\dot{z}^{\nu} + eA_{\mu}\delta z^{\mu}\Big|_{\partial\Gamma}$$

$$\delta I = 0 \Rightarrow F_{\mu\nu}(z)\dot{z}^{\nu} = 0$$

The classical orbits are those with zero Lorentz force:

$$\vec{E} + \vec{v}\times\vec{B} = 0$$

the electric and magnetic forces cancel each other out.

N.B.: In order to obtain the equation of motion, z must satisfy periodic boundary conditions.

$$A_{\mu}(z)\delta z^{\mu}\Big|_{\partial\Gamma} = 0$$

A bit about the quantum theory

$$Z[z] = \int_{PBC} [\mu(z)] \exp\left(\frac{i}{\eta} I[z]\right) = \int_{PBC} [\mu(z)] \exp\left(\frac{ie}{\eta} \oint A(z)\right)$$

Thus, the integral is dominated by those orbits for which the holonomies are quantized:

$$\frac{e}{\eta} \oint_{S^1} A(z) = 2n\pi \Rightarrow e \int_{D^2} F = 2n\pi\eta$$

Flux quantization

Does this describe a physically sensible system?

What are the degrees of freedom?

$$I[z] = e\int_\Gamma A_\mu(z)\dot{z}^\mu d\tau$$

Let, $z^\mu=(z^0=t,\ z^i)$, $i=1,2,\cdots,2s$.

$$I[z] = e\int A(z) = e\int_\Gamma [A_0(z) + A_i(z)\dot{z}^i]dt,$$

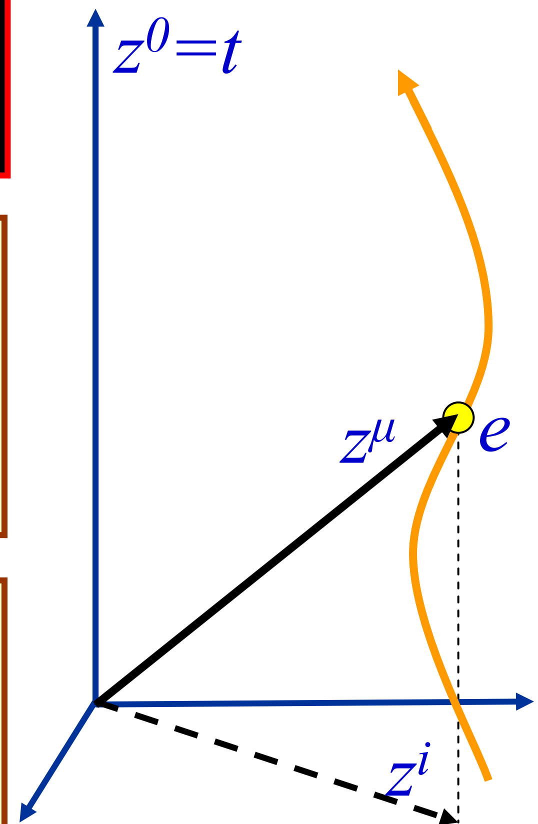
This action describes a mechanical system:

$$I[z] = \int_\Gamma [p_i \dot{q}^i - H(z)]dt,$$

where $eA_i(z)= p_i$ (2nd class constraints),
and $eA_0(z)= -H$

The equations of motion are Hamilton's

$$F_{ij}(z)\dot{z}^j = E_i(z) \Rightarrow \varepsilon_{ij}\dot{z}^j = \partial_i H(z)$$



- Any mechanical system with s degrees of freedom can be described by a 0+1 C-S action in a ($2s+1$)-dimensional target space. (Jackiw-Percacci '87)

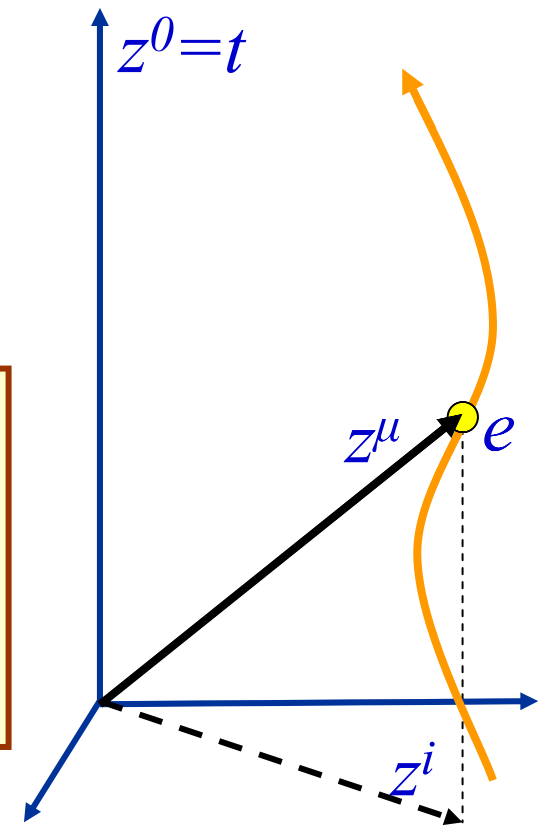
What is the meaning of the flux quantization?

$$e\oint_{S^1} A(z) = 2n\pi\eta$$

Substituting $eA_0(z) = -H$ and $eA_i(z) = p_i$,

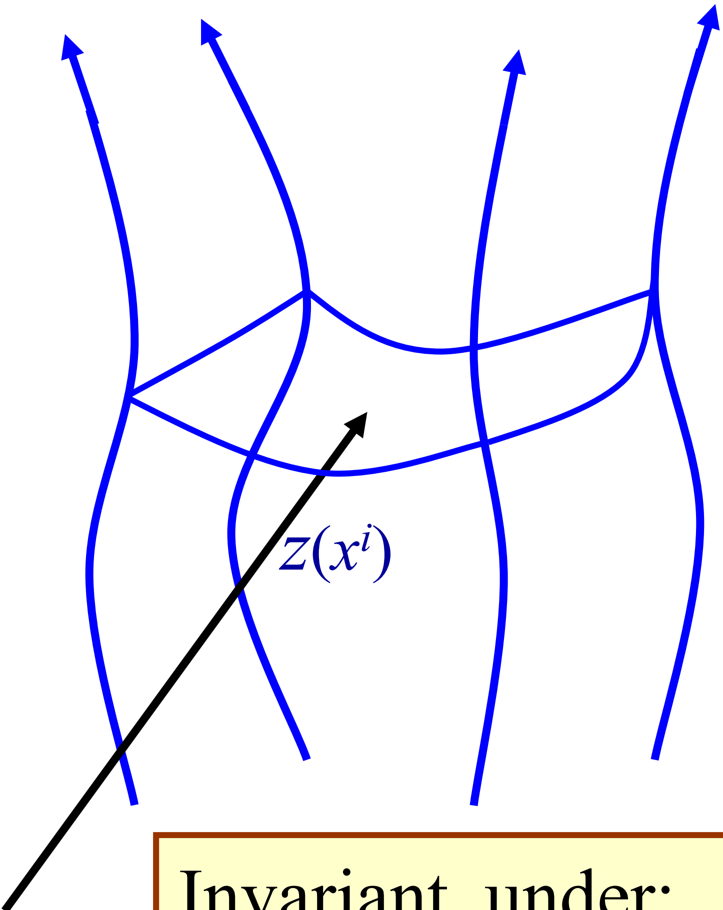
$$\oint_{S^1} [p_i dq^i - H dt] = 2n\pi\eta$$

Bohr-Sommerfeld quantization rule



0+1Chern-Simons		Classical mechanics
Vanishing Lorentz force	⟷	Hamilton’s equations
Gauge invariance	⟷	Invariance under canonical transf.
Gen. coordinate transformations	⟷	Invariance under time reparametrizations
Flux/holonomy quantization	⟷	Bohr-Sommerfeld quantization

More dimensions…



A CS action in 2+1 dimensions can be viewed as a coupling between a brane and an external gauge field

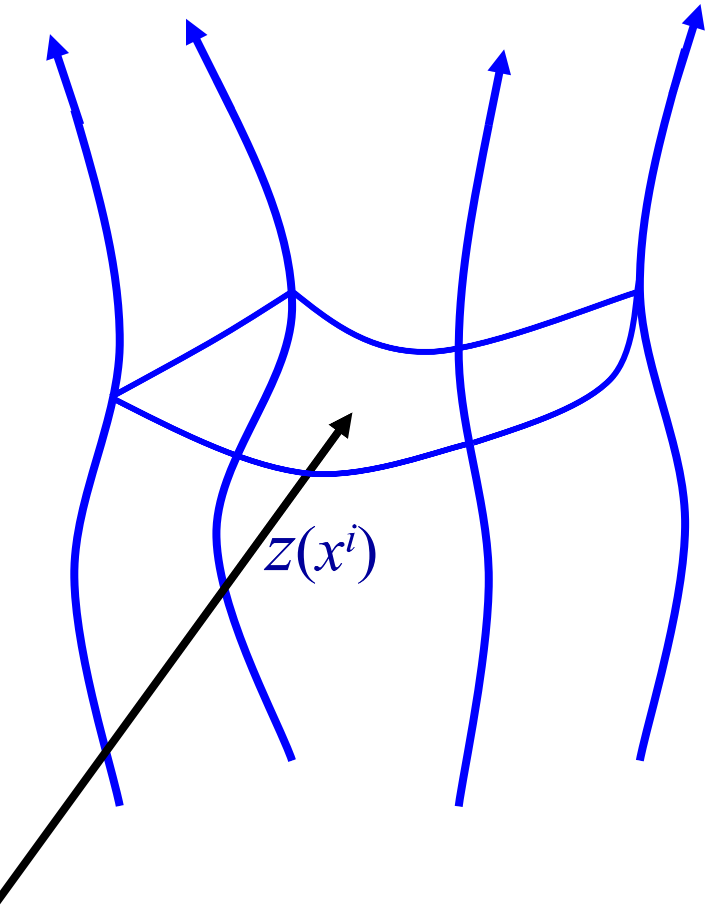
$$I[z] = \int j^{\mu\nu\lambda}(A_\mu \partial_\nu A_\lambda) d^n x$$

$$= \kappa \int_{\Gamma^3} A(z) dA(z)$$

Invariant under:

- Gen. coord. transf. on the worldvolume $z^\mu \rightarrow z'^\mu(z)$
- Lorentz transformations on the target space ($\Lambda^\mu{}_\nu$)
- Gauge transformations $A \rightarrow A + d\Omega$ [quasi-invariant]

For $D=3, 5, 7,\ldots$ New possibilities arise:



- Nonabelian algebras

- $A(z)$ can be dynamical (propagating in the worldvolume)

- Worldvolume dynamics (gravity)

- Degeneracy (for $D\geq 5$)

- Quantization? (open problem)

2. CS action in *2n+1* dimensions

Non abelian CS action in *2n+1* dimensions

$$I[A] = \kappa \int_{M^{2n+1}} \left\langle A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1} \right\rangle$$

The coefficients $c_1, \ldots c_n$ are fixed rational numbers,

$$L_{2n+1}^{CS}(A) = (n+1)\kappa \int_0^1 dt \left\langle A \wedge F_t^{n+1} \right\rangle,$$

where $F_t = t dA + t^2 A^2$

Invariant under gauge transformations (up to boundary terms)

$$A' = g^{-1} A g + g^{-1} dg, \quad g(x) \in G$$

Classical CS dynamics

$$I[A] = \int_{\Gamma^{2n+1}} L(A)$$

L	$\delta L=0$	A
0+1 eA	$\delta A = d\Omega$	*Arbitrary connection*
2+1 $\kappa \langle AdA + \frac{2}{3} A \wedge A \wedge A \rangle$	$F = 0$	*Pure gauge, nonpropagating, nondegenerate*
2n+1 $\kappa \langle A(dA)^n + c_1 A^3 (dA)^{n-1} + \cdots + c_n A^{2n+1} \rangle$	$F^{n+1} = 0$	*Nontrivial, propagating, degenerate*

Degeneracy of CS theories ($D=2n+1 \geq 5$)

The problem arises from the fact that for $D=2n+1$ with $n \geq 2$, the field equations are nonlinear in the curvature,

$$\langle G_k F^n \rangle = 0$$

where G_k are the generators of the Lie algebra.

The linearized perturbations around a given classical configuration F_0, obey

$$\langle G_k F_0{}^{n-1} \delta F \rangle = 0$$

The dynamics depends on the form of F_0 .

Consequences of the degeneracy

- Unpredictability of evolution
- Freezing out of degrees of freedom
- Loss of information about the initial data
- Irreversibility of evolution

D/source	*Quantization*	*Comment*
$0+1$ *point particle (0-brane)*	$e\oint A = e\oiint F = 2k\pi\eta$	*Holonomy/Flux quant.* *Dirac quant. condition* *Bohr-Sommerfeld rule*
$2+1$ *membrane*	$A = g_k^{-1} dg_k, \quad g \in \text{Top. class}$	*Finite, power- counting renormalizable*
$2n+1$ *2n-brane*	Holonomies + local deg. of f. How?	*Anybody's guess*

3. CS Gravity actions in *2n+1* dimensions

1. <u>Equivalence Principle:</u>

- Spacetime is locally approximated by Minkowski space and has the same (local) Lorentz symmetry.

- GR is the oldest known nonabelian gauge theory; gauge group $SO(3,1)$.

2. Gravitation should be a theory whose output is the spacetime geometry. Therefore, it is best to start with a theory that makes no assumptions about the local geometry.

1&2 → Chern-Simons theory is probably a better choice

There are two characteristic classes associated to the rotation groups $SO(s,t)$: the Euler and the Pontryagin classes. Associated with each of them there are the corresponding CS actions

Action (first order formalism):

$$I[e,\omega]=\kappa\int_{M^D} L_D\left(e,de,\omega,d\omega\right)$$

Vielbein (metricity) → e

Connection (parallelism) → ω

$$R^{ab}=d\omega^{ab}+\omega^{a}{}_{c}\wedge\omega^{cb}$$ = Curvature 2-form

$$T^{a}=de^{a}+\omega^{a}{}_{b}\wedge e^{b}$$ = Torsion 2-form

Steps for constructing CS gravities:

1. Combine the vielbein and spin connection into a connection for the *dS*, *AdS*, or *Poincaré* group:

$$A = l^{-1} e^{a} \mathrm{J}_{a} + \tfrac{1}{2}\omega^{ab}\mathrm{J}_{ab},$$

2. Select the bracket that corresponds to the invariant characteristic class to be used (Euler or Pontryagin), e.g.,

$$\left\langle \mathrm{J}_{a_1}\mathrm{J}_{a_2a_3}\cdots\mathrm{J}_{a_{D-1}a_D}\right\rangle = \varepsilon_{a_1\cdots a_D}$$

3. Write down the lagrangian

$$L(e,\omega) = (n+1)\int_0^1 dt \left\langle A\wedge F_t^{\,n}\right\rangle$$

CS gravities (*D=2n-1*) (summary)

- No dimensionful constants (scale invariant)
- No arbitrary adjustable/renormalizable constants
- Possess black hole solutions $\Lambda = -\# l^{-D/2} < 0$
- Admit $\Lambda \neq 0$ and $\Lambda \to 0 \;\; (l \to \infty)$ limit
- Admit SUSY extensions for $\Lambda \leq 0$ and any odd *D*,
  and yield field theories with *spins* $\leq$ *2* only
- Give rise to acceptable *D=4* effective theories

4. Coupling to matter sources

It is a beautiful feature of CS theories the fact that they possess no free adjustable coupling constants.

…and it is also one of the difficulties when trying to make sense of them

$$I[A] = \kappa \int_{M^{2n+1}} \left\langle A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1} \right\rangle$$

Quantized (κ)

Fixed rational numbers ~ O(1) ($c_1, \ldots, c_n$)

But doesn't mean one cannot have interactions

Nothing prevents putting together CS actions of different dimensions,

$$I[A] = \sum_{r=0} \kappa_r \int_{M^{2r+1}} \left\langle A \wedge (dA)^r + c_1 A^3 \wedge (dA)^{r-1} + \cdots + c_n A^{2r+1} \right\rangle$$

But, what would this mean?

Consider the simplest case:

$$I[A] = \kappa_3 \int_{M^{2+1}} \left\langle A \wedge dA + \frac{2}{3} A^3 \right\rangle + \kappa_1 \int_{M^{0+1}} \overline{A} \qquad A \in \mathcal{G}$$

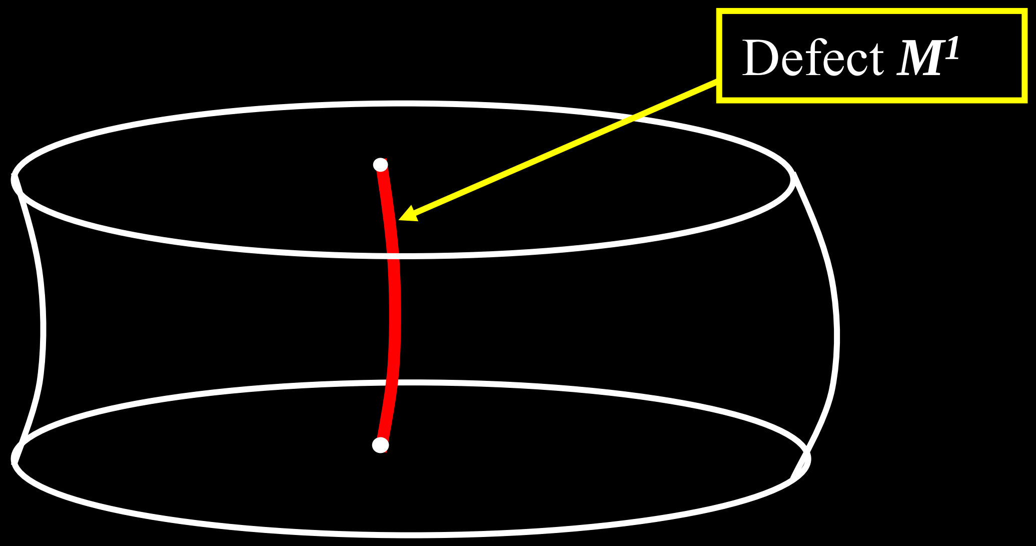
where $\overline{A}$ is the restriction of A to an abelian subalgebra

$$\overline{A} \in \mathcal{G}_0 \subset \mathcal{G}$$

The effects of coupling to this 0-brane are

1. Breaking the symmetry: $\mathcal{G} \rightarrow \mathcal{G}_0$

2. Introducing a topological defect.



For example, if $\mathcal{G} = SO(2,2)$ the 3d part describes 2+1 gravity, and the defect is a sort of 'conical' singularity…

Under closer scrutiny, it can be seen that the defect is the result of an identification by an abelian subgroup of the *AdS_3* group: a 2+1 black hole!

The mass and angular momentum are related to the strength of the coupling constant (κ_1) and the particular subgroup of $SO(2,2)$ that is used.

Coupling more *2n*-branes in this way, more complex structures can be produced (black holes, branes, ...?)

5. Summary

- CS actions have been used in physics much longer than one usually thinks: e-m coupling, all of classical mechanics!
- CS theories can be viewed as boundary theories coming from topological field theories in even $D = 2n$ manifolds.
- They have no free adjustable parameters and require no metric structure.
- Degeneracy for $D \geq 5$: limited predictability, irreversible loss of degrees of freedom, dynamical dimensional reduction.
- There exist CS (super-) gravities with dimensionless couplings, all fields have spins ≤ 2 and the metric is not a fundamental field but a condensate…

- The natural way to couple CS theories is to *2n*-branes.
- The different branes produce topological defects of co-dimension 2, 4,…
- They break supersymmetry down to ½, ¼, … of the one in the highest dimensional CS form.
- …..

CS theories are so exceptional, it's not only worth studying them. It is also understandable if one looses his mind because of them…

Thanks!