Worldline Path Integral Formalism new results and applications

Davis, 27 February 2007

Olindo Corradini

Dipartimento di Fisica e INFN, Bologna

Outline of the talk

Mainly based on 0205182, 0211134, 0312064, 0503155, 0510010, 0612236, 0701055

Bastianelli, Benincasa, OC, Giombi, Latini, Pisani, Zirotti

1.

Outline of the talk

Mainly based on 0205182, 0211134, 0312064, 0503155, 0510010, 0612236, 0701055

Bastianelli, Benincasa, OC, Giombi, Latini, Pisani, Zirotti

1. Introduction

Outline of the talk

Mainly based on 0205182, 0211134, 0312064, 0503155, 0510010, 0612236, 0701055

Bastianelli, Benincasa, OC, Giombi, Latini, Pisani, Zirotti

1. Introduction
2. Worldline formalism in flat space
   - the case of scalar QED
     $\rightarrow$ 1D path integral in flat space

Outline of the talk

Mainly based on 0205182, 0211134, 0312064, 0503155, 0510010, 0612236, 0701055

Bastianelli, Benincasa, OC, Giombi, Latini, Pisani, Zirotti

1. Introduction
2. Worldline formalism in flat space
   - the case of scalar QED
     $\rightarrow$ 1D path integral in flat space
3. Worldline formalism in curved space
   - 1-loop effective action for a scalar field
     $\rightarrow$ 1D path integral in curved space
   - UV regularization of the path integral
   - IR aspects: zero modes on the circle S^1

Outline of the talk

4. Worldline formalism with local SUSY's
   - Spinning particle w/ N=1 $\Rightarrow$ spin-$\frac{1}{2}$ field
   - Spinning particle w/ N=2 $\Rightarrow$ spin-1 field
     $\rightarrow$ coupling to gravity OK

Outline of the talk

4. Worldline formalism with local SUSY's
   - Spinning particle w/ N=1 $\Rightarrow$ spin-$\frac{1}{2}$ field
   - Spinning particle w/ N=2 $\Rightarrow$ spin-1 field
     $\rightarrow$ coupling to gravity OK
5. $SO(N)$ Spinning particle w/ N>2 $\Rightarrow$ spin-$\frac{N}{2}$ field
   - one-loop qzn in flat space
   - Dof's from orthogonal polynomials method $\forall D,\ N$

Outline of the talk

4. Worldline formalism with local SUSY's
   - Spinning particle w/ N=1 $\Rightarrow$ spin-$\frac{1}{2}$ field
   - Spinning particle w/ N=2 $\Rightarrow$ spin-1 field
     $\rightarrow$ coupling to gravity OK
5. $SO(N)$ Spinning particle w/ N>2 $\Rightarrow$ spin-$\frac{N}{2}$ field
   - one-loop qzn in flat space
   - Dof's from orthogonal polynomials method $\forall D,\ N$
6. Manifolds with boundary
   - Method of the "image charge"

Outline of the talk

4. Worldline formalism with local SUSY's
   - Spinning particle w/ N=1 $\Rightarrow$ spin-$\frac{1}{2}$ field
   - Spinning particle w/ N=2 $\Rightarrow$ spin-1 field
     $\rightarrow$ coupling to gravity OK
5. $SO(N)$ Spinning particle w/ N>2 $\Rightarrow$ spin-$\frac{N}{2}$ field
   - one-loop qzn in flat space
   - Dof's from orthogonal polynomials method $\forall D,\ N$
6. Manifolds with boundary
   - Method of the "image charge"
7. Outlook

Introduction

Worldline method
QFT results from QM path integrals

$\Rightarrow$ no need to compute momentum integrals and Dirac traces

- Alternative way to compute correlation functions

Introduction

Worldline method
QFT results from QM path integrals

$\Rightarrow$ no need to compute momentum integrals and Dirac traces

- Alternative way to compute correlation functions
- Effective actions of quantum fields coupled to external fields (gravity, vector), chiral and conformal anomalies

Worldline formalism in flat space

- Case of scalar contribution to QED at 1-loop

Worldline formalism in flat space

- Case of scalar contribution to QED at 1-loop
- Classical action:

$$S[\phi, \phi^*, A] = \int d^D x \, (\, |(\partial_\mu + ieA_\mu)\phi|^2 + m^2 |\phi|^2 \,)$$

Worldline formalism in flat space

- Case of scalar contribution to QED at 1-loop
- Classical action:

$$S[\phi, \phi^*, A] = \int d^D x \, (\, |(\partial_\mu + ieA_\mu)\phi|^2 + m^2|\phi|^2 \,)$$

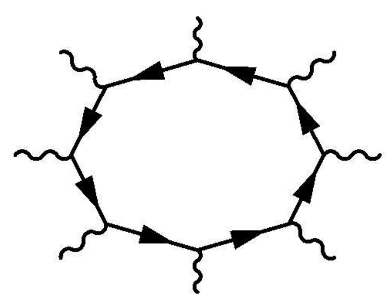
- The corresponding 1-loop effective action is

$$\mathrm{e}^{-\Gamma[A]} = \int D\phi D\phi^* \, \mathrm{e}^{-S[\phi,\phi^*,A]} = \mathrm{Det}^{-1}(-\nabla_A^2 + m^2)$$

Worldline formalism in flat space

- Thus

$$
\begin{aligned}
\Gamma[A] &= \mathrm{Tr}\,\log\left(-\nabla_A^2 + m^2\right) \\
&= -\int_0^\infty \frac{dT}{T}\,\mathrm{Tr}\;\mathrm{e}^{-(-\nabla_A^2+m^2)T} \\
&= -\int_0^\infty \frac{dT}{T}\int_{PBC} Dx\;\mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+ieA_\mu(x)\dot{x}^\mu+m^2\right)} \\
&= \sum
\end{aligned}
$$



Worldline formalism in flat space

- Thus

$$\Gamma[A] = \mathrm{Tr}\,\log\left(-\nabla_A^2 + m^2\right)$$

$$= -\int_0^\infty \frac{dT}{T}\,\mathrm{Tr}\,\mathrm{e}^{-(-\nabla_A^2+m^2)T}$$

$$= -\int_0^\infty \frac{dT}{T} \boxed{\int_{PBC} Dx\,\mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+ieA_\mu(x)\dot{x}^\mu+m^2\right)}}$$

quantum mechanical path integrals

Worldline formalism in flat space

- Expand in powers of A_μ (sum of plane waves)

$$A_\mu = \sum_{i=1}^{N} \varepsilon_{i,\mu}\, e^{ip_i \cdot x}$$

Worldline formalism in flat space

- Expand in powers of A_μ (sum of plane waves)

$$A_\mu = \sum_{i=1}^{N} \varepsilon_{i,\mu}\, e^{ip_i \cdot x}$$

- get averages of "photon vertex operators"

$$\left\langle \varepsilon_{1,\mu_1}\, \dot{x}^{\mu_1}(\tau_1) e^{ip_1 \cdot x(\tau_1)} \cdots \varepsilon_{N,\mu_N}\, \dot{x}^{\mu_N}(\tau_N) e^{ip_N \cdot x(\tau_N)} \right\rangle$$

Worldline formalism in flat space

- Expand in powers of A_μ (sum of plane waves)

$$A_\mu = \sum_{i=1}^{N} \varepsilon_{i,\mu}\, e^{ip_i \cdot x}$$

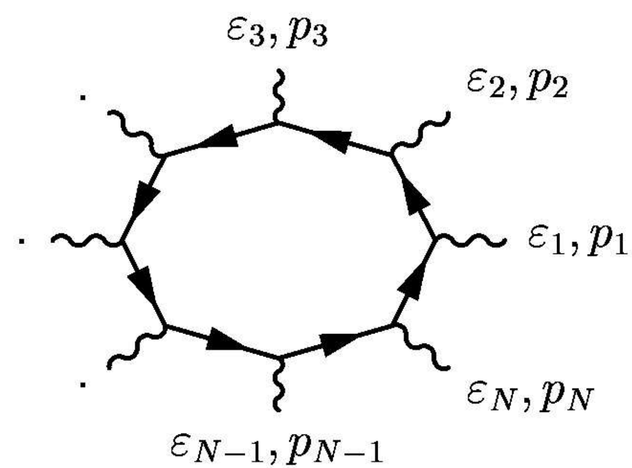
- get averages of "photon vertex operators"

$$\left\langle \varepsilon_{1,\mu_1}\, \dot{x}^{\mu_1}(\tau_1) e^{ip_1\cdot x(\tau_1)} \cdots \varepsilon_{N,\mu_N}\, \dot{x}^{\mu_N}(\tau_N) e^{ip_N\cdot x(\tau_N)} \right\rangle$$

- and obtain the "Bern-Kosower master formula"

Bern-Kosower master formula

$$\Gamma[p_1, \varepsilon_1; \ldots ; p_N, \varepsilon_N] =$$



Bern-Kosower master formula

$$\Gamma[p_1,\varepsilon_1;\dots;p_N,\varepsilon_N] = -(-ie)^N(2\pi)^D\delta^D\left(\sum_{i=1}^N p_i\right)$$

$$\int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2T}}{(4\pi T)^{\frac{D}{2}}}\,\prod_{i=1}^N\int_0^T d\tau_i$$

$$\exp\sum_{i,j=1}^N\left[\frac{1}{2}\Delta_{ij}\,p_i\cdot p_j - i\,{}^{\bullet}\Delta_{ij}\,\varepsilon_i\cdot p_j + \frac{1}{2}\,{}^{\bullet\bullet}\Delta_{ij}\,\varepsilon_i\cdot\varepsilon_j\right]\Bigg|_{\mathrm{lin}\,\varepsilon_{\mathrm{i}}}$$

Bern-Kosower master formula

$$\Gamma[p_1,\varepsilon_1;\ldots;p_N,\varepsilon_N] = -(-ie)^N (2\pi)^D \delta^D\left(\sum_{i=1}^N p_i\right)$$

$$\int_0^\infty \boxed{\frac{dT}{T}} \frac{e^{-m^2T}}{\boxed{(4\pi T)^{\frac{D}{2}}}} \prod_{i=1}^N \int_0^T d\tau_i$$

$$\exp \sum_{i,j=1}^N \left[\frac{1}{2}\Delta_{ij}\, p_i \cdot p_j - i\,{}^\bullet\Delta_{ij}\,\varepsilon_i \cdot p_j + \frac{1}{2}\,{}^{\bullet\bullet}\Delta_{ij}\,\varepsilon_i \cdot \varepsilon_j\right]\Bigg|_{\mathrm{lin}\,\varepsilon_\mathrm{i}}$$

integral over the modulus of the circle

one-loop determinant for the free path integral

Worldline formalism in curved space

- A real scalar field coupled to gravity

$$S[\phi, g] = \int d^D x \sqrt{g}\, \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2)$$

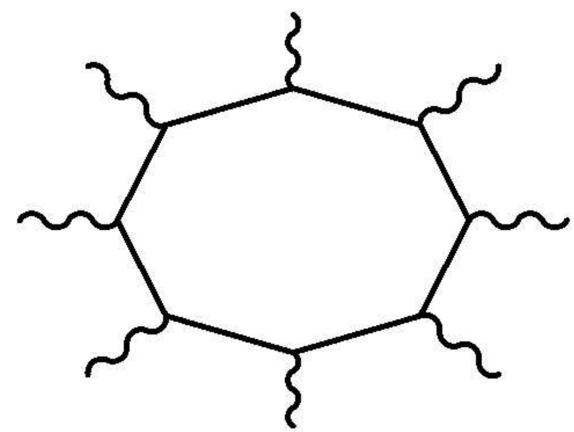
Worldline formalism in curved space

- A real scalar field coupled to gravity

$$S[\phi, g] = \int d^D x \sqrt{g}\, \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2)$$

- produces an effective action ($e^{-\Gamma[g]} = \int \mathcal{D}\phi\, e^{-S[\phi,g]}$)

$$\Gamma[g] = \frac{1}{2}\mathrm{Tr}\log(-\nabla^2 + m^2 + \xi R) =$$



Worldline formalism in curved space

- which can be represented as

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int_{S^1} \mathcal{D}x \; e^{-S[x^\mu]}$$

with

$$S[x^\mu] = \int_0^1 d\tau \left(\frac{1}{4T} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + T(m^2 + \xi R(x)) \right)$$

Worldline formalism in curved space

- which can be represented as

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int_{S^1} \mathcal{D}x\; e^{-S[x^\mu]}$$

with

$$S[x^\mu] = \int_0^1 d\tau \left(\frac{1}{4T} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + T(m^2 + \xi R(x))\right)$$

- 1d non-linear sigma model

Worldline formalism in curved space

- The effective action

$$\Gamma[g] = \frac{1}{2}\mathrm{Tr}\log(-\nabla^2 + m^2 + \xi R)$$

can be obtained directly from 1^{st} qzn.

Worldline formalism in curved space

- The effective action

$$\Gamma[g] = \frac{1}{2}\mathrm{Tr}\log(-\nabla^2 + m^2 + \xi R)$$

can be obtained directly from 1$^{\mathrm{st}}$ qzn.

- start from the relativistic point-particle action

$$S[e, x^\mu] = \int_0^1 d\tau \frac{1}{2}[e^{-1} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + e(m^2 + \xi R(x))]$$

Worldline formalism in curved space

- The effective action

$$\Gamma[g] = \frac{1}{2}\mathrm{Tr}\log(-\nabla^2 + m^2 + \xi R)$$

can be obtained directly from 1$^{\mathrm{st}}$ qzn.

- start from the relativistic point-particle action

$$S[e, x^\mu] = \int_0^1 d\tau \frac{1}{2}[e^{-1} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + e(m^2 + \xi R(x))]$$

- gauge fix the diffeomorphisms $e = 2T$

Worldline formalism in curved space

- The effective action

$$\Gamma[g] = \frac{1}{2}\mathrm{Tr}\log(-\nabla^2 + m^2 + \xi R)$$

can be obtained directly from 1$^{\mathrm{st}}$ qzn.

- start from the relativistic point-particle action

$$S[e, x^\mu] = \int_0^1 d\tau \frac{1}{2}[e^{-1} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + e(m^2 + \xi R(x))]$$

- gauge fix the diffeomorphisms $e = 2T$
- divide out the length of the circle

Worldline formalism in curved space

1. UV regularization of the non-linear σ model

   3 regularization schemes have been studied

   - Mode Regularization (Bastianelli, OC, Schalm, van Niuewenhuizen)
   - Time Slicing (de Boer, Peeters, Skenderis, van Niuewenhuizen)
   - Dimensional Regularization (Bastianelli, OC, van Nieuwenhuizen)

Worldline formalism in curved space

1. UV regularization of the non-linear σ model

   3 regularization schemes have been studied

   - Mode Regularization (Bastianelli, OC, Schalm, van Niuewenhuizen)
   - Time Slicing (de Boer, Peeters, Skenderis, van Niuewenhuizen)
   - Dimensional Regularization (Bastianelli, OC, van Nieuwenhuizen)

   DR allows for covariant counterterms

$$V_{CT} = -\frac{\hbar}{8}R$$

Worldline formalism in curved space

1. UV regularization of the non-linear σ model

   3 regularization schemes have been studied

   - Mode Regularization (Bastianelli, OC, Schalm, van Niuewenhuizen)
   - Time Slicing (de Boer, Peeters, Skenderis, van Niuewenhuizen)
   - Dimensional Regularization (Bastianelli, OC, van Nieuwenhuizen)

   DR allows for covariant counterterms

$$V_{CT} = -\frac{\hbar}{8} R$$

2. Factorization of zero modes

   - non-covariant total derivatives
   - treated with BRST methods

Effective action from DR worldline

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int DxDaDbDc\, e^{-S}$$

with

$$S = \int_0^1 d\tau \left(\frac{1}{4T} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + T(m^2 + \bar{\xi} R) \right)$$

where $\bar{\xi} = \xi - \frac{1}{4}$ includes the DR counterterm.

Effective action from DR worldline

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int DxDaDbDc\, e^{-S}$$

with

$$S = \int_0^1 d\tau \left(\frac{1}{4T} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + T(m^2 + \bar{\xi} R) \right)$$

where $\bar{\xi} = \xi - \frac{1}{4}$ includes the DR counterterm.

- bosonic ghosts a and fermionic ghosts $b,\ c$ provide the non-trivial path integral measure

Effective action from DR worldline

Expand in $h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$, substitute the h^N term with

$$h_{\mu\nu} = \sum_{i=1}^{N} \epsilon_{\mu\nu}^{(i)} e^{ip_i \cdot x}$$

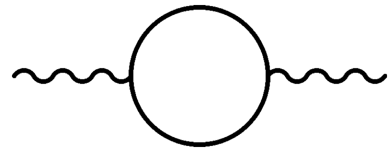
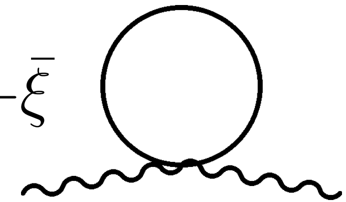
and pick terms linear in $\epsilon^{(i)} \Rightarrow$

N-graviton amplitude in momentum space $\tilde{\Gamma}^{\epsilon_1,..,\epsilon_N}_{(p_1,..,p_N)}$.

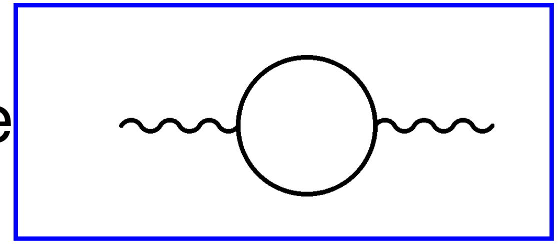
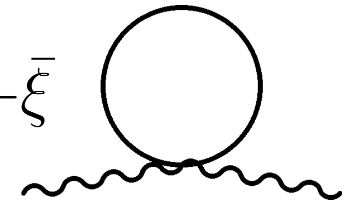
- Get quantum mechanical correlators of the form

$$\Big\langle \underbrace{(\dot{x}_1^{\mu_1}\dot{x}_1^{\nu_1} + a_1^{\mu_1}a_1^{\nu_1} + b_1^{\mu_1}c_1^{\nu_1})e^{ip_1\cdot x_1}}_{\text{graviton vertex operator}} \cdots \underbrace{(\dot{x}_N^{\mu_N}\dot{x}_N^{\nu_N} + a_N^{\mu_N}a_N^{\nu_N} + b_N^{\mu_N}c_N^{\nu_N})e^{ip_N\cdot x_N}}_{\text{graviton vertex operator}} \Big\rangle$$

Explicit computation

E.g. Two-graviton amplitude  $+\bar{\xi}$ 

Explicit computation

E.g. Two-graviton amplitude  $+\bar{\xi}$ 

Case $\bar{\xi} = 0$ (i.e. $\xi = \frac{1}{4}$).
Quadratic part in $h_{\mu\nu}$

$$\tilde{\Gamma}^{\epsilon_1,\epsilon_2}_{(p_1,p_2)} = \frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T} \frac{1}{(4\pi T)^{\frac{D}{2}}} \int d^D x_0$$

$$\times \left\langle \frac{1}{2}\left[\int_0^1 d\tau\, \frac{1}{4T}(h_{\mu\nu}(\dot{y}^\mu \dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu))\right]^2 \right\rangle\Big|_{lin\ \epsilon_1,\epsilon_2}$$

where $h_{\mu\nu} = \epsilon^{(1)}_{\mu\nu} e^{ip_1\cdot x} + \epsilon^{(2)}_{\mu\nu} e^{ip_2\cdot x} \quad x = x_0 + y$

Explicit computation

Use Wick contractions and get

$$\Gamma^{\epsilon_1 \epsilon_2}_{(p,-p)} = \frac{1}{8}\frac{1}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2 T}$$

$$\times \left(r_1 I_1 + r_2 I_2 + 2Tp^2(r_3 I_3 + r_4 I_4) + 4T^2 p^4 r_5 I_5\right)$$

where $r_i = \epsilon^{(1)}_{\mu\nu} R_i^{\mu\nu\alpha\beta} \epsilon^{(2)}_{\alpha\beta}$ and

$$R_1^{\mu\nu\alpha\beta} = \delta^{\mu\nu}\delta^{\alpha\beta}, \quad R_2^{\mu\nu\alpha\beta} = \delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}$$

$$R_3^{\mu\nu\alpha\beta} = \frac{1}{p^2}\left(\delta^{\mu\alpha}p^\nu p^\beta + \delta^{\nu\alpha}p^\mu p^\beta + \delta^{\mu\beta}p^\nu p^\alpha + \delta^{\nu\beta}p^\mu p^\alpha\right)$$

$$R_4^{\mu\nu\alpha\beta} = \frac{1}{p^2}\left(\delta^{\mu\nu}p^\alpha p^\beta + \delta^{\alpha\beta}p^\mu p^\nu\right), \quad R_5^{\mu\nu\alpha\beta} = \frac{1}{p^4} p^\mu p^\nu p^\alpha p^\beta$$

Explicit computation

$$I_1 = \int_0^1 d\tau \int_0^1 d\sigma \, ({}^\bullet\Delta^\bullet + \Delta_{gh})|_\tau \, ({}^\bullet\Delta^\bullet + \Delta_{gh})|_\sigma \; e^{-2Tp^2\Delta_0}$$

$$I_2 = \int_0^1 d\tau \int_0^1 d\sigma \, ({}^\bullet\Delta^{\bullet 2} - \Delta_{gh}^2) \; e^{-2Tp^2\Delta_0}$$

$$I_3 = \int_0^1 d\tau \int_0^1 d\sigma \, {}^\bullet\Delta \, {}^\bullet\Delta^\bullet \, \Delta^\bullet \, e^{-2Tp^2\Delta_0}$$

$$I_4 = \int_0^1 d\tau \int_0^1 d\sigma \, ({}^\bullet\Delta^\bullet + \Delta_{gh})|_\tau \, (\Delta^\bullet)^2 \, e^{-2Tp^2\Delta_0}$$

$$I_5 = \int_0^1 d\tau \int_0^1 d\sigma \, ({}^\bullet\Delta)^2 \, (\Delta^\bullet)^2 \, e^{-2Tp^2\Delta_0}$$

Explicit computation

Use (WL) dimensional regularization when necessary

Translation invariance can be used to fix $\sigma = 0$

$$I_1 = \int_0^1 d\tau \, e^{-Tp^2(\tau-\tau^2)}$$

$$I_2 = \frac{1}{4}Tp^2 - 2 + I_1 \qquad I_3 = \frac{1}{8} - \frac{1}{2Tp^2}(1 - I_1)$$

$$I_4 = \frac{1}{2Tp^2}(1 - I_1) \qquad I_5 = \frac{1}{8Tp^2} - \frac{3}{4T^2p^4}(1 - I_1)$$

Explicit computation

Proper time integral can be carried out at complex D

$$
\begin{aligned}
(4\pi)^{\frac{D}{2}}\Gamma_{(p,-p)} &= -\frac{1}{8}\Gamma\biggl(-\frac{D}{2}\biggr)\Bigl[(m^2)^{\frac{D}{2}}(R_1-R_2) \\
&+ \Bigl((P^2)^{\frac{D}{2}}-(m^2)^{\frac{D}{2}}\Bigr)(S_1+S_2)\Bigr] \\
&- \frac{1}{32}\Gamma\biggl(1-\frac{D}{2}\biggr)p^2(m^2)^{\frac{D}{2}-1}S_2
\end{aligned}
$$

where

$$
(P^2)^a = \int_0^1 d\tau\,(m^2+p^2(\tau-\tau^2))^a\,, \qquad S_i \text{ transverse}
$$

Explicit computation

Additional term for the case $\bar{\xi} \neq 0$ (i.e. $\xi \neq \frac{1}{4}$)

$$
\begin{aligned}
(4\pi)^{\frac{D}{2}} \Delta\Gamma_{(p,-p)} = & -\frac{\bar{\xi}}{8} \Gamma\left(1 - \frac{D}{2}\right) p^2 \Big[(m^2)^{\frac{D}{2}-1} (2S_1 + S_2) \\
& - 4(P^2)^{\frac{D}{2}-1} S_1 \Big] \\
& - \frac{\bar{\xi}^2}{2} \Gamma\left(2 - \frac{D}{2}\right) p^4 (P^2)^{\frac{D}{2}-2} S_1
\end{aligned}
$$

Explicit computation

Additional term for the case $\bar{\xi} \neq 0$ (i.e. $\xi \neq \frac{1}{4}$)

$$
\begin{aligned}
(4\pi)^{\frac{D}{2}} \Delta\Gamma_{(p,-p)} = & -\frac{\bar{\xi}}{8}\Gamma\left(1-\frac{D}{2}\right) p^2 \Big[(m^2)^{\frac{D}{2}-1}(2S_1+S_2) \\
& - 4(P^2)^{\frac{D}{2}-1} S_1 \Big] \\
& - \frac{\bar{\xi}^2}{2}\Gamma\left(2-\frac{D}{2}\right) p^4 (P^2)^{\frac{D}{2}-2} S_1
\end{aligned}
$$

Ward Identity from general coordinate invariance

$$
\nabla_\mu^{(x)} \frac{1}{\sqrt{g(x)}} \frac{\delta\Gamma[g]}{\delta g_{\mu\nu}(x)} = 0 \qquad \surd
$$

Extensions

Effective action for spin 1/2 coupled to gravity

- Obtained by considering N=1 supersymmetric extension of previous path integral.

Extensions

Effective action for spin 1/2 coupled to gravity

- Obtained by considering N=1 supersymmetric extension of previous path integral.
- The supersymmetric partners $\psi^\mu = e^\mu{}_a \, \psi^a$ of the coordinates x^μ generate the gamma matrices. One can use either ψ^a or ψ^μ.

Extensions

Effective action for spin 1/2 coupled to gravity

- Obtained by considering N=1 supersymmetric extension of previous path integral.
- The supersymmetric partners $\psi^\mu = e^\mu{}_a \, \psi^a$ of the coordinates x^μ generate the gamma matrices. One can use either ψ^a or ψ^μ.
- Using ψ^μ there is no need of introducing the vielbein $e^\mu{}_a$: one can work directly with the metric $g_{\mu\nu}$.

Extensions

Effective action for spin 1/2 coupled to gravity

- Obtained by considering N=1 supersymmetric extension of previous path integral.
- The supersymmetric partners $\psi^\mu = e^\mu{}_a \, \psi^a$ of the coordinates x^μ generate the gamma matrices. One can use either ψ^a or ψ^μ.
- Using ψ^μ there is no need of introducing the vielbein $e^\mu{}_a$: one can work directly with the metric $g_{\mu\nu}$.
- Dimensional regularization can be extended to this model as well: DR is a supersymmetric regularization.

Extensions

- Path integral with ψ^μ has additional bosonic ghosts α^μ

$$\Gamma[g_{\mu\nu}] = \frac{1}{2}\int_0^\infty \frac{dT}{T} \oint_{PBC} DxDaDbDc \oint_{ABC} \mathcal{D}\psi\mathcal{D}\alpha\; e^{-S}$$

with

$$S = \int_0^1 d\tau\, \frac{1}{4T}\Big[g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + g_{\mu\nu}(x)(\psi^\mu\dot{\psi}^\nu + \alpha^\mu\alpha^\nu) - \partial_\mu g_{\nu\lambda}(x)\psi^\mu\psi^\nu\dot{x}^\lambda\Big] + Tm^2$$

Extensions

- Path integral with ψ^μ has additional bosonic ghosts α^μ

$$\Gamma[g_{\mu\nu}] = \frac{1}{2}\int_0^\infty \frac{dT}{T} \oint_{PBC} DxDaDbDc \oint_{ABC} \mathcal{D}\psi\mathcal{D}\alpha\, e^{-S}$$

with

$$\begin{aligned} S \;=\; & \int_0^1 d\tau\, \frac{1}{4T}\Big[g_{\mu\nu}(x)(\dot{x}^\mu\dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) \\ & + \; g_{\mu\nu}(x)(\psi^\mu\dot{\psi}^\nu + \alpha^\mu\alpha^\nu) - \partial_\mu g_{\nu\lambda}(x)\psi^\mu\psi^\nu\dot{x}^\lambda \Big] + Tm^2 \end{aligned}$$

- Linear in $g_{\mu\nu}$ (only vertices with a single graviton emission)!

Extensions

Effective action for spin 1 (Bastianelli, Benincasa, Giombi, '05)

- Obtained by considering N=2 locally supersymmetric extension of previous path integral.

Extensions

Effective action for spin 1 (Bastianelli, Benincasa, Giombi, ’05)

- Obtained by considering N=2 locally supersymmetric extension of previous path integral.
- $SO(2)$ gauge symmetry: yields a new constraint

Extensions

Effective action for spin 1 (Bastianelli, Benincasa, Giombi, '05)

- Obtained by considering N=2 locally supersymmetric extension of previous path integral.
- $SO(2)$ gauge symmetry: yields a new constraint
- Can introduce a Chern Simons coupling $q \int_0^1 d\tau a$

Extensions

Effective action for spin 1 (Bastianelli, Benincasa, Giombi, '05)

- Obtained by considering N=2 locally supersymmetric extension of previous path integral.
- $SO(2)$ gauge symmetry: yields a new constraint
- Can introduce a Chern Simons coupling $q \int_0^1 d\tau a$
- It projects-in $p-$forms, with $p = \frac{D}{2} - q - 1$

$$D = 4, \ q = 0 \quad \Rightarrow \quad p = 1 \text{ vector field}$$

Extensions

Effective action for spin 1 (Bastianelli, Benincasa, Giombi, '05)

- Obtained by considering N=2 locally supersymmetric extension of previous path integral.
- $SO(2)$ gauge symmetry: yields a new constraint
- Can introduce a Chern Simons coupling $q\int_0^1 d\tau a$
- It projects-in $p-$forms, with $p = \frac{D}{2} - q - 1$

$$D = 4,\ q = 0 \quad \Rightarrow \quad p = 1 \text{ vector field}$$

- Coupling to ext gravity $\surd$

Extensions

Effective action for spin 1 (Bastianelli, Benincasa, Giombi, '05)

- Obtained by considering N=2 locally supersymmetric extension of previous path integral.
- $SO(2)$ gauge symmetry: yields a new constraint
- Can introduce a Chern Simons coupling $q \int_0^1 d\tau a$
- It projects-in $p-$forms, with $p = \frac{D}{2} - q - 1$

$$D = 4,\ q = 0 \quad \Rightarrow \quad p = 1 \text{ vector field}$$

- Coupling to ext gravity $\surd$
- Massive spin 1 by KK reduction

Higher spin fields

- Obtained by considering N>2 locally supersymmetric extension of previous models
  $SO(N)$ spinning particle models

Higher spin fields

- Obtained by considering N>2 locally supersymmetric extension of previous models
  $SO(N)$ spinning particle models
- $SO(N)$ gauge symmetry –> new constraints
  Bargmann-Wigner EoM's

Higher spin fields

- Obtained by considering N>2 locally supersymmetric extension of previous models
  $SO(N)$ spinning particle models
- $SO(N)$ gauge symmetry –> new constraints
  Bargmann-Wigner EoM's
- In 4D projects-in a spin-$\frac{N}{2}$ field
  $N = 4$ in 4D $\rightarrow$ graviton
  $N = 3$ in 4D $\rightarrow$ gravitino

Higher spin fields

- Obtained by considering N>2 locally supersymmetric extension of previous models
  $SO(N)$ spinning particle models
- $SO(N)$ gauge symmetry –> new constraints
  Bargmann-Wigner EoM's
- In 4D projects-in a spin-$\frac{N}{2}$ field
  $N = 4$ in 4D $\rightarrow$ graviton
  $N = 3$ in 4D $\rightarrow$ gravitino
- In generic D, massless rep.'s of the conformal group $SO(D, 2)$ (Siegel)

Higher spin fields

- Starting point

$$S[X] = \int dt \left(p_\mu \dot{x}^\mu + \frac{1}{2}\psi_{i,\mu}\dot{\psi}^{i,\mu} - \frac{1}{2}\delta^{\mu\nu} p_\mu p_\nu \right)$$

Higher spin fields

- Starting point

$$S[X] = \int dt \left(p_\mu \dot{x}^\mu + \frac{1}{2}\psi_{i,\mu}\dot{\psi}^{i,\mu} - \frac{1}{2}\delta^{\mu\nu}p_\mu p_\nu \right)$$

- Symmetry algebra

$$H = \frac{1}{2}\delta^{\mu\nu}p_\mu p_\nu \quad Q_i = p \cdot \psi_i \quad \boxed{J_{ij} = \psi_i \cdot \psi_j}$$

$SO(N)$ generators

Higher spin fields

- Starting point

$$S[X] = \int dt \left(p_\mu \dot{x}^\mu + \frac{1}{2}\psi_{i,\mu}\dot{\psi}^{i,\mu} - \frac{1}{2}\delta^{\mu\nu}p_\mu p_\nu \right)$$

- Symmetry algebra

$$H = \frac{1}{2}\delta^{\mu\nu}p_\mu p_\nu \quad Q_i = p\cdot\psi_i \quad \boxed{J_{ij} = \psi_i \cdot \psi_j}$$

$SO(N)$ generators

- It can be gauged: add gauge fields $G = (e, \chi_i, a_{ij})$

$$L = p_\mu \dot{x}^\mu + \frac{1}{2}\psi_{i,\mu}\dot{\psi}^{i,\mu} - \frac{e}{2}\delta^{\mu\nu}p_\mu p_\nu - \chi_i\, p\cdot\psi^i - a_{ij}\,\psi^i\cdot\psi^j$$

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

- wave function is a multispinor $\Psi_{\alpha_1 \cdots \alpha_N}(x)$

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

- wave function is a multispinor $\Psi_{\alpha_1 \cdots \alpha_N}(x)$
- Hamiltonian constraint $\rightarrow$ K-G equation

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

- wave function is a multispinor $\Psi_{\alpha_1 \cdots \alpha_N}(x)$
- Hamiltonian constraint $\rightarrow$ K-G equation
- SUSY constraints $\rightarrow$ N Dirac eqn.'s $\quad \partial_{\alpha_i}{}^{\tilde{\alpha}_i} \Psi_{\ldots \tilde{\alpha}_i \ldots}(x) = 0$

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

- wave function is a multispinor $\Psi_{\alpha_1 \cdots \alpha_N}(x)$
- Hamiltonian constraint $\rightarrow$ K-G equation
- SUSY constraints $\rightarrow$ N Dirac eqn.'s $\quad \partial_{\alpha_i}{}^{\tilde{\alpha}_i} \Psi_{\ldots\tilde{\alpha}_i\ldots}(x) = 0$
- vector constraints select propagating DoF's

$$\psi_{i,\mu}\psi_j{}^\mu \approx 0 \quad \Longrightarrow \quad (\gamma^\mu)_{\alpha_i}{}^{\tilde{\alpha}_i} (\gamma_\mu)_{\alpha_j}{}^{\tilde{\alpha}_j} \Psi_{\ldots\tilde{\alpha}_i\ldots\tilde{\alpha}_j\ldots}(x) = 0$$

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

- wave function is a multispinor $\Psi_{\alpha_1 \cdots \alpha_N}(x)$
- Hamiltonian constraint $\rightarrow$ K-G equation
- SUSY constraints $\rightarrow$ N Dirac eqn.'s $\quad \partial_{\alpha_i}{}^{\tilde{\alpha}_i} \Psi_{\ldots \tilde{\alpha}_i \ldots}(x) = 0$
- vector constraints select propagating DoF's

$$\left(\gamma_\mu^T \Gamma \gamma^\mu\right)^{\alpha_i \alpha_j} \Psi_{\ldots \alpha_i \ldots \alpha_j \ldots}(x) = 0\,, \quad \Gamma \in \{C, C\gamma_*, C\gamma_\mu, \ldots\}$$

Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

$$[\psi_{i,\mu}, \psi^{j,\nu}] = \delta_i{}^j \delta_\mu{}^\nu \quad \text{set of N Clifford albegras}$$

- wave function is a multispinor $\Psi_{\alpha_1 \cdots \alpha_N}(x)$
- Hamiltonian constraint $\rightarrow$ K-G equation
- SUSY constraints $\rightarrow$ N Dirac eqn.'s $\quad \partial_{\alpha_i}{}^{\tilde{\alpha}_i} \Psi_{\ldots \tilde{\alpha}_i \ldots}(x) = 0$
- vector constraints select propagating DoF's

$$\left(\gamma_\mu^T \Gamma \gamma^\mu\right)^{\alpha_i \alpha_j} \Psi_{\ldots \alpha_i \ldots \alpha_j \ldots}(x) = 0\,, \quad \Gamma \in \{C, C\gamma_*, C\gamma_\mu, \ldots\}$$

$$\gamma_\mu^T \Gamma^{(n)} \gamma^\mu \sim (n - \frac{D}{2}) \Gamma^{(n)} \Rightarrow \; n = \frac{D}{2} \text{ acts trivially}$$

Higher spin fields

- E.g. $D = 4$, $N = 3$, trivial constraint $\gamma_\mu^T \Gamma^{(2)} \gamma^\mu$

$$\Psi_{\alpha_1 \alpha_2 \alpha} = \left(\Gamma^{\mu\nu}\right)_{\alpha_1 \alpha_2} \chi_{\mu\nu\,\alpha}$$

$$\Rightarrow \chi_{\mu\nu\,\alpha} = -\chi_{\nu\mu\,\alpha}$$

Higher spin fields

- E.g. $D=4$, $N=3$, trivial constraint $\gamma^T_\mu \Gamma^{(2)} \gamma^\mu$

$$\Psi_{\alpha_1\alpha_2\alpha} = \left(\Gamma^{\mu\nu}\right)_{\alpha_1\alpha_2} \chi_{\mu\nu\,\alpha}$$
$$\Rightarrow \chi_{\mu\nu\,\alpha} = -\chi_{\nu\mu\,\alpha}$$

- non-trivial constraints

$$\begin{cases} \partial^\mu \chi_{\mu\nu\,\alpha} = \partial_{[\sigma}\chi_{\mu\nu]\,\alpha} = 0 \\ \partial_\alpha{}^{\tilde\alpha}\chi_{\mu\nu\,\tilde\alpha} = \left(\gamma^\mu\right)_\alpha{}^{\tilde\alpha}\chi_{\mu\nu\,\tilde\alpha} = 0 \end{cases} \Longrightarrow \begin{cases} \chi_{\mu\nu\,\alpha} = \partial_\mu\phi_{\nu\,\alpha} - \partial_\nu\phi_{\mu\,\alpha} \\ \partial_\mu \not\phi - \not\partial\phi_\mu = 0 \end{cases}$$

$$\text{Rarita} - \text{Schwinger eq.} \Rightarrow \text{spin}-\frac{3}{2}\text{ field}$$

Higher spin fields

- E.g. $D = 4$, $N = 3$, trivial constraint $\gamma_\mu^T \Gamma^{(2)} \gamma^\mu$

$$\Psi_{\alpha_1 \alpha_2 \alpha} = \left(\Gamma^{\mu\nu}\right)_{\alpha_1 \alpha_2} \chi_{\mu\nu\,\alpha}$$

$$\Rightarrow \chi_{\mu\nu\,\alpha} = -\chi_{\nu\mu\,\alpha}$$

- non-trivial constraints

$$\begin{cases} \partial^\mu \chi_{\mu\nu\,\alpha} = \partial_{[\sigma} \chi_{\mu\nu]\,\alpha} = 0 \\ \partial_\alpha{}^{\tilde{\alpha}} \chi_{\mu\nu\,\tilde{\alpha}} = \left(\gamma^\mu\right)_\alpha{}^{\tilde{\alpha}} \chi_{\mu\nu\,\tilde{\alpha}} = 0 \end{cases} \Longrightarrow \begin{cases} \chi_{\mu\nu\,\alpha} = \partial_\mu \phi_{\nu\,\alpha} - \partial_\nu \phi_{\mu\,\alpha} \\ \partial_\mu \not{\phi} - \not{\partial} \phi_\mu = 0 \end{cases}$$

$$\text{Rarita} - \text{Schwinger eq.} \;\Rightarrow\; \text{spin}-\frac{3}{2} \text{ field}$$

- Similarly $D = 4$, $N = 4 \Rightarrow$ spin-2 field

Higher spin fields

Goal: path integral qzn.

- Start from free fields

Higher spin fields

Goal: path integral qzn.

- Start from free fields
- Configuration space action

$$S[X,G] = \int_0^1 d\tau \left[\frac{1}{2} e^{-1} (\dot{x}^\mu - \chi_i \psi_i^\mu)^2 + \frac{1}{2} \psi_i^\mu (\delta_{ij} \partial_\tau - a_{ij}) \psi_{j\mu} \right]$$

Higher spin fields

Goal: path integral qzn.

- Start from free fields
- Configuration space action

$$S[X,G] = \int_0^1 d\tau \left[\frac{1}{2}e^{-1}(\dot{x}^\mu - \chi_i\psi_i^\mu)^2 + \frac{1}{2}\psi_i^\mu(\delta_{ij}\partial_\tau - a_{ij})\psi_{j\mu}\right]$$

- Need to find the correct integration measure

Higher spin fields

Goal: path integral qzn.

- Start from free fields
- Configuration space action

$$S[X,G] = \int_0^1 d\tau \left[\frac{1}{2}e^{-1}(\dot{x}^\mu - \chi_i\psi_i^\mu)^2 + \frac{1}{2}\psi_i^\mu(\delta_{ij}\partial_\tau - a_{ij})\psi_{j\mu}\right]$$

- Need to find the correct integration measure
- Gauge transf. on sugra multiplet

$$\begin{aligned} \delta e &= \dot{\xi} + 2\chi_i\epsilon_i \quad \delta\chi_i = \dot{\epsilon}_i - a_{ij}\epsilon_j + \alpha_{ij}\chi_j \\ \delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im}a_{mj} + \alpha_{jm}a_{im} \end{aligned}$$

Higher spin fields

- One-loop partition function on S_1

$$Z \sim \int_{T^1} \frac{\mathcal{D}X\mathcal{D}G}{\mathrm{Vol}\,(\mathrm{Gauge})}\, e^{-S[X,G]}$$

PBC (ABC) for bosonic (fermionic) fields

Higher spin fields

- One-loop partition function on S_1

$$Z \sim \int_{T^1} \frac{\mathcal{D}X\mathcal{D}G}{\mathrm{Vol}\,(\mathrm{Gauge})}\, e^{-S[X,G]}$$

PBC (ABC) for bosonic (fermionic) fields

- gauge fix local symmetries

$$\begin{aligned} e &= \beta\,, \quad \text{modulus of the circle} \\ \chi_i &= 0\,, \quad \text{no zero} - \text{mode for } \epsilon_i \quad \text{b/c ABC} \\ a_{ij} &= \hat{a}_{ij}(\theta_k)\,, \quad k = 1, ..., r = \text{rank of } SO(N) \end{aligned}$$

Higher spin fields

- One-loop partition function on S_1

$$Z \sim \int_{T^1} \frac{\mathcal{D}X\mathcal{D}G}{\mathrm{Vol\,(Gauge)}}\, e^{-S[X,G]}$$

  PBC (ABC) for bosonic (fermionic) fields

- gauge fix local symmetries

$$\begin{aligned} e &= \beta\,, \quad \text{modulus of the circle} \\ \chi_i &= 0\,, \quad \text{no zero} - \text{mode for } \epsilon_i \quad \text{b/c ABC} \\ a_{ij} &= \hat{a}_{ij}(\theta_k)\,, \quad k = 1, ..., r = \text{rank of } SO(N) \end{aligned}$$

- Obtain the correct measure via FP trick

Higher spin fields

$$Z = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \; K_N \left[\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\right]$$

$$\times \; \left(\mathrm{Det}\left(\partial_\tau - \hat{a}_{vec}\right)_{ABC}\right)^{\frac{D}{2}-1} \mathrm{Det}'\left(\partial_\tau - \hat{a}_{adj}\right)_{PBC}$$

Higher spin fields

$$Z = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\; K_N \left[\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\right]$$

$$\times \underbrace{\Big(\mathrm{Det}\left(\partial_\tau - \hat{a}_{vec}\right)_{ABC}\Big)^{\frac{D}{2}-1}}\;\underbrace{\mathrm{Det}'\left(\partial_\tau - \hat{a}_{adj}\right)_{PBC}}$$

one-loop fermionic determinant + susy ghosts

gauge symmetry ghosts

Higher spin fields

$$Z = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\; K_N\left[\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\right]$$
$$\times \left(\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC}\right)^{\frac{D}{2}-1}\mathrm{Det}'\,(\partial_\tau - \hat{a}_{adj})_{PBC}$$

$$Dof(D,N) = K_N\left[\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\right]$$
$$\times \left(\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC}\right)^{\frac{D}{2}-1}\mathrm{Det}'\,(\partial_\tau - \hat{a}_{adj})_{PBC}$$

Computes the # of degrees of freedom. $Dof(D,0) = 1$

Higher spin fields

- $N = 2r$, $r =$ rank of the group

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 & . & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & . & 0 & 0 \\ 0 & 0 & -\theta_2 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & \theta_r \\ 0 & 0 & 0 & 0 & . & -\theta_r & 0 \end{pmatrix}$$

Higher spin fields

- $N = 2r$, $r =$ rank of the group

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 & . & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & . & 0 & 0 \\ 0 & 0 & -\theta_2 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & \theta_r \\ 0 & 0 & 0 & 0 & . & -\theta_r & 0 \end{pmatrix}$$

- θ's are angles: large gauge transf.'s $\Rightarrow\ \theta_i \cong \theta_i + 2\pi n$

Higher spin fields

- $\dfrac{1}{K_{2r}} = \dfrac{2^r r!}{2}$, # copies of fundamental domain

  different regions identified up to constant gauge transf.'s
  - $r!$, permutation of r θ's
  - 2^r, Z_2-symmetry on $O(N)$

Higher spin fields

- $\frac{1}{K_{2r}} = \frac{2^r r!}{2}$, # copies of fundamental domain

  different regions identified up to constant gauge transf.'s
  - $r!$, permutation of r θ's
  - 2^r, Z_2-symmetry on $O(N)$

$$\begin{aligned}
\mathrm{Det}\left(\partial_\tau - \hat{a}_{vec}\right)_{PBC} &= \prod_{k=1}^{r} \mathrm{Det}\left(\partial_\tau + i\theta_r\right) \mathrm{Det}\left(\partial_\tau - i\theta_r\right) \\
&= \prod_{k=1}^{r} \left(2\cos\frac{\theta_k}{2}\right)^2
\end{aligned}$$

Higher spin fields

$$
\begin{aligned}
\mathrm{Det}'\left(\partial_\tau - \hat{a}_{adj}\right)_{_{PBC}} &= \prod_{k=1}^{r} \mathrm{Det}'\left(\partial_\tau\right) \\
&\times \prod_{k<l} \mathrm{Det}\left(\partial_\tau + i(\theta_k + \theta_l)\right) \mathrm{Det}\left(\partial_\tau - i(\theta_k + \theta_l)\right) \\
&\times \prod_{k<l} \mathrm{Det}\left(\partial_\tau + i(\theta_k - \theta_l)\right) \mathrm{Det}\left(\partial_\tau - i(\theta_k - \theta_l)\right) \\
&= \prod_{k<l} \left(2\sin\frac{\theta_k + \theta_l}{2}\right)^2 \left(2\sin\frac{\theta_k - \theta_l}{2}\right)^2
\end{aligned}
$$

Higher spin fields

$$
\begin{aligned}
Dof(D,N) &= \frac{2}{2^r r!}\left[\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2}\right] \\
&\times \prod_{k<l}\left(2\sin\frac{\theta_k+\theta_l}{2}\right)^2\left(2\sin\frac{\theta_k-\theta_l}{2}\right)^2 \\
&= \frac{2}{2^r r!}\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2} \\
&\times \prod_{k<l}\left[\left(2\cos\frac{\theta_k}{2}\right)^2-\left(2\cos\frac{\theta_l}{2}\right)^2\right]^2 \\
Dof(2d+1,N) &= 0
\end{aligned}
$$

Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$
\begin{aligned}
Dof(2d, 2r) &= \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \\
&\times \prod_{k=1}^{r} \int_0^1 dx_k \; x_k^{-1/2} (1 - x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2
\end{aligned}
$$

Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$
\begin{aligned}
Dof(2d, 2r) &= \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \\
&\times \prod_{k=1}^{r} \int_0^1 dx_k \, x_k^{-1/2} (1 - x_k)^{d-3/2} \boxed{\prod_{k<l} (x_l - x_k)^2}
\end{aligned}
$$

(Van der Monde determinant)^2: matrix models

Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$
\begin{aligned}
Dof(2d, 2r) &= \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \\
&\times \prod_{k=1}^{r} \int_0^1 dx_k \, x_k^{-1/2} (1-x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2(x_i) &= \det \begin{pmatrix} p_0(x_1) & \cdots & p_{r-1}(x_1) \\ p_0(x_2) & \cdots & p_{r-1}(x_2) \\ \vdots & & \vdots \\ p_0(x_r) & \cdots & p_{r-1}(x_r) \end{pmatrix} \begin{pmatrix} p_0(x_1) & \cdots & p_0(x_r) \\ p_1(x_1) & \cdots & p_1(x_r) \\ \vdots & & \vdots \\ p_{r-1}(x_1) & \cdots & p_{r-1}(x_r) \end{pmatrix} \\
&= \det K(x_i, x_j) \,, \quad p_k(x) = x^k + a_{k-1} x^{k-1} + \cdots \,, \quad \forall a_i
\end{aligned}
$$

Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$
\begin{aligned}
Dof(2d, 2r) &= \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \\
&\times \prod_{k=1}^{r} \int_0^1 dx_k \, x_k^{-1/2} (1-x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2
\end{aligned}
$$

$$
K(x_i, x_j) = \sum_{k=0}^{r-1} p_k(x_i) p_k(x_j) \qquad \int_0^1 dx \, w(x) p_n(x) p_m(x) = h_n \, \delta_{nm}
$$

$$
w(x) = x^{-\frac{1}{2}} (1-x)^{d-\frac{3}{2}} \quad \text{Jacobi polynomials}
$$

Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$Dof(2d, 2r) = \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \times \prod_{k=1}^{r} \int_0^1 dx_k\; x_k^{-1/2} (1-x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2$$

$$\frac{1}{r!} \int_0^1 dx_r\; w(x_r) \cdots \int_0^1 dx_1\; w(x_1)\, \Delta^2(x_i) = \prod_{k=0}^{r-1} h_k$$

Final results

- Even dimension

$$Dof(2d,2r) = 2^{r-1}\frac{(2d-2)!}{[(d-1)!]^2}\prod_{k=1}^{r-1}\frac{k\,(2k-1)!\,(2k+2d-3)!}{(2k+d-2)!\,(2k+d-1)!}$$

Final results

- Even dimension

$$Dof(2d,2r) = 2^{r-1}\frac{(2d-2)!}{[(d-1)!]^2}\prod_{k=1}^{r-1}\frac{k\,(2k-1)!\,(2k+2d-3)!}{(2k+d-2)!\,(2k+d-1)!}$$

- Similarly $Dof(2d,2r+1)$

Final results

- Even dimension

$$Dof(2d,2r) = 2^{r-1}\frac{(2d-2)!}{[(d-1)!]^2}\prod_{k=1}^{r-1}\frac{k\,(2k-1)!\,(2k+2d-3)!}{(2k+d-2)!\,(2k+d-1)!}$$

- Similarly $Dof(2d,2r+1)$

Few interesting special cases

- $Dof(2,N)=1, \quad \forall N \quad \surd$
- $Dof(4,N)=2, \quad \forall N \quad \surd$

Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic N and D

Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic N and D
- In 4d, higher spin fields propagate into the loop

Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic N and D
- In 4d, higher spin fields propagate into the loop
- Couple it to maximally symmetric backgrounds

Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic N and D
- In 4d, higher spin fields propagate into the loop
- Couple it to maximally symmetric backgrounds
- More general couplings?

Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic N and D
- In 4d, higher spin fields propagate into the loop
- Couple it to maximally symmetric backgrounds
- More general couplings?
  - covariantization for $N > 2$ problematic for a generic background; algebra doesn't seem to close

Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic N and D
- In 4d, higher spin fields propagate into the loop
- Couple it to maximally symmetric backgrounds
- More general couplings?
  - covariantization for $N > 2$ problematic for a generic background; algebra doesn't seem to close
- More general symmetry group (Hallowell, Waldron '07)

Manifolds with boundary

- Given the differential operator

$$H_x = -\frac{1}{2}\nabla_x^2 + V(x)$$

Manifolds with boundary

- Given the differential operator

$$H_x = -\frac{1}{2}\nabla_x^2 + V(x)$$

- Heat kernel is the solution of the heat equation (Schroedinger eq. in euclidean time)

$$\begin{aligned} -\frac{\partial}{\partial\beta}K(x,y;\beta) &= H_x K(x,y;\beta) \\ \text{with b.c.}'\text{s} \quad K(x,y;0) &= \delta^D(x-y) \end{aligned}$$

Manifolds with boundary

- Given the differential operator

$$H_x = -\frac{1}{2}\nabla_x^2 + V(x)$$

- Heat kernel is the solution of the heat equation (Schroedinger eq. in euclidean time)

$$\begin{aligned} -\frac{\partial}{\partial\beta}K(x,y;\beta) &= H_x K(x,y;\beta) \\ \text{with b.c.'s} \quad K(x,y;0) &= \delta^D(x-y) \end{aligned}$$

- it can be perturbatively solved via DeWitt ansatz

Manifolds with boundary

- DeWitt ansatz

$$K(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} e^{-S_0[\bar{x}]}\,\Omega(x,y;\beta),$$

$$\Omega(x,y;\beta) \;\sim\; \sum_{n=0}^{\infty} a_n(x,y)\beta^n$$

Manifolds with boundary

- DeWitt ansatz

$$K(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} e^{-S_0[\bar{x}]}\,\Omega(x,y;\beta),$$

$$\Omega(x,y;\beta) \sim \sum_{n=0}^{\infty} a_n(x,y)\beta^n$$

Feynman measure

Manifolds with boundary

- DeWitt ansatz

$$K(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} e^{-S_0[\bar{x}]}\,\Omega(x,y;\beta),$$

$$\Omega(x,y;\beta) \sim \sum_{n=0}^{\infty} a_n(x,y)\beta^n$$

Feynman measure

Action for the classical configuration $\bar{x}(\tau)$

Manifolds with boundary

- DeWitt ansatz

$$K(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} e^{-S_0[\bar{x}]}\,\Omega(x,y;\beta),$$

$$\Omega(x,y;\beta) \sim \sum_{n=0}^{\infty} a_n(x,y)\beta^n$$

Feynman measure

Action for the classical configuration $\bar{x}(\tau)$

Seeley-DeWitt coefficients

Manifolds with boundary

- DeWitt ansatz

$$K(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} e^{-S_0[\bar{x}]}\,\Omega(x,y;\beta),$$

$$\Omega(x,y;\beta) \sim \sum_{n=0}^{\infty} a_n(x,y)\beta^n$$

Feynman measure

Action for the classical configuration $\bar{x}(\tau)$

Seeley-DeWitt coefficients

It's a short-time perturbative expansion

Manifolds with boundary

- Path integral representation of the Heat kernel

$$K(x, y; \beta) = \int_{x(0)=x}^{x(\beta)=y} Dx\, e^{-S[x]} = \langle y | e^{-\beta \hat{H}} | x \rangle$$

Manifolds with boundary

- Path integral representation of the Heat kernel

$$K(x,y;\beta) = \int_{x(0)=x}^{x(\beta)=y} Dx\, e^{-S[x]} = \langle y|e^{-\beta\hat{H}}|x\rangle$$

$$S[x] = S_0[x] + S_{int}[x]$$

$$S_0[x] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2}\delta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad S_{int}[x] = \frac{1}{\beta}\int_0^1 d\tau\, \beta^2 V(x)$$

Manifolds with boundary

- Path integral representation of the Heat kernel

$$K(x,y;\beta) = \int_{x(0)=x}^{x(\beta)=y} Dx\, e^{-S[x]} = \langle y|e^{-\beta\hat{H}}|x\rangle$$

$$S[x] = S_0[x] + S_{int}[x]$$

$$S_0[x] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2}\delta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \qquad S_{int}[x] = \frac{1}{\beta}\int_0^1 d\tau\, \beta^2 V(x)$$

Flat target-space metric

Manifolds with boundary

- Path integral representation of the Heat kernel

$$K(x,y;\beta) = \int_{x(0)=x}^{x(\beta)=y} Dx\, e^{-S[x]} = \langle y|e^{-\beta\hat{H}}|x\rangle$$

$$S[x] = S_0[x] + S_{int}[x]$$

$$S_0[x] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2}\delta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \qquad S_{int}[x] = \frac{1}{\beta}\int_0^1 d\tau\, \beta^2 V(x)$$

Flat target-space metric

- Path integral for generic V not known
- Treat V as perturbation $\rightarrow$ vertices from Taylor expanding about initial/final point

Manifolds with boundary

- Path integral representation of the Heat kernel

$$K(x,y;\beta) = \int_{x(0)=x}^{x(\beta)=y} Dx\, e^{-S[x]} = \langle y|e^{-\beta\hat{H}}|x\rangle$$

$$S[x] = S_0[x] + S_{int}[x]$$

$$S_0[x] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2}\delta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \qquad S_{int}[x] = \frac{1}{\beta}\int_0^1 d\tau\, \beta^2 V(x)$$

Flat target-space metric

- Path integral for generic V not known
- Treat V as perturbation $\rightarrow$ vertices from Taylor expanding about initial/final point

Manifolds with boundary

- Boundaryless flat space

Manifolds with boundary

- Boundaryless flat space
  - Split $x(\tau) = \bar{x}(\tau) + q(\tau)$

Manifolds with boundary

- Boundaryless flat space
  - Split $x(\tau) = \bar{x}(\tau) + q(\tau)$
  - Measure

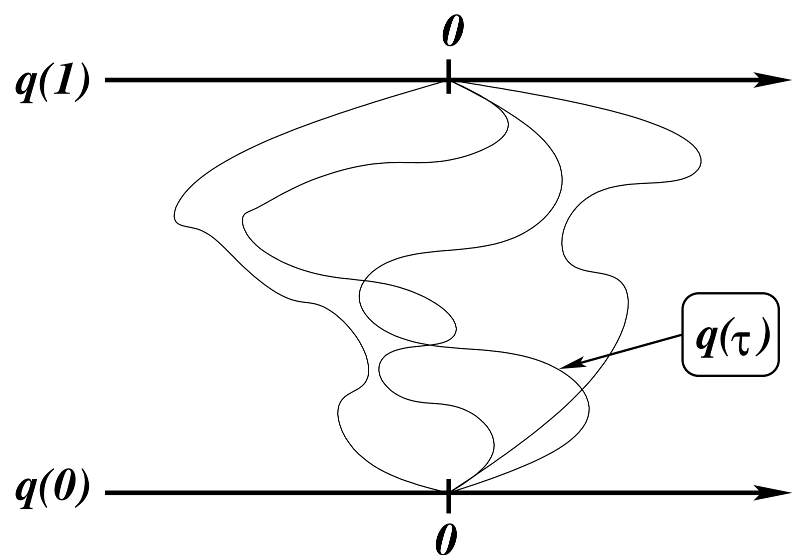
$$Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau$$

Manifolds with boundary

- Boundaryless flat space
  - Split $x(\tau) = \bar{x}(\tau) + q(\tau)$
  - Measure

$$Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau$$

  - Pictorially



Manifolds with boundary

- Boundaryless flat space
  - Split $x(\tau) = \bar{x}(\tau) + q(\tau)$
  - Measure

$$Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau$$

  - $\frac{1}{(2\pi\beta)^{D/2}} \longrightarrow$ Semiclassical (one-loop) contribution

Manifolds with boundary

- Boundaryless flat space
  - Split $x(\tau) = \bar{x}(\tau) + q(\tau)$
  - Measure

$$Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau$$

  - $\frac{1}{(2\pi\beta)^{D/2}} \longrightarrow$ Semiclassical (one-loop) contribution
  - $a_n(x, y; \beta) \longrightarrow (n+1)-$loop contribution in the worldline path-integral via Wick's theorem

Manifolds with boundary

- Flat 1D space with boundary: particle on the half line

Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  - $M = \mathbb{R}_+ \quad \partial M = \{0\}$

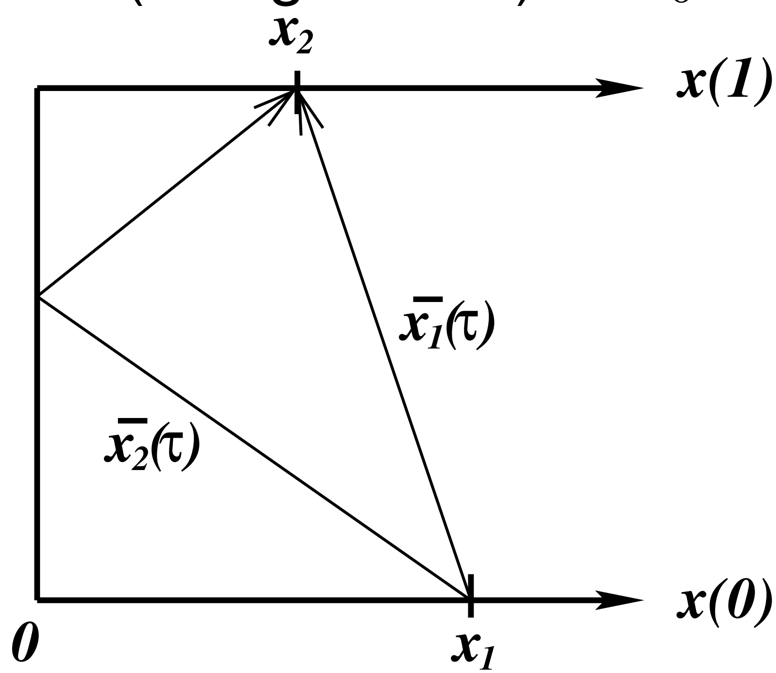
Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  - $M = \mathbb{R}_+ \quad \partial M = \{0\}$
  - Dirichlet - Neumann b.c.'s

    $K(x, x_2; \beta) = 0, \ \partial_n K(x, x_2; \beta) = 0$ for $x \in \partial M$

Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  - $M = \mathbb{R}_+ \quad \partial M = \{0\}$
  - Dirichlet - Neumann b.c.'s
    $K(x, x_2; \beta) = 0, \ \partial_n K(x, x_2; \beta) = 0$ for $x \in \partial M$
  - Two classical paths (straight lines) of S_0 now



Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  - $M = \mathbb{R}_+ \quad \partial M = \{0\}$
  - Dirichlet - Neumann b.c.'s

    $K(x, x_2; \beta) = 0, \ \partial_n K(x, x_2; \beta) = 0$ for $x \in \partial M$
  - Write the ansatz as McAvity-Osborn '91

$$K(x_1, x_2; \beta) \ = \ \frac{1}{(2\pi\beta)^{\frac{1}{2}}} \Big(e^{-S_0[\bar{x}_1]} \, \Omega_1(x_1, x_2; \beta)$$
$$+\gamma \, e^{-S_0[\bar{x}_2]} \, \Omega_2(x_1, x_2; \beta) \Big), \quad \gamma = -1/+1$$

Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  - $M = \mathbb{R}_+ \quad \partial M = \{0\}$
  - Dirichlet - Neumann b.c.'s

    $K(x, x_2; \beta) = 0, \; \partial_n K(x, x_2; \beta) = 0$ for $x \in \partial M$
  - Write the ansatz as McAvity-Osborn '91

$$K(x_1, x_2; \beta) = \frac{1}{(2\pi\beta)^{\frac{1}{2}}} \Big(e^{-S_0[\bar{x}_1]} \, \Omega_1(x_1, x_2; \beta) + \gamma \, e^{-S_0[\bar{x}_2]} \, \Omega_2(x_1, x_2; \beta) \Big), \quad \gamma = -1/+1$$

  - How to implement it in the path integral?

Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  - $M = \mathbb{R}_+ \quad \partial M = \{0\}$
  - Dirichlet - Neumann b.c.'s

    $K(x, x_2; \beta) = 0, \ \partial_n K(x, x_2; \beta) = 0$ for $x \in \partial M$
  - Write the ansatz as McAvity-Osborn '91

$$K(x_1, x_2; \beta) = \frac{1}{(2\pi\beta)^{\frac{1}{2}}} \Big(e^{-S_0[\bar{x}_1]} \, \Omega_1(x_1, x_2; \beta) + \gamma \, e^{-S_0[\bar{x}_2]} \, \Omega_2(x_1, x_2; \beta) \Big), \quad \gamma = -1/+1$$

  - How to implement it in the path integral? i.e. How to implement the constraint $x(\tau) \geq 0$ in the path integral?

Manifolds with boundary

- Two classes of paths

Manifolds with boundary

- Two classes of paths
  1. paths bounded to the interior of M

Manifolds with boundary

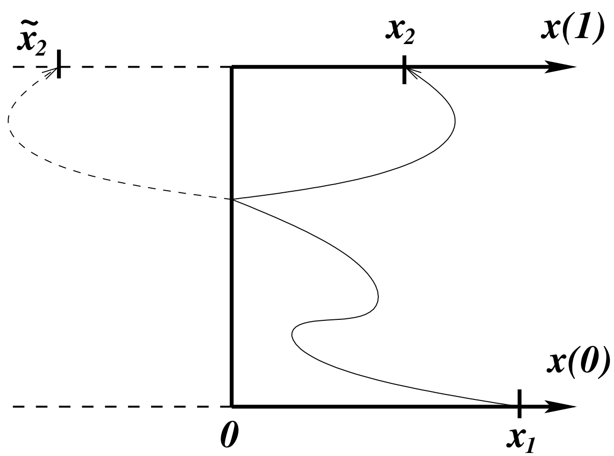
- Two classes of paths
  1. paths bounded to the interior of M
  2. paths that hit the boundary

Manifolds with boundary

- Two classes of paths
  1. paths bounded to the interior of M
  2. paths that hit the boundary
- Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary

Manifolds with boundary

- Two classes of paths
  1. paths bounded to the interior of M
  2. paths that hit the boundary
- Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary



Manifolds with boundary

- Two classes of paths
  1. paths bounded to the interior of M
  2. paths that hit the boundary
- Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary

1. Extend the potential to the whole $\mathbb{R}$ as an even function

$$V(x(\tau)) \longrightarrow \tilde{V}(x(\tau)) = \theta(x(\tau))V(x(\tau)) + \theta(-x(\tau))V(-x(\tau))$$

Manifolds with boundary

- Two classes of paths
  1. paths bounded to the interior of M
  2. paths that hit the boundary
- Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary

1. Extend the potential to the whole $\mathbb{R}$ as an even function

$$V(x(\tau)) \longrightarrow \tilde{V}(x(\tau)) = \theta(x(\tau))V(x(\tau)) + \theta(-x(\tau))V(-x(\tau))$$

2. For Dirichlet/Neumann bc's write the kernel as

$$K_M(x_1, x_2; \beta) = K_{\mathbb{R}}(x_1, x_2; \beta) \mp K_{\mathbb{R}}(x_1, -x_2; \beta) \quad x_{1,2} \in M$$

Manifolds with boundary

- For Dbc the heuristics is quite clear: the second term $K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum$ paths that hit the boundary

Manifolds with boundary

- For Dbc the heuristics is quite clear: the second term $K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum$ paths that hit the boundary

  $\Rightarrow K_M(x_1, x_2; \beta) = \sum$ paths that do NOT hit the boundary

$$K_M(x_1, 0; \beta) = 0$$

Manifolds with boundary

- For Dbc the heuristics is quite clear: the second term $K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum$ paths that hit the boundary

  $\Rightarrow K_M(x_1, x_2; \beta) = \sum$ paths that do NOT hit the boundary

$$K_M(x_1, 0; \beta) = 0$$

- For Nbc: by direct inspection

$$\partial_2 K_M(x_1, 0; \beta) = 0$$

Manifolds with boundary

- For Dbc the heuristics is quite clear: the second term $K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum$ paths that hit the boundary

  $\Rightarrow K_M(x_1, x_2; \beta) = \sum$ paths that do NOT hit the boundary

$$K_M(x_1, 0; \beta) = 0$$

- For Nbc: by direct inspection

$$\partial_2 K_M(x_1, 0; \beta) = 0$$

- Left w/ computation of whole-line path integrals w/ $\tilde{V}$

Manifolds with boundary

- For Dbc the heuristics is quite clear: the second term $K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum$ paths that hit the boundary

  $\Rightarrow K_M(x_1, x_2; \beta) = \sum$ paths that do NOT hit the boundary

$$K_M(x_1, 0; \beta) = 0$$

- For Nbc: by direct inspection

$$\partial_2 K_M(x_1, 0; \beta) = 0$$

- Left w/ computation of whole-line path integrals w/ $\tilde{V}$
- Not yet straightforward: naive perturbation theory w/ $\theta(x(\tau))$ is problematic

Manifolds with boundary

- Task: computation of

Manifolds with boundary

- Task: computation of

$$\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \; e^{-S_2[q]} e^{-\beta \int d\tau \; \tilde{V}(x_{cl}(\tau)+q(\tau))}$$

Manifolds with boundary

- Task: computation of

$$\langle x_2, \beta | x_1, 0\rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq\, e^{-S_2[q]} e^{-\beta \int d\tau\ \tilde{V}(x_{cl}(\tau)+q(\tau))}$$

$$= e^{-\frac{1}{2\beta}(x_2-x_1)^2} \int_{q(0)=0}^{q(1)=0} Dq\, e^{-S_2[q]} \Bigg(1 - \beta \int_0^1 d\tau\, \tilde{V}(x_{cl}(\tau) + q(\tau))$$

$$+\frac{\beta^2}{2!} \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \tilde{V}(x_{cl}(\tau_1) + q(\tau_1))\tilde{V}(x_{cl}(\tau_2) + q(\tau_2)) + \cdots$$

Manifolds with boundary

- Task: computation of

$$\langle x_2, \beta | x_1, 0\rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq\, e^{-S_2[q]} e^{-\beta \int d\tau\ \tilde{V}(x_{cl}(\tau)+q(\tau))}$$

- Single$-V$ insertion: extract the worldline integral out

$$\int_0^1 d\sigma \int_{q(0)=0}^{q(1)=0} Dq\, e^{-S_2[q]} \Big[\theta(x_{cl}(\sigma)+q(\sigma)) V(x_{cl}(\sigma)+q(\sigma)) + \theta(-x_{cl}(\sigma)-q(\sigma)) V(-x_{cl}(\sigma)-q(\sigma)) \Big]$$

Manifolds with boundary

- Task: computation of

$$\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \; e^{-S_2[q]} e^{-\beta \int d\tau \; \tilde{V}(x_{cl}(\tau)+q(\tau))}$$

- Single$-V$ insertion: extract the worldline integral out

$$\int_0^1 d\sigma \int_{q(0)=0}^{q(1)=0} Dq \; e^{-S_2[q]} \Big[\theta(x_{cl}(\sigma) + q(\sigma)) V(x_{cl}(\sigma) + q(\sigma)) + \theta(-x_{cl}(\sigma) - q(\sigma)) V(-x_{cl}(\sigma) - q(\sigma)) \Big]$$

- For fixed σ constraints act only on $q(\sigma)$

Manifolds with boundary

- Task: computation of

$$\langle x_2,\beta|x_1,0\rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]}\int_{q(0)=0}^{q(1)=0} Dq\; e^{-S_2[q]} e^{-\beta\int d\tau\;\tilde{V}(x_{cl}(\tau)+q(\tau))}$$

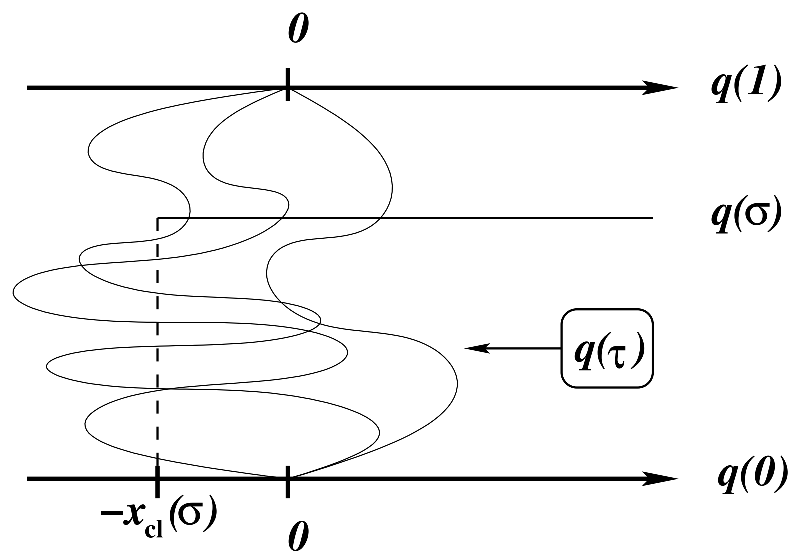
- Single$-V$ insertion: extract the worldline integral out

$$\int_0^1 d\sigma \int_{q(0)=0}^{q(1)=0} Dq\; e^{-S_2[q]}\Big[\theta(x_{cl}(\sigma)+q(\sigma))V(x_{cl}(\sigma)+q(\sigma)) + \theta(-x_{cl}(\sigma)-q(\sigma))V(-x_{cl}(\sigma)-q(\sigma))\Big]$$

$$q(\sigma)\geq -x_{cl}(\sigma)$$

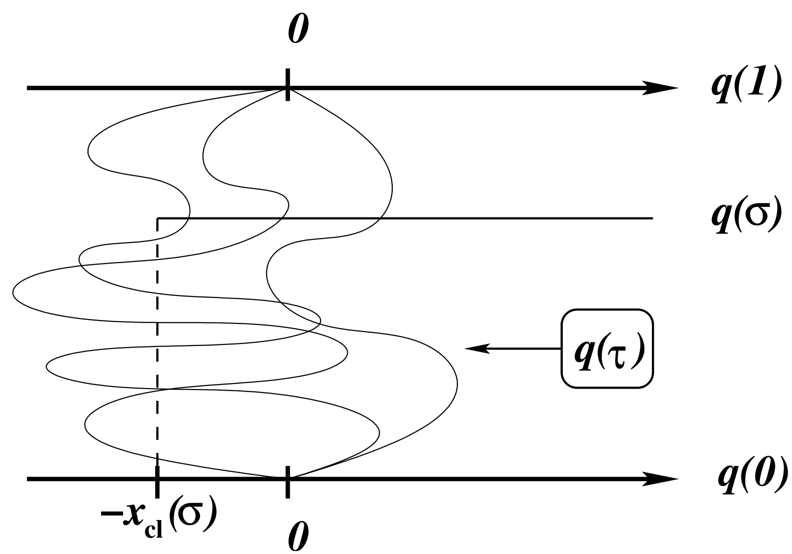
Manifolds with boundary

- Pictorially



Manifolds with boundary

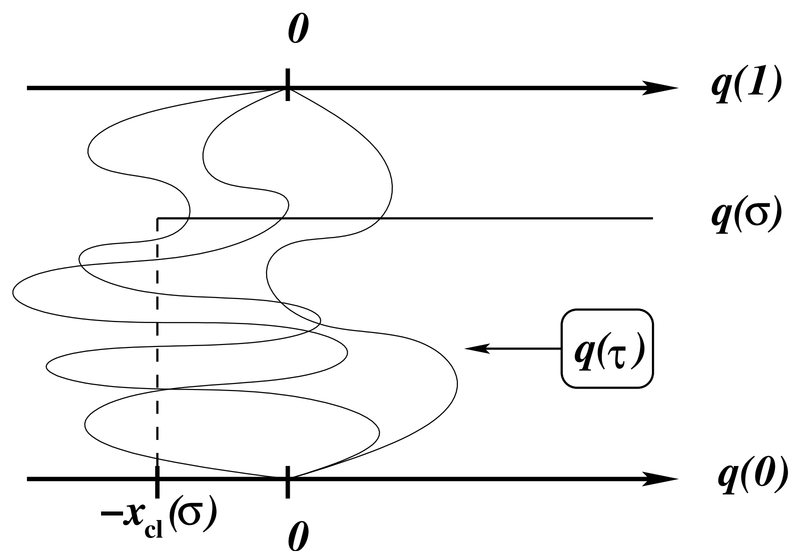
- Pictorially



- Split the integral in two pieces $[0, \sigma], [\sigma, 1]$; then, for fixed σ

Manifolds with boundary

- Pictorially



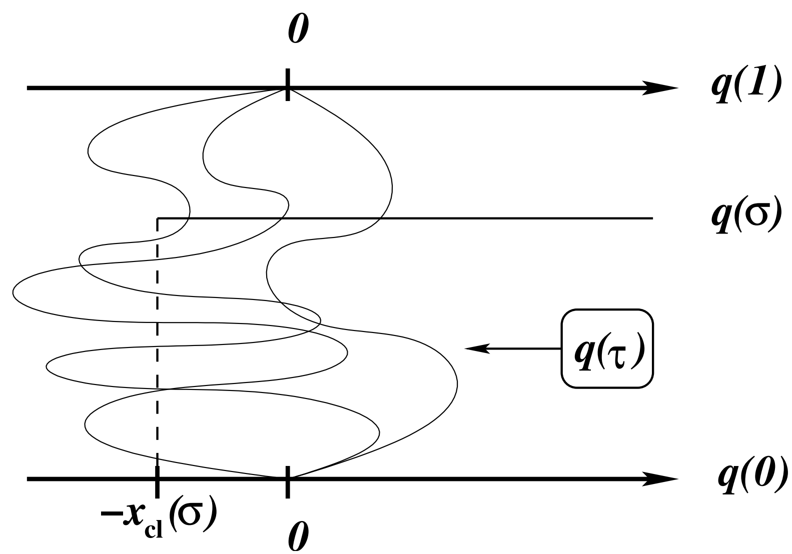
- Split the integral in two pieces $[0,\sigma]$, $[\sigma,1]$; then, for fixed σ

$$\int_{q(0)=0}^{q(1)=0} Dq\, e^{-S_2[q]} \theta(x_{cl}(\sigma)+q(\sigma)) V(x_{cl}(\sigma)+q(\sigma))$$

$$= \int_{-x_{cl}(\sigma)}^{\infty} dy \int_{q(0)=0}^{q(\sigma)=y} Dq\, e^{-S_2[q]} \int_{q(\sigma)=y}^{q(1)=0} Dq\, e^{-S_2[q]}\, V(x_{cl}(\sigma)+y)$$

Manifolds with boundary

- Pictorially



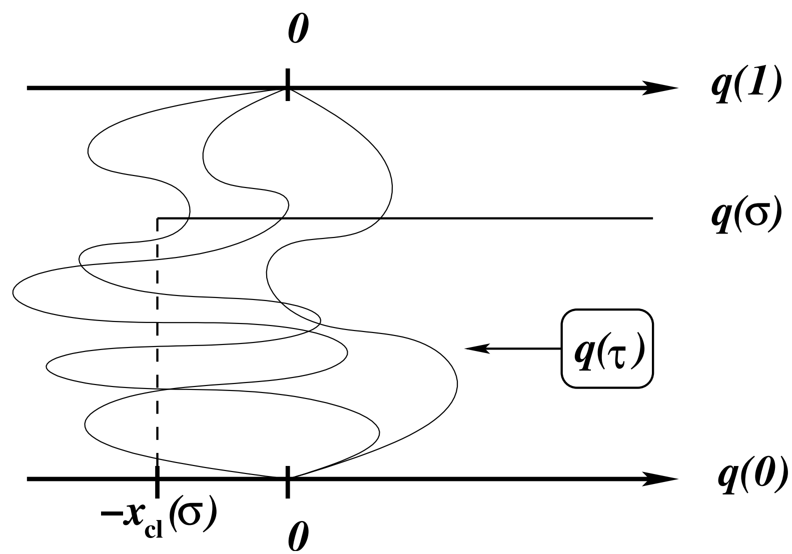
- Split the integral in two pieces $[0,\sigma]$, $[\sigma,1]$; then, for fixed σ

$$\int_{q(0)=0}^{q(1)=0} Dq\, e^{-S_2[q]}\theta(x_{cl}(\sigma)+q(\sigma))V(x_{cl}(\sigma)+q(\sigma))$$

$$= \int_{-x_{cl}(\sigma)}^{\infty} dy \int_{q(0)=0}^{q(\sigma)=y} Dq\, e^{-S_2[q]} \int_{q(\sigma)=y}^{q(1)=0} Dq\, e^{-S_2[q]}\; V(x_{cl}(\sigma)+y)$$

Manifolds with boundary

- Pictorially

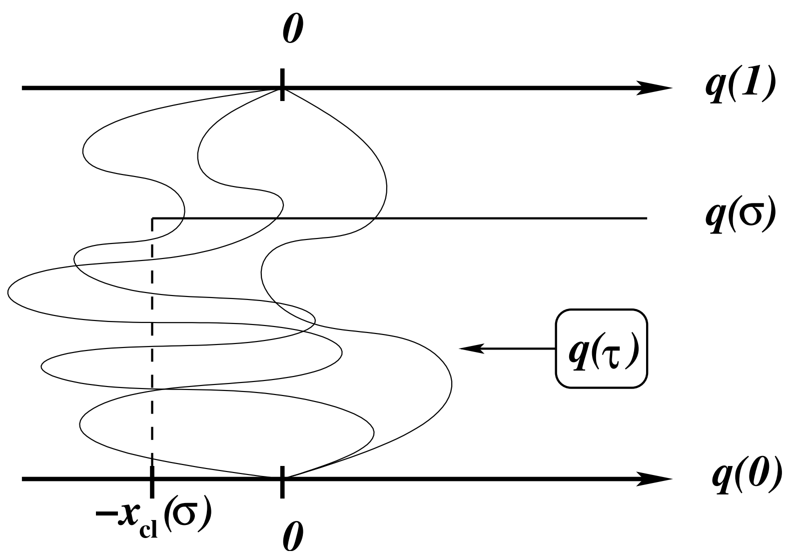


- Split the integral in two pieces $[0,\sigma],[\sigma,1]$; then, for fixed σ

$$\int_{-x_{cl}(\sigma)}^{\infty} dy\, \frac{1}{2\pi\beta\sqrt{\sigma(1-\sigma)}} e^{-\frac{1}{2\beta\sigma(1-\sigma)}y^2}\, V(x_{cl}(\sigma)+y)$$

Manifolds with boundary

- Pictorially



- Split the integral in two pieces $[0,\sigma],[\sigma,1]$; then, for fixed σ

$$\int_{-x_{cl}(\sigma)}^{\infty} dy\, \frac{1}{2\pi\beta\sqrt{\sigma(1-\sigma)}} e^{-\frac{1}{2\beta\sigma(1-\sigma)}y^2}\, V(x_{cl}(\sigma)+y)$$

- Similarly $\displaystyle\int_{-\infty}^{-x_{cl}(\sigma)} dy\, \frac{1}{2\pi\beta\sqrt{\sigma(1-\sigma)}} e^{-\frac{1}{2\beta\sigma(1-\sigma)}y^2}\, V(-x_{cl}(\sigma)-y)$

Results

- Single-V whole-line heat kernel

$$\langle x_2, \beta | x_1, 0\rangle_{\mathbb{R}} = \frac{e^{-\frac{1}{2\beta}(x_2-x_1)^2}}{(2\pi\beta)^{1/2}}\Bigg(1$$

$$-\int_0^1 d\sigma\, l_\sigma \int_{-\infty}^{+\infty} dy\, e^{-\frac{y^2}{2\beta\sigma(1-\sigma)}}\, V(x_{cl}(\sigma)+y)$$

$$+\int_0^1 d\sigma\, l_\sigma \int_{-\infty}^{-x_{cl}(\sigma)} dy\, e^{-\frac{y^2}{2\beta\sigma(1-\sigma)}} \Big[V(x_{cl}(\sigma)+y) - V(-x_{cl}(\sigma)-y)\Big]\Bigg)$$

Results

- Even potential V ($\Rightarrow \tilde{V} = V$): third line vanishes

$$
\begin{aligned}
\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}_+} &= \frac{e^{-\frac{1}{2\beta}(x_2 - x_1)^2}}{(2\pi\beta)^{1/2}} \left[1 - \beta \bar{V} - \frac{\beta^2}{2 \cdot 3!} \bar{V}'' \left(1 - \frac{(x_2 - x_1)^2}{\beta} \right) + O(\beta^3) \right] \\
&+ \frac{e^{-\frac{1}{2\beta}(x_2 + x_1)^2}}{(2\pi\beta)^{1/2}} \left[1 - \beta \bar{V} - \frac{\beta^2}{2 \cdot 3!} \bar{V}'' \left(1 - \frac{(x_2 + x_1)^2}{\beta} \right) + O(\beta^3) \right]
\end{aligned}
$$

Results

- Even potential V ($\Rightarrow \tilde{V} = V$): third line vanishes

$$
\begin{aligned}
\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}_+} &= \frac{e^{-\frac{1}{2\beta}(x_2 - x_1)^2}}{(2\pi\beta)^{1/2}} \left[1 - \beta \bar{V} - \frac{\beta^2}{2 \cdot 3!} \bar{V}'' \left(1 - \frac{(x_2 - x_1)^2}{\beta} \right) + O(\beta^3) \right] \\
&+ \frac{e^{-\frac{1}{2\beta}(x_2 + x_1)^2}}{(2\pi\beta)^{1/2}} \left[1 - \beta \bar{V} - \frac{\beta^2}{2 \cdot 3!} \bar{V}'' \left(1 - \frac{(x_2 + x_1)^2}{\beta} \right) + O(\beta^3) \right]
\end{aligned}
$$

- McAvity-Osborn ansatz, w/ $\Omega_i' s$ integer power series in β and $|x_2 - x_1|$ $\quad \surd$

Results

- Even potential V ($\Rightarrow \tilde{V} = V$): third line vanishes

$$\begin{aligned}\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}_+} &= \frac{e^{-\frac{1}{2\beta}(x_2-x_1)^2}}{(2\pi\beta)^{1/2}} \left[1 - \beta\bar{V} - \frac{\beta^2}{2\cdot 3!}\bar{V}'' \left(1 - \frac{(x_2-x_1)^2}{\beta} \right) + O(\beta^3) \right] \\ &+ \frac{e^{-\frac{1}{2\beta}(x_2+x_1)^2}}{(2\pi\beta)^{1/2}} \left[1 - \beta\bar{V} - \frac{\beta^2}{2\cdot 3!}\bar{V}'' \left(1 - \frac{(x_2+x_1)^2}{\beta} \right) + O(\beta^3) \right]\end{aligned}$$

- McAvity-Osborn ansatz, w/ $\Omega_i's$ integer power series in β and $|x_2 - x_1| \quad \surd$

- Result coincides w/ conventional expansion (Wick's theorem): no θ's involved $\surd$

Results

- Generic potential: additional boundary contributions

Results

- Generic potential: additional boundary contributions
- Can obtain the perturbative expansion (in β) of the partition function $\mathrm{Tr}_{\mathbb{R}_+} e^{-\beta\hat{H}}$

$$
\begin{aligned}
\mathrm{Tr}_{\mathbb{R}_+}\, e^{-\beta\hat{H}} &= \int_0^\infty dx\, \langle x,\beta|x,0\rangle_{\mathbb{R}_+} \\
&= \frac{1}{(2\pi\beta)^{1/2}}\Bigg[\int_0^\infty dx\left(1-\beta V(x)+\beta^2\left(\frac{1}{2}V^2(x)-\frac{1}{12}V''(x)\right)\right) \\
&\mp \sqrt{\frac{\pi\beta}{8}}\left(1-\beta V(0)+\beta^2\left(\frac{1}{2}V^2(0)-\frac{1}{8}V''(0)\right)\right) \\
&- \frac{\beta^2}{6}(2_+,-1_-)V'(0)+O(\beta^3)\Bigg]
\end{aligned}
$$

Results

- Generic potential: additional boundary contributions
- Can obtain the perturbative expansion (in β) of the partition function $\mathrm{Tr}_{\mathbb{R}_+} e^{-\beta\hat{H}}$

$$
\begin{aligned}
\mathrm{Tr}_{\mathbb{R}_+}\, e^{-\beta\hat{H}} &= \int_0^\infty dx\, \langle x,\beta|x,0\rangle_{\mathbb{R}_+} \\
&= \frac{1}{(2\pi\beta)^{1/2}}\left[\int_0^\infty dx\left(1-\beta V(x)+\beta^2\left(\frac{1}{2}V^2(x)-\frac{1}{12}V''(x)\right)\right)\right. \\
&\mp \sqrt{\frac{\pi\beta}{8}}\left(1-\beta V(0)+\beta^2\left(\frac{1}{2}V^2(0)-\frac{1}{8}V''(0)\right)\right) \\
&- \left.\frac{\beta^2}{6}(2_+,-1_-)V'(0)+O(\beta^3)\right]
\end{aligned}
$$

- It’s the needed object in anomaly computations and worldline formalism

Outlook

- Developed worldline path integral method in (flat) manifolds with boundary

Outlook

- Developed worldline path integral method in (flat) manifolds with boundary
- Carried on expansion up to order $\beta^{\frac{7}{2}}$

Outlook

- Developed worldline path integral method in (flat) manifolds with boundary
- Carried on expansion up to order $\beta^{\frac{7}{2}}$
- Higher-order increasingly difficult

Outlook

- Developed worldline path integral method in (flat) manifolds with boundary
- Carried on expansion up to order $\beta^{\frac{7}{2}}$
- Higher-order increasingly difficult

- Simplify the implementation of the constraints into the path integral

Outlook

- Developed worldline path integral method in (flat) manifolds with boundary
- Carried on expansion up to order $\beta^{\frac{7}{2}}$
- Higher-order increasingly difficult

- Simplify the implementation of the constraints into the path integral
- Mixed (Robin) boundary conditions: need to add $\delta-$function potential