The Seiberg–Witten theory

Coulomb phase of $\mathcal{N}=1$ $SO(N)$

$\mathcal{N}=1$, $SO(N)$ $F=N-2$:

	$SO(N)$	$SU(F=N-2)$	$U(1)_R$
Φ	$\square$	$\square$	0

generic point in the classical moduli space $SO(N)\to SO(2)\approx U(1)$
homomorphic $U(1)$ coupling

$$\tau=\frac{\theta_{\rm YM}}{2\pi}+\frac{4\pi i}{g^2}\ ,$$

transforms under electric–magnetic duality $(E^i\to B^i,\, B^i\to -E^i)$ as:

$$S:\tau\to-\frac{1}{\tau}$$

not a symmetry, exchanges two equivalent descriptions
one weakly coupled, one strongly coupled

Coulomb phase of $\mathcal{N} = 1$ $SO(N)$

shifting $\theta_{\rm YM}$ by 2π is a symmetry

$$T : \tau \to \tau + 1$$

in general

$$\tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$$

where $\alpha, \beta, \gamma, \delta$ are integers $(\alpha, \beta, \gamma, \delta \in \mathbb{Z})$ and $\alpha\delta - \beta\gamma = 1$
S and T generate $SL(2, \mathbb{Z})$
gives a set of equivalent $U(1)$ gauge theories
different holomorphic couplings

Coulomb phase of $\mathcal{N} = 1$ $SO(N)$

as a function on the moduli space, τ depends on flavor invariant

$$z = \det \Phi\Phi$$

for large z the theory is weakly coupled and we know that the holomorphic $SO(N)$ gauge coupling is

$$\tau_{SO} \approx \frac{i}{2\pi} \ln\left(\frac{z}{\Lambda^b}\right)$$

where $b = 3(N-2) - F = 2(N-2)$
$SO(N) \to SO(4) \approx SU(2) \times SU(2) \to SU(2)_D \to U(1)$
so the $U(1)$ gauge coupling g is related to the $SO(N)$ coupling by

$$\frac{1}{g^2} = \frac{1}{g^2_{SO}} + \frac{1}{g^2_{SO}}$$

$$\tau \approx \frac{i}{\pi} \ln\left(\frac{z}{\Lambda^b}\right)$$

Coulomb phase of $\mathcal{N} = 1$ $SO(N)$

τ has a singularity in the complex variable z at $z = \infty$
as $z \longrightarrow e^{2\pi i} z$, τ is shifted by -2
called a *monodromy*
monodromy of τ at $z = \infty$

$$\mathcal{M}_\infty = T^{-2}$$

$$consider \Phi_i \Phi_j \longrightarrow e^{2\pi i} \Phi_i \Phi_j \quad z \longrightarrow e^{F \cdot 2\pi i} z$$

$$\tau \longrightarrow \tau - 2F$$

and the the monodromy of τ at ∞ on moduli space is

$$\mathcal{M}^F_\infty = T^{-2F}$$

τ is not a single-valued function on the moduli space
even at weak coupling

Coulomb phase of $\mathcal{N} = 1$ $SO(N)$

$$\frac{4\pi}{g^2} = \mathrm{Im}\,\tau$$

is invariant under $\mathcal{M}_\infty$ (single-valued at weak coupling)
single-valued everywhere $\Rightarrow$ derivatives would be well-defined, by holomorphy

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \mathrm{Im}\,\tau = 0$$

where $z = x + iy$
$\Rightarrow$ $\mathrm{Im}\,\tau$ *harmonic function*, < 0 somewhere, $\Rightarrow$ g is imaginary
$\mathrm{Im}\,\tau$ is not single-valued everywhere
moduli space has complicated topology or additional singular points

Singular points

some particles become massless

singular points have their own monodromies

at least two monodromies that do not commute with $\mathcal{M}_\infty$

otherwise $\mathrm{Im}\,\tau$ single-valued and $g^2 < 0$

with only one other monodromy, circling one is equivalent to circling around the other, and hence the two monodromies commute

monodromy is determined by the perturbative β function

Singular points

imagine a weakly coupled dual $U(1)$ gauge theory near a singular point with k light flavors

$$W_i = (z - z_i) \sum_{j=1}^{k} c_j \phi^{+j} \phi^{-j} + \mathcal{O}(z - z_i)^2$$

perturbative holomorphic dual coupling is

$$\tilde{\tau}_i \approx \frac{i\tilde{b}}{2\pi} \ln(z - z_i) + \text{const.}$$

$$\tilde{b} = -\sum_j \tfrac{4}{3} Q_{fj}^2 + \tfrac{2}{3} Q_{sj}^2$$

if all k light flavors have unit charges $\tilde{b} = -2k$
monodromy in $\tilde{\tau}_i$ is T^{2k}
"duality transformation" $\widetilde{\tau}_i = D_{z_i} \tau$
monodromy in τ at the singularity z_i is

$$\mathcal{M}_{z_i} = D_{z_i}^{-1} T^{2k} D_{z_i}$$

Singular points

we need

$$[\mathcal{M}_0, \mathcal{M}_{z_i}] \neq 0$$

D_{z_i} must be nontrivial, and thus contain an odd power of S (and possibly some power of T).

S interchanges electric and magnetic fields

the dual quarks must have magnetic charge!

Dual with one more Flavor

dual of $SO(N)$ with $N-1$ flavors is (for $N>3$)

	$SO(3)$	$SU(F=N-1)$	$U(1)_R$
ϕ	$\square$	$\overline{\square}$	$\frac{N-2}{N-1}$
M'	$\mathbf{1}$	$\square\square$	$\frac{2}{N-1}$

with a superpotential

$$W = \frac{M'_{ji}}{2\mu}\phi^j\phi^i - \frac{1}{64\Lambda^{2N-5}_{N,N-1}}\det M'$$

integrate out one flavor with a mass term $\frac{1}{2}mM'_{N-1,N-1}$
M: mesons composed of the remaining light flavors
eqm give

$$\phi^{N-1}\phi^{N-1} = \frac{\mu \det M}{32\Lambda^{2N-5}_{N,N-1}} - \mu m$$

near $\det M = 0$, $SO(3) \to U(1)$

Dual with $F = N - 2$

effective superpotential is:

$$W_{\mathrm{eff}} = \frac{1}{2\mu}\, f\left(\frac{\det M}{\Lambda_{N,N-2}^{2N-4}}\right) M_{ij}\, \phi^{+i}\phi^{-j} \ .$$

dual holomorphic gauge coupling is

$$\widetilde{\tau} = -\frac{i}{\pi}\ln\left(\det M\right) + \mathrm{const.}$$

at strong coupling for large $\det M$

Monodromy at $\det M = 0$

$r = \mathrm{rank}(M)$, $F - r = N - 2 - r$ massless flavors at $\det M = 0$

Consider M_0 such that $\det M_0 = 0$, and take

$$M_0 \to e^{2\pi i} M_0$$

then

$$\widetilde{\tau} \to \widetilde{\tau} + 2(F - r)$$

a shift for each zero eigenvalue

monodromy of τ at the singular point M_0 is

$$\mathcal{M}_0^{F-r} = D_0^{-1} T^{2(F-r)} D_0$$

corresponding to a monodromy in τ on the z-plane

$$\mathcal{M}_0 = D_0^{-1} T^2 D_0 \ ,$$

because of the electric–magnetic duality, $\phi^{\pm}$ are magnetically charged

$\widetilde{\tau} \to 0 \Rightarrow \tau \to \infty$

strong and weak coupling interchanged

Dual of the Dual

magnetic dual of $SO(N)$ with $F = N - 1$ is $SO(3)$
To get the correct dual of the dual, the $SO(F+1)$ dual of $SO(3)$ with F flavors must have a dual superpotential

$$\widetilde{W} = \frac{M_{ji}}{2\mu}\phi^j\phi^i + \epsilon\alpha\,\det(\phi^j\phi^i)$$

α determined by consistency
$\epsilon = \pm 1$ since $SO(3)$ theory has a discrete axial Z_{4F} symmetry

$$Q \to e^{\frac{2\pi i}{4F}}Q$$

while $SO(F+1)$ theory only has a Z_{2F} symmetry (for $F > 2$). Under the full Z_{4F} the $\det(\phi^j\phi^i)$ term changes sign, and $\theta_{\rm YM}$ is shifted

$$\theta_{\rm YM} \to \theta_{\rm YM} + \pi$$

Dual of the Dual of $SO(N)$

with $F = N - 1$, dual dual superpotential

$$\widetilde{\widetilde{W}} = \frac{M_{ji} N^{ij}}{2\mu} + \frac{N^{ij}}{2\widetilde{\mu}} d_j d_i - \frac{\det M}{64 \Lambda_{N,N-1}^{2N-5}} + \epsilon \alpha \det(d_j d_i)$$

couples the dual meson $N^{ij} = \phi^i \phi^j$ to the dual–dual quarks d_j.
with $\widetilde{\mu} = -\mu$, the eqm for N^{ij} sets $M_{ji} = d_j d_i$ as we expect

for $\epsilon = 1$, $\widetilde{\widetilde{W}} = 0$ if

$$\alpha = \frac{1}{64 \Lambda_{N,N-1}^{2N-5}}$$

dual of the dual is the original theory for $\epsilon = 1$, what about $\epsilon = -1$?

The dyonic dual: $\epsilon = -1$

	$SO(N)$	$SU(F = N-1)$	$U(1)_R$
d	$\square$	$\square$	$\frac{1}{F}$

$$W_{\text{dyonic}} = -\frac{\det(d_i d_j)}{32\Lambda_{N,N-1}^{2N-5}}$$

$$d = (d_i, d_F),\ i = 1, \ldots, N-2$$

add a mass term $\frac{1}{2}m\, d_F d_F$, integrate out one flavor
eqm for d_i gives $d_i d_F = 0$
For $\det(d_i d_j) \neq 0$, $SO(N) \Rightarrow U(1)$
we have (using $\Lambda_{N,N-2}^{2N-4} = m\Lambda_{N,N-1}^{2N-5}$)

$$W_{\text{eff}} = \frac{1}{2}m\left(1 - \frac{\det(d_i d_j)}{16\Lambda_{N,N-2}^{2N-4}}\right) d_F^+ d_F^-$$

The dyonic dual: $\epsilon = -1$

Near

$$\det(d_i d_j) = 16\Lambda_{N,N-2}^{2N-4} \equiv z_d \ ,$$

the fields d_F^+ and d_F^- are light
duals of monopoles with $\theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} + \pi$, are dyons
electric and magnetic charge
one light field, monodromy of the dyonic coupling must be

$$\widetilde{\widetilde{\mathcal{M}}}_{z_d} = T^2$$

charges are such that

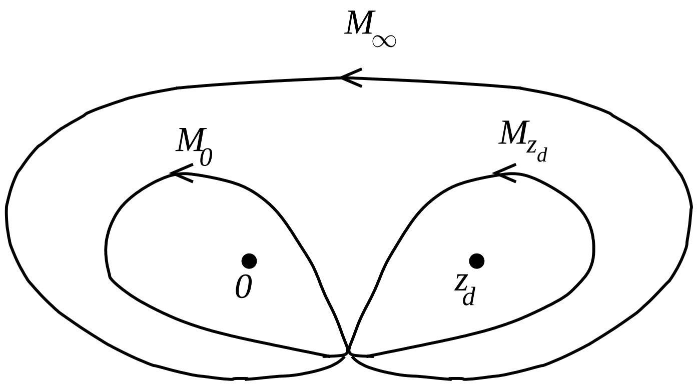
$$\phi^{\pm i}\Phi_i \sim d_F^{\pm}$$

$m \to 0 \Rightarrow \Lambda_{N,N-2} \to 0$, light dyon point $\to$ light monopole point
at $m = 0$ $SO(3)$ dual with IR fixed point

Monodromies

assuming two singular points in the interior of the moduli space monodromy of τ at z_d is determined

$$\mathcal{M}_0 \mathcal{M}_{z_d} = \mathcal{M}_\infty$$



Web of Three Dualities: mass term

three points where different particles are light and weakly interacting

Integrating out a flavor in electric theory gives $SO(N)$ with $F = N-3$
two branches: runaway vacuum and confinement
magnetic dual: monopole VEV, dual Meissner effect $\leftrightarrow$ confinement
light monopoles $\leftrightarrow$ hybrids $h^i = W_\alpha W^\alpha Q^{N-4}$.
dyonic dual: dyon VEV $\leftrightarrow$ "oblique" confinement
(in terms of light meson M'')

$$\langle d_F^+ d_F^- \rangle = \frac{16\, m_2 \Lambda_{N,N-2}^{2N-4}}{m \det M''} = \frac{16\, \Lambda_{N,N-3}^{2N-3}}{m \det M''}$$

yields a runaway superpotential

$$W_{\text{eff}} = \frac{8\, \Lambda_{N,N-3}^{2N-3}}{\det M''}$$

similar to $\mathcal{N} = 2$ Seiberg–Witten theory
difference is that the $\mathcal{N} = 1$ monopoles, dyons are not BPS states

Elliptic curves

τ not a single-valued function, transforms under $SL(2,\mathbb{Z})$
τ is a section of an $SL(2,\mathbb{Z})$ bundle
$SL(2,\mathbb{Z})$ is the *modular symmetry* group of a torus
section $\leftrightarrow$ *modular parameter* of a torus

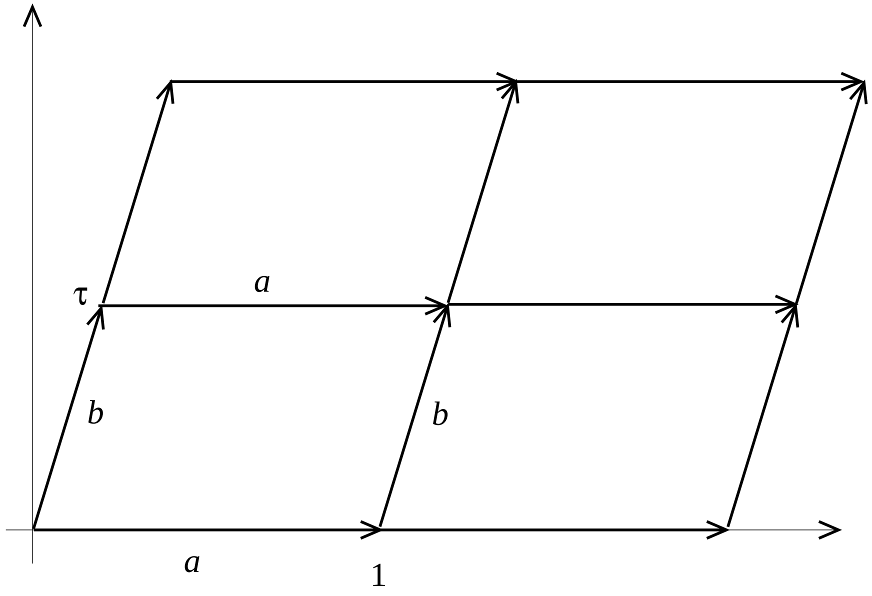
torus is the solution of a cubic (elliptic) complex equation in two complex dimensions:

$$y^2 = x^3 + Ax^2 + Bx + C \equiv (x - x_1)(x - x_2)(x - x_3)$$

where $x, y \in \mathbb{C}$, A, B, C single-valued functions of the moduli and parameters of the gauge theory

Modular parameter of a torus

making a lattice of points in $\mathbb{C}$ using τ and 1 as basis vectors



identify opposite sides $\rightarrow$ torus with modular parameter τ

Equivalent Lattice

using new basis vectors $\alpha\tau + \beta$ and $\gamma\tau + \delta$
If $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta - \beta\gamma = 1$ then the new lattice contained in old
transformation is invertible with another set of integers
$\alpha\delta - \beta\gamma = 1$ ensures new parallelogram encloses one basic parallelogram

Rescaling second basis vector to 1, the rescaled first basis vector is

$$\tau \longrightarrow \tfrac{\alpha\tau+\beta}{\gamma\tau+\delta}$$

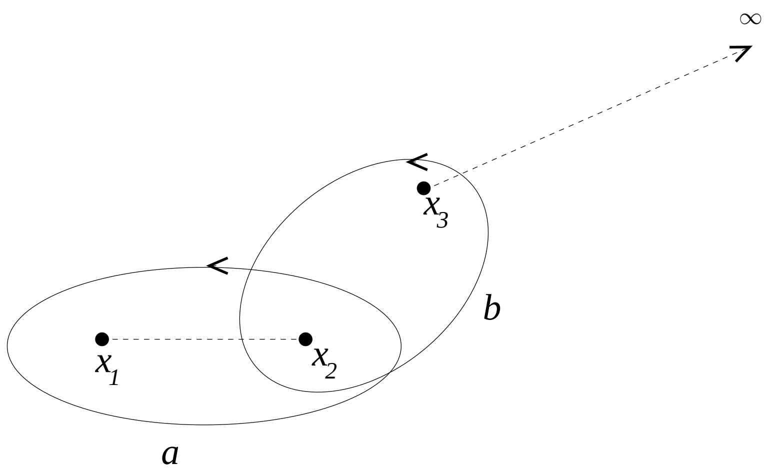
$SL(2, \mathbb{Z})$ of torus $\leftrightarrow$ $SL(2, \mathbb{Z})$ of the $U(1)$ gauge theory

Elliptic Curve and the Torus

$$y^2 = x^3 + Ax^2 + Bx + C \equiv (x - x_1)(x - x_2)(x - x_3)$$

y is square root, x plane two sheets that meet along branch cuts

cubic has three zeroes, one branch cut between two of the zeroes, other branch cut between the third zero and ∞



including point at ∞ , cut plane is topologically $\sim$ two spheres connected by two tubes $\sim$ torus, a and b cycles of torus

Modular Parameter of the Torus

given by the ratio of the periods, ω_1 and ω_2, of the torus:

$$\omega_1 = \int_a \frac{dx}{y} \ , \ \ \omega_2 = \int_b \frac{dx}{y} \ , \ \ \tau(A,B,C) = \frac{\omega_2}{\omega_1}$$

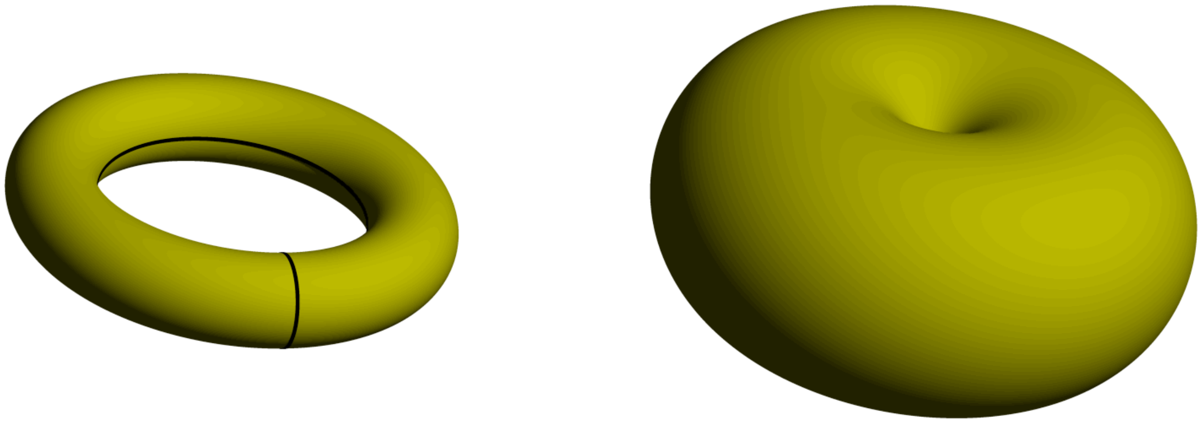
where a and b are basis of cycles around the torus
cycles $\leftrightarrow$ two sides of the parallelogram

holomorphic coupling τ is singular when a cycle shrinks to zero, i.e. when two roots meet or one roots goes to ∞, branch cuts disappears, torus is singular

Singular Tori

Two roots are equal if the discriminant vanishes

$$\Delta = \Pi_{i<j}(x_i - x_j)^2 = 4A^3C - B^2A^2 - 18ABC + 4B^3 + 27C^2 = 0$$



single-valued A, B, C easier to determine than the multi-valued τ

given A, B, and C we can calculate τ

$SO(N)$ with $F = N - 2$

singular points in the $z = \det M$ plane at $z = 0$ and $z = 16\Lambda_{N,N-2}^{2N-4}$ at these points the charged massless particles drive the dual photon coupling to zero

dual holomorphic coupling is singular

$$y^2 = x(x - 16\Lambda_{N,N-2}^{2N-4})(x - z)$$

weak coupling limit $\Lambda_{N,N-2} \to 0$ the curve becomes

$$y^2 = x^2(x - z)$$

which is singular for all $z = \det M$ as required by the fact that in an asymptotically free theory the gauge coupling runs to zero in the UV

Consistency Checks

A, B, and C must be holomorphic functions of the moduli and $\Lambda_{N,N-2}$ so that τ is holomorphic

curve must be compatible with the global symmetries for example, $\det M$ and $\Lambda^b_{N,N-2}$ have R-charge and anomalous axial charge $(0, 2F)$ which is consistent with charge assignments for x and y of $(0, 2F)$ and $(0, 3F)$

Consistency Checks: Monodromies

near a singularity z_0, $z = z_0 + \epsilon$, two roots approach each other:
$x_0 \pm a\epsilon^{n/2} \leftrightarrow \Delta \sim \epsilon^n$

$$y^2 = (x - x_1)(x - x_0 - a\epsilon^{n/2})(x - x_0 + a\epsilon^{n/2})$$

After shifting x by x_0 and rescaling x and y

$$y^2 = (x - \tilde{x})(x^2 - \epsilon^n)$$

$$\begin{aligned} \omega_1 &= \int_{-\epsilon^{n/2}}^{\epsilon^{n/2}} \frac{dx}{y} \approx \int_{-\epsilon^{n/2}}^{\epsilon^{n/2}} \frac{dx}{i\sqrt{\tilde{x}}\,\sqrt{x^2-\epsilon^n}} \approx -\frac{\pi}{\sqrt{\tilde{x}}} \\ \omega_2 &= \int_{\epsilon^{n/2}}^{\tilde{x}} \frac{dx}{y} \approx \int_{\epsilon^{n/2}}^{\tilde{x}} \frac{dx}{\sqrt{(x-\tilde{x})(x^2-\epsilon^n)}} \approx \frac{i}{\sqrt{\tilde{x}}} \ln \epsilon^{n/2} \end{aligned}$$

$$\tau = \frac{\omega_2}{\omega_1} \approx \frac{1}{2\pi i} \ln \epsilon^n$$

monodromy at the singular point z_0 is T^n

$SO(N)$ with $F = N - 2$

$z = \det M$, near zero $\Delta \sim z^2 \leftrightarrow \mathcal{M}_0 \sim T^2$ (up to a duality transformation $D^{-1}T^2D$)
monodromy in τ on moduli space, with $\operatorname{rank} M = r$, is $\mathcal{M}_0 \sim T^{2(F-r)}$
since we encircle a singular point for each zero eigenvalue ($\ln\det = \operatorname{Tr}\ln$)

near $z = z_d$, $\Delta \sim (z - z_d)^2$ corresponding to $\mathcal{M}_{z_d} \sim T^2$, and the monodromy over M is also $\mathcal{M}_{z_d}$

$SO(N)$ with $F = N - 2$

monodromy at ∞ have to be more careful since for large z the roots are approximately $(0, 16\Lambda_{N,N-2}^{4N-8}/z, z)$, so two sets of singular points are approaching each other simultaneously

rescale the coordinates so that only two roots approach each other

$$x \to x'(8\Lambda_{N,N-2}^{2N-4} - z), \;\; y \to y'(8\Lambda_{N,N-2}^{2N-4} - z)^{3/2}$$

which gives the curve

$$y'^2 = x'^3 + x'^2 + \frac{16\Lambda_{N,N-2}^{4N-8}}{(8\Lambda_{N,N-2}^{2N-4} - z)^2} x'$$

near $z = \infty$, $\Delta \sim z^{-2} \leftrightarrow \mathcal{M}_\infty \sim T^{-2}$ while the monodromy in the moduli space is $\mathcal{M}_\infty \sim T^{-2F}$

back in original $x - y$ plane the change of variables gives a factor $\sim 1/\sqrt{z}$ in $dx/y \Rightarrow$ an additional sign flip in τ

$\mathcal{M}_\infty = -T^{-2}$

$SO(N)$ with $F = N - 2$

Assuming $\mathcal{M}_0 = S^{-1}T^2S$, then the simplest solution of

$$\mathcal{M}_0\mathcal{M}_{z_d} = \mathcal{M}_\infty$$

gives $\mathcal{M}_{z_d} = (ST^{-1})^{-1}T^2ST^{-1}$

Aside from popping up in the analysis of $U(1)$ theories with monopoles, elliptic curves are also now used for factoring large numbers and for encryption in cell phones.

proof of Fermat's last theorem crucially involved proof of conjecture relating elliptic curves over rationals to modular forms

$\mathcal{N} = 2$: Seiberg–Witten

consider $\mathcal{N} = 1$ SUSY $SO(3)$ gauge theory with one flavor
since the adjoint = vector, theory has $\mathcal{N} = 2$ SUSY
classical D-term potential:

$$V = \frac{1}{g^2}\mathrm{Tr}\left[\phi, \phi^\dagger\right]^2$$

where ϕ is the scalar component of the adjoint chiral superfield
classical moduli space where $\left[\phi, \phi^\dagger\right] = 0$
parameterize the moduli space by gauge invariant $u = \mathrm{Tr}\phi^2$
up to gauge transformations take $\phi = \frac{1}{2}a\sigma^3$, classically $u = \frac{1}{2}a^2$
generic point in the moduli space $SO(3) \to U(1)$
$SU(2)_R \times U(1)_R$ R-symmetry, fermion superpartner of ϕ must have the same $U(1)_R$ charge as the λ^a, R-charge of ϕ is 2
$\Rightarrow U(1)_R$ is anomalous and instantons break $U(1)_R \to Z_4$
VEV for u breaks $Z_4 \to Z_2$ which acts on u by taking $u \to -u$

$\mathcal{N} = 2$: Seiberg–Witten

$\mathcal{N} = 2$ SUSY $\Rightarrow$ superpotential and the leading (up to two-derivative, or four fermion) terms from the Kähler function are related to a prepotential low-energy effective $U(1)$ theory can be written as

$$\mathcal{L} = \tfrac{1}{8\pi i} \int d^4\theta \tfrac{\partial P}{\partial A} \overline{A} + \tfrac{1}{16\pi i} \int d^2\theta \tfrac{\partial^2 P}{\partial A \partial A} W^\alpha W_\alpha + h.c.$$

where the $\mathcal{N} = 2$ supermultiplet contains the $\mathcal{N} = 1$ chiral supermultiplet A with scalar component a

$$\tau = \tfrac{\partial^2 P}{\partial A \partial A}$$

Perturbatively, the prepotential is completely determined by the anomaly (or equivalently the β function) however, it can receive nonperturbative corrections from instantons

$$P(A) = \tfrac{i}{2\pi} A^2 \ln \tfrac{A^2}{\Lambda^2} + A^2 \textstyle\sum_{k=1}^{\infty} p_k \left(\tfrac{\Lambda}{A}\right)^{4k}$$

$\mathcal{N}=2$: dual description

Taking W_α in the $d^2\theta$ term as an independent field we can impose the superspace Bianchi identity $\mathrm{Im} D^\alpha W_\alpha = 0$ (the analog of $\partial^\mu \tilde{F}_{\mu\nu} = 0$) by using a vector multiplet V_D as a Lagrange multiplier:

$$\begin{aligned}\frac{1}{4\pi}\mathrm{Im}\int d^4x d^4\theta V_D D^\alpha W_\alpha &= \frac{1}{4\pi}\mathrm{Re}\int d^4x d^4\theta i D^\alpha V_D W_\alpha \\ &= -\frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta W_D^\alpha W_\alpha\end{aligned}$$

Performing the path integral over W_α we arrive at a dual $d^2\theta$ term:

$$\frac{1}{16\pi i}\int d^2\theta \left(-\frac{1}{\tau(A)}\right) W_D^\alpha W_{D\alpha} + h.c.$$

$\mathcal{N}=2$: dual description

Defining

$$A_D \equiv h(A) \equiv \frac{\partial P}{\partial A}$$

(with scalar component a_D) we can rewrite the $d^4\theta$ term as

$$\frac{1}{8\pi i}\int d^4\theta\, h_D(A_D)\overline{A}_D + h.c.$$

where h_D is defined implicitly by its inverse:

$$h_D(-A)^{-1} = h(A)\ .$$

Thus, since $\tau(A) = h'(A)$, we have

$$\frac{-1}{\tau(A)} = \frac{-1}{h'(A)} = h'_D(A_D) \equiv \tau_D(A_D)$$

the duality just implements the S transformation
shift symmetry T is a symmetry of this theory
there is a full $SL(2,\mathbb{Z})$ acting on τ

$SL(2,\mathbb{Z})$

$$\tau = \frac{\partial^2 P}{\partial A \partial A} = \frac{\partial a_D}{\partial a}$$

since T^n shifts τ by n we require for consistency that

$$a_D \longrightarrow a_D + n\, a \ , \quad a \longrightarrow a \ .$$

represent the $SL(2,\mathbb{Z})$ generators S and T acting on the scalar fields (a_D, a) as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

BPS Central Charge

monopole and dyon states have masses M given by a central charge $M^2 = 2|Z|^2$
classical analysis gives

$$Z_{\rm cl} = a\, n_e + a\, \tau_{\rm cl}\, n_m$$

where n_e and n_m are the electric and magnetic charges of the soliton adding an $\mathcal{N}=2$ hypermultiplet (two conjugate $\mathcal{N}=1$ chiral multiplets Q and $\overline{Q}$) with $U(1)$ charge n_e to the theory requires a superpotential

$$W_{\rm hyper} = \sqrt{2} n_e A Q \overline{Q}$$

this state we must have $Z = a n_e$, by S-duality a monopole must have a central charge $Z = a_D n_m$, and in general we have

$$Z = a n_e + a_D n_m$$

invariant under any $SL(2,\mathbb{Z})$ transformation $\mathcal{M}$, since $\vec{s} = (a_D, a)^T$ transforms to $\mathcal{M}\vec{s}$ while $\vec{c} = (n_m, n_e)$ transforms to $\vec{c}\mathcal{M}^{-1}$

Stability

dyon with charges (n_m, n_e) that are not relatively prime is only marginally stable
there are lighter dyons whose charges and masses add up to (n_m, n_e) and $\sqrt{2}|an_e + a_D n_m|$

If n_m and n_e are relatively prime then dyon is absolutely stable

Weak Coupling

$$P(A) = \tfrac{i}{2\pi} A^2 \ln \tfrac{A^2}{\Lambda^2} + \ldots$$

for large $|a|$, weak coupling

$$a = \sqrt{2u}\ , \quad a_D = \frac{\partial P}{\partial a} = \frac{2ia}{\pi} \ln\left(\frac{a}{\Lambda}\right) + \frac{2ia}{\pi}$$

traversing a loop in u around ∞ where $\ln u \longrightarrow \ln u + 2\pi i$

$$\begin{array}{rcl} \ln a & \longrightarrow & \ln a + i\pi \\ a & \longrightarrow & -a \\ a_D & \longrightarrow & -a_D + 2a \end{array}$$

monodromy matrix acting on $(a_D, a)^T$ at ∞ is

$$\mathcal{M}_\infty = -T^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

Singular Points

for $\mathrm{Im}\,\tau = 1/g^2$ to be positive we need at least two more singular points with monodromies that do not commute with $\mathcal{M}_\infty$

suppose there is a singular point u_j where a BPS state with electric charge, $(n_m, n_e) = (0, 1)$, becomes massless: $a(u) \approx c_j(u - u_j)$ near u_j near this point the $U(1)$ gauge coupling flows to zero in the IR, β function gives

$$\tau(a(u)) \approx \tfrac{-i}{\pi} \ln \tfrac{a(u)}{\Lambda}$$

monodromy of $(u - u_j) \to e^{2\pi i}(u - u_j)$

$$a_D(u) \to a_D(u) + 2a(u) \;,\;\; a(u) \to a(u)$$

$$\mathcal{M}_{u_j} = T^2$$

Singular Points

consider a dyon with charge (n_m, n_e) massless at $u = u_k$
find an $SL(2,\mathbb{Z})$ transformation D_{u_k} that converts this to charge $(0,1)$

$$\begin{aligned}
\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} &= D_{u_k} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix} \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} \delta n_m - \gamma n_e \\ -\beta n_m + \alpha n_e \end{pmatrix}
\end{aligned}$$

monodromy in the original variables is

$$\begin{aligned}
\mathcal{M}_{u_k} = D_{u_k}^{-1}\, T^2\, D_{u_k} &= \begin{pmatrix} 1 + 2\gamma\delta & 2\delta^2 \\ -2\gamma^2 & 1 - 2\gamma\delta \end{pmatrix} \\
&= \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{pmatrix}
\end{aligned}$$

Two Singular Points

simplest possibility: two singular points at finite u related by the Z_2 symmetry $u \to -u$
consider two singular points u_1 and u_{-1} where BPS states with charges (m, n) and(p, q), respectively become massless, then we must have

$$\mathcal{M}_{u_1}\mathcal{M}_{u_{-1}} = \mathcal{M}_\infty$$

Assuming massless monopole with charge $(1, 0)$ at u_1, we have

$$\mathcal{M}_{u_1} = \left(\begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array}\right) , \ \mathcal{M}_{u_{-1}} = \left(\begin{array}{cc} -1 & 2 \\ -2 & 3 \end{array}\right)$$

$\Rightarrow$ massless BPS state at u_{-1} is a dyon with charge $(-1, 1)$ or $(1, -1)$ related by the $SL(2,\mathbb{Z})$ transformation $-I$
since $\mathcal{M}_\infty$ changes the electric charge by 2, we can obtain all the classical dyons with charge $(\pm 1, 2n + 1)$ from phase redefinitions of u.
$\mathcal{M}_{u_1} = S^{-1}T^2S$, and $\mathcal{M}_{u_{-1}} = (ST^{-1})^{-1}T^2ST^{-1}$
same as the monodromies we saw for $SO(N)$ with $N - 2$ flavors

Consistency Check

consider the point u_1, where a_D vanishes.
low-energy effective theory has monopoles and dual photon
If we add a mass term $m\mathrm{Tr}\phi^2$, the effective $\mathcal{N}=1$ superpotential for the dual adjoint and monopoles:

$$W_{\text{eff}} = \sqrt{2} A_D M \overline{M} + m f(A_D)$$

eqm give

$$\sqrt{2} M \overline{M} + m f'(A_D) = 0 \ , \ \ a_D M = 0 \ , \ \ a_D \overline{M} = 0$$

For $m = 0 \Rightarrow \mathcal{N} = 2$ moduli space: $M = 0$, $\overline{M} = 0$, a_D arbitrary
For $m \neq 0$, $a_D = 0$, $M^2 = \overline{M}^2 = -mf'(0)/\sqrt{2}$. Since M is charged, gives a mass to the dual photon and hence electric charge confinement through the dual Meissner effect

agrees with gaugino condensation and confinement with a mass gap

The Seiberg–Witten curve

gives the complete solution for τ and BPS masses:

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u)$$

singularities at $u = \pm\Lambda^2$, which are related by a Z_2 symmetry as required
Near these points Δ is quadratic in $u \pm \Lambda^2$ so the $\mathcal{M}_{\pm\Lambda^2} \sim T^2$
singularity at ∞ is subtle since the roots are approximately given by $(0, \Lambda^4/(4u), u)$, so two sets of points are approaching; rescale by

$$x \to x'(\Lambda^2 - u), \;\; y \to y'(\Lambda^2 - u)^{3/2}$$

roots at large u given by $(\pm\Lambda/u, -1)$, one pair of branch points converge
For large u, $\Delta \sim u^{-2}$ so the monodromy is $\sim T^{-2}$
back in $x - y$ plane change of variables gives a factor of $\sqrt{u}$ to dx/y, odd under $u \to e^{2\pi i}u$, so $\mathcal{M}_\infty = -T^{-2}$
curve has the appropriate singularities and associated monodromies

Holomorphic Coupling

$$\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a/\partial u} = \frac{\omega_2}{\omega_1}$$

identify the derivatives of a and a_D with the periods of the torus

$$\frac{\partial a_D}{\partial u} = f(u)\,\omega_2 = f(u)\int_b \frac{dx}{y}\ , \quad \frac{\partial a}{\partial u} = f(u)\,\omega_1 = f(u)\int_a \frac{dx}{y}$$

$f(u)$ is chosen so as to reproduce the correct weak coupling limit
Defining

$$\frac{d\lambda}{du} \equiv f(u)\frac{dx}{y}$$

we have

$$a_D = \int_b \lambda\ , \quad a = \int_a \lambda\ (*)$$

adding arbitrary constants in (*) would destroy $SL(2,\mathbb{Z})$ transformation properties of a and a_D

Periods

Using

$$\int_0^1 dx\,(1-zx)^{-\alpha}x^{\beta-1}(1-x)^{\gamma-\beta-1} = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}F\,(\alpha,\beta,\gamma;z)$$

where $F(\alpha,\beta,\gamma,z)$ is the hypergeometric function

$$\begin{aligned}\omega_1 &= 2\int_{-\Lambda^2}^{\Lambda^2}\frac{dx}{\sqrt{y}} = \frac{2\pi}{\Lambda\sqrt{1+\frac{u}{\Lambda^2}}}F\left(\frac{1}{2},\frac{1}{2},1;\frac{2}{1+\frac{u}{\Lambda^2}}\right)\\ \omega_2 &= 2\int_{u}^{\Lambda^2}\frac{dx}{\sqrt{y}} = \frac{-\pi i}{\sqrt{2}\Lambda}F\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{2}(1-\frac{u}{\Lambda^2})\right)\end{aligned}$$

for large $|u|$ periods are approximated by

$$\omega_1 = \frac{2\pi}{\sqrt{u}} \quad \omega_2 = \frac{i}{\sqrt{u}}\ln\left(\frac{u}{\Lambda^2}\right)$$

from weak coupling result we must choose

$$f(u) = \frac{\sqrt{2}}{2\pi}$$

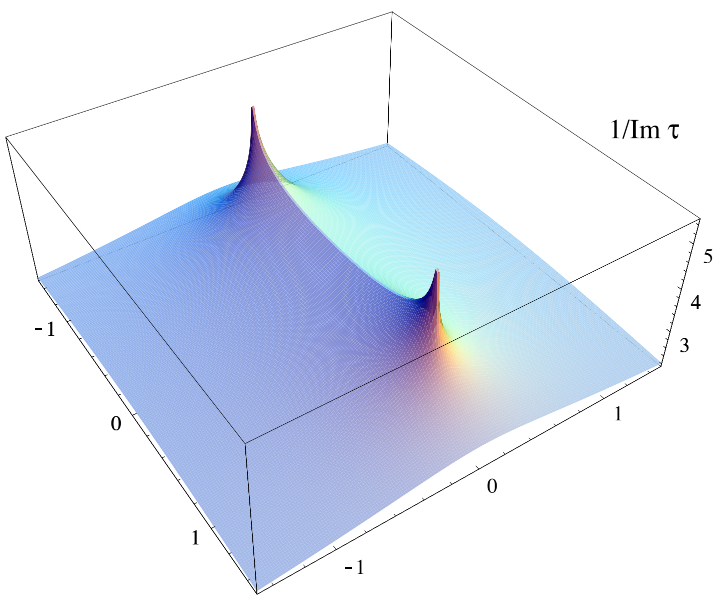
Holomorphic Coupling

$$
\begin{aligned}
a(u) &= -\frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}\frac{dx\,\sqrt{x-u}}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)}} \\
&= -\sqrt{2(\Lambda^2+u)}\,F\left(-\tfrac{1}{2},\tfrac{1}{2},1;\tfrac{2}{1+\frac{u}{\Lambda^2}}\right) \\
a_D(u) &= -\frac{\sqrt{2}}{\pi}\int_{u}^{\Lambda^2}\frac{dx\,\sqrt{x-u}}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)}} \\
&= -i\tfrac{1}{2}\left(\tfrac{u}{\Lambda}-\Lambda\right)F\left(\tfrac{1}{2},\tfrac{1}{2},2;\tfrac{1}{2}\left(1-\tfrac{u}{\Lambda^2}\right)\right)
\end{aligned}
$$

a_D vanishes at $u=\Lambda^2$ as expected for a vanishing monopole mass, and at $u=-\Lambda^2$, $a=a_D$

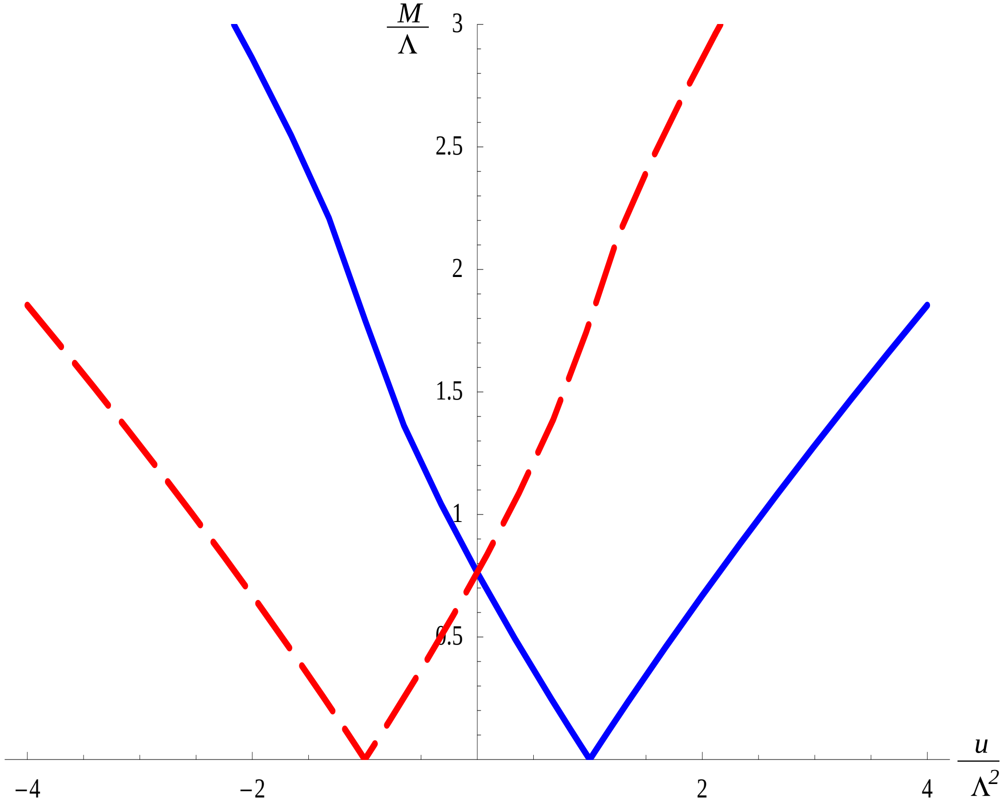
different choice of cycles yields an $SL(2,\mathbb{Z})$ transformed a and a_D

Holomorphic Coupling



gauge coupling g^2 over the complex u/Λ^2 plane

BPS Masses



The mass (in units of Λ) of the monopole (solid line) and dyon (dashed line) as a function of real u/Λ^2

Donaldson theory

Poincarè knew compact 2-manifolds classified by number of handles
conjectured that the same situation holds in 3D
generalized to n-manifolds and proven for $n \neq 3$
for $n = 3$ Thurston conjectured a classification of all 3-manifolds
Perelman seems to have proven Thurston's conjecture (using RG analog)
$\Rightarrow$ Poincaré conjecture

no proposed classification of 4-manifolds
study topological invariants: different invariants $\Rightarrow$ different manifolds
Donaldson constructed invariants by studying instantons
Seiberg–Witten theory allows for simpler invariants
monopoles, unlike instantons, cannot shrink to arbitrarily small size

Adding flavors to Seiberg–Witten

hypermultiplets in the spinor representation, $SU(2)$ gauge theory in $\mathcal{N} = 1$ language, a superpotential is required:

$$W = \sqrt{2}\widetilde{Q}^i A Q_i$$

$\square$ of $SU(2)$ is pseudo-real $\Rightarrow$ "parity" symmetry interchanges $\widetilde{Q}^i$ and Q_i
superpotential $\Rightarrow$ squark $U(1)_R$ charge to be zero
$U(1)_R$ symmetry is anomalous, assign scale Λ_1 a spurious R-charge of 2
u has R-charge 4, weak coupling ($\Lambda_1 \to 0$) limit, where $y^2 = x^2(x - u)$,
$\Rightarrow$ x has R-charge 4 and y has R-charge 6
one flavor with mass m, assign m a spurious R-charge of 2
n-instanton corrections $\propto \Lambda_1^{bn} = \Lambda_1^{3n}$; only n even respects "parity"
m is odd under "parity", n odd comes with an odd power of m
most general form of the elliptic curve is

$$y^2 = x^3 - ux^2 + t\Lambda_1^6 + m\Lambda_1^3(ax + bu) + cm^3\Lambda_1^3$$

a, b, c, and t must be determined

Adding flavors to Seiberg–Witten

theory with doublets now has particles with half-integral electric charge, rescales n_e by 2 and a by $\frac{1}{2} \Rightarrow \tau$ by 2.
corresponding elliptic curve

$$y^2 = x^3 - ux^2 + \tfrac{1}{4}\Lambda^4 x$$

decouple the single flavor by taking m large
matching condition $\Lambda^4 = m\Lambda_1^3$
taking $m \to \infty$ with Λ held fixed the curve must reduce to no flavor case
$\Rightarrow a = \frac{1}{4}$, $b = c = 0$
in this limit see one singularity moves to ∞, singularity at $u \approx -m^2/(64t)$
since there is a singularity when the flavor becomes massless at $u = m^2$
$\Rightarrow t = -1/64$
correct curve is

$$y^2 = x^3 - ux^2 + \tfrac{m}{4}\Lambda_1^3 x - \tfrac{1}{64}\Lambda_1^6$$

Massless flavor

$$y^2 = x^3 - ux^2 + \frac{m}{4}\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6$$

$m \to 0$ two roots coincide when

$$u = e^{w\pi in/3}\,\frac{3\,\Lambda^{4/3}}{4\,2^{2/3}}\ ,$$

so there is a Z_3 symmetry on the moduli space
monodromies at these points are conjugate to T

Adding flavors to Seiberg–Witten

curves for arbitrary F obtained similarly

for F flavors the monodromy at ∞ is determined by the β function

$$\mathcal{M}_\infty = -T^{F-4}$$

central charge Z with $F \neq 0$ is more complicated

depends on the masses of the the flavors as well as on global $U(1)$ charges

Massless Flavors

F	monodromies	BPS charges (n_m, n_e)
0	STS^{-1}, $D_2TD_2^{-1}$	$(1,0)$, $(1,2)$
1	STS^{-1}, $D_1TD_1^{-1}$, $D_2TD_2^{-1}$	$(1,0)$, $(1,1)$, $(1,2)$
2	ST^2S^{-1}, $D_1T^2D_1^{-1}$	$(1,0)$, $(1,1)$
3	ST^4S^{-1}, $(ST^2S)T(ST^2S)^{-1}$	$(1,0)$, $(2,1)$

where $D_n = T^nS$
monodromy $D_nT^kD_n^{-1} \leftrightarrow k$ massless dyons with charge $(1,n)$

product of the monodromies satisfies

$$\mathcal{M}_{u_1}\mathcal{M}_{u_{-1}} = \mathcal{M}_\infty$$