Exercises for Chapter 1

1. Construct the massive supermultiplet of $\mathcal{N}=3$ SUSY for the lowest weight state (Clifford "vacuum" $|\Omega_0\rangle$) having spin 0. (Use the notation where the raising operator on this state produces a state $(\square, 2)$ where the $\square$ indicates a 3 of the $SU(3)$ R-symmetry and 2 denotes a spin half doublet. You can use the following $SU(3)$ group theory results (keep in mind that $\overline{\square} = \begin{smallmatrix}\square\\\square\end{smallmatrix} \leftrightarrow \overline{3}$)

$$\begin{aligned}
\square \times \square &= \begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square \quad \leftrightarrow \quad 3 \times 3 = \overline{3}_A + 6_S \ , && (1)\\
\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \square &= 1 + \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \quad \leftrightarrow \quad \overline{3} \times 3 = 1 + 8 \ . && (2)
\end{aligned}$$

Check that there are an equal number of bosonic and fermionic states in the supermultiplet. Is this state equivalent to a massive supermultiplet of $\mathcal{N}=4$?

2. Consider $\mathcal{N}=4$ SUSY with a 4×4 central charge matrix $\mathbf{Z}$. In a skew diagonal basis we can write

$$Z = \begin{pmatrix} Z_1\epsilon^{ab} & 0 \\ 0 & Z_2\epsilon^{ab} \end{pmatrix} \tag{3}$$

where $a=1,2$ and $b=1,2$. In this basis the SUSY algebra can be written as

$$\begin{aligned}
\{Q^{aL}_\alpha, Q^\dagger_{\dot\alpha bN}\} &= 2\sigma^\mu_{\alpha\dot\alpha}P_\mu \delta^a_b \delta^L_N \ , && (4)\\
\{Q^{aL}_\alpha, Q^{bN}_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab}\delta^{LN} Z_N \ , && (5)\\
\{Q^\dagger_{\dot\alpha aL}, Q^\dagger_{\dot\beta bN}\} &= 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\epsilon_{ab}\delta_{LN} Z_N \ , && (6)
\end{aligned}$$

where $L=1,2$; $N=1,2$; and the repeated index N is not summed over. Defining

$$\begin{aligned}
A^L_\alpha &= \frac{1}{2}\left[Q^{1L}_\alpha + \epsilon_{\alpha\beta}\left(Q^{2L}_\beta\right)^\dagger\right] \ , && (7)\\
B^L_\alpha &= \frac{1}{2}\left[Q^{1L}_\alpha - \epsilon_{\alpha\beta}\left(Q^{2L}_\beta\right)^\dagger\right] \ , && (8)
\end{aligned}$$

reduces the algebra in the rest frame to

$$\begin{aligned}
\{A^L_\alpha, A^\dagger_{\beta N}\} &= \delta_{\alpha\beta}\delta^L_N (M + \sqrt{2}Z_L) \ , && (9)\\
\{B^L_\alpha, B^\dagger_{\beta N}\} &= \delta_{\alpha\beta}\delta^L_N (M - \sqrt{2}Z_L) \ , && (10)
\end{aligned}$$

Consider a massive state with $M = \sqrt{2}Z_1 = \sqrt{2}Z_2$ and construct the short multiplet starting with the spin 0 Clifford "vacuum" $|\Omega_0\rangle$. How many raising operators are there? Label the elements of the multiplet with dimension of the R-symmetry representation, d_R, and the spin degneracy $2j+1$.