Gauging nonlocal Lagrangians

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We describe a method for introducing gauge fields into nonlocal Lagrangians, and for deriving the resulting Feynman rules. The method is applied in detail to the nonlocal chiral quark model. In particular we describe how to calculate coupling constants of the effective chiral Lagrangian that results when the quarks are integrated out of the theory.

I. INTRODUCTION

Recently the nonlocal chiral quark model [1,2] (NCQM) has been used to derive an effective chiral Lagrangian for the pseudoscalar mesons (the π's, K's, and η), and the results are in surprisingly good agreement with experiment. Since such calculations require the evaluation of Feynman graphs with external gauge fields, it is necessary to be able to introduce gauge fields into the nonlocal Lagrangian so as to produce a gauge-invariant nonlocal Lagrangian. A method for doing this is well known [3], but a method for determining the Feynman rules for the nonlocal gauge interactions seems not to have been discussed in the literature.

In this paper we will describe a method for gauging nonlocal Lagrangians, and for deriving the resulting Feynman rules. In Sec. II we discuss how the method works for the simple case of a local Lagrangian. In Secs. III and IV we will apply the method to the NCQM and derive Feynman rules for diagrams involving one gauge field line. In Sec. V we discuss Feynman vertices involving two gauge field couplings. Section VI covers the derivation of the effective chiral Lagrangian for the model. In the Appendix we present some mathematical results that are required in the previous sections.

II. PATH EXPONENTIALS

As a warm-up exercise we will gauge a local theory of free quarks in a slightly unusual way. In Euclidean space, the Lagrangian for quarks interacting with photons can be written down directly (assuming minimal coupling):

$$\mathcal{L}=\bar{\psi}(x)\gamma^{\mu}[\partial_{\mu}-ieQV_{\mu}(x)]\psi(x)+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} , \tag{1}$$

where Q is the charge matrix of the quarks. This Lagrangian is invariant under the infinitesimal transformations

$$\begin{aligned}\psi(x)&=[1+i\alpha(x)Q]\psi'(x) ,\\ \bar{\psi}(x)&=\bar{\psi}'(x)[1-i\alpha(x)Q] ,\\ V_{\mu}(x)&=V'_{\mu}(x)+\frac{1}{e}\partial_{\mu}\alpha(x) .\end{aligned} \tag{2}$$

We will now try to obtain this result with a method that can be generalized to the nonlocal case. We will use the path exponential introduced by Bloch [4], and the path-ordered exponential introduced by Wilson [3]. The path ordering is not essential for electromagnetic gauge fields, since U(1) is an Abelian group, but it will be necessary for including external weak gauge fields.

Consider the free quark action

$$\begin{aligned}S&=\int d^4x\, d^4y\, \bar{\psi}(x)\delta(x-y)\partial\!\!\!/\psi(y)\\ &=\int d^4x\, d^4y\,\bar{\psi}(x)[-\partial\!\!\!/_y\delta(x-y)]\psi(y) .\end{aligned} \tag{3}$$

This action can be gauged by the introduction of an exponential of a line integral of the gauge field. The useful feature of the path exponential for nonlocal interactions is its transformation property. Under the gauge transformation defined in Eq. (2), the path exponential transforms as

$$\begin{aligned}\exp\left[-ieQ\int_x^y V_\nu dw^\nu\right]&=[1+i\alpha(x)Q]\\ &\quad\times\exp\left[-ieQ\int_x^y V'_\nu dw^\nu\right]\\ &\quad\times[1-i\alpha(y)Q] .\end{aligned} \tag{4}$$

(This simple transformation property is maintained in the non-Abelian case by path ordering.) Thus, ignoring the $F_{\mu\nu}F^{\mu\nu}$ term, we have the gauge-invariant action

$$\begin{aligned}S_g&=\int d^4x\, d^4y\, \bar{\psi}(x)[-\partial\!\!\!/_y\delta(x-y)]\\ &\quad\times\exp(-ieQ\int_x^y V_\nu dw^\nu)\psi(y)\\ &=\int d^4x\, d^4y\, \bar{\psi}(x)\delta(x-y)\partial\!\!\!/_y\\ &\quad\times\exp\left[-ieQ\int_x^y V_\nu\, dw^\nu\right]\psi(y) .\end{aligned} \tag{5}$$

This looks very nice but the derivative of the line integral is ambiguous, and must be carefully defined. Following Mandelstam [5] we introduce the notation

$$I(x,y,p)\equiv\int_x^y V_\nu dw^\nu , \tag{6}$$

where p explicitly denotes the dependence on the path

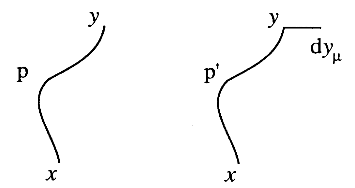

FIG. 1. Extension of the path used in calculating the derivative of the line integral.

taken from x to y. Then the derivative may be defined by [5]

$$\lim_{dy_\mu \to 0} dy_\mu \frac{\partial}{\partial y^\mu} I(x,y,p) = \lim_{dy_\mu \to 0} I(x,y+dy_\mu,p') - I(x,y,p) \ , \quad (7)$$

where p' is the path obtained from p by adding the extension dy_μ to the y end, as shown in Fig. 1.

Using the definition in Eq. (7), we have

$$\frac{\partial}{\partial y^\mu} I(x,y,p) = V_\mu(y) \ , \quad (8)$$

the important point being that the derivative of the line integral does not depend on the path used in defining it. Expanding the exponential in Eq. (5), it is easy to see that if $\delta(x-y)I(x,y,p)=0$ (i.e., if, as $x \to y$, the path p shrinks to a point, and not to a closed loop), then

$$S_g = \int d^4x\, d^4y\, \bar{\psi}(x)\delta(x-y)\gamma^\mu(\partial_\mu - ieQV_\mu)\psi(y) \ . \quad (9)$$

Thus, we recover minimal coupling from the introduction of the path exponential if we use the correct prescriptions for derivatives and for zero-length paths. We will use path-ordered exponentials, with these same prescriptions, to introduce external electroweak gauge fields to the NCQM.

III. THE GAUGED NONLOCAL MASS

For the purposes of this paper, the relevant aspects of the NCQM may be summarized by the following nonlocal Lagrangian [1,2] for N flavors of quarks and N^2-1 Goldstone boson (GB) fields:

$$\mathcal{L}_{\mathrm{NCQM}}(x,y) = \bar{\psi}(x)\delta(x-y)\not{\partial}\psi(y) + \bar{\psi}(x)\Sigma_\pi(x,y)\psi(y) \ , \quad (10)$$

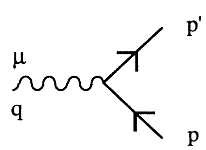

FIG. 2. The photon-quark-quark vertex: $ie\Gamma^\mu(p,q,p')$. Straight lines represent quarks; the wavy line designates the photon.

where

$$\Sigma_\pi(x,y) = \Sigma(x-y)\left[1 - \frac{i}{F_0}\gamma_5[\pi(x)+\pi(y)] - \frac{1}{2F_0^2}[\pi^2(x)+\pi^2(y)+\pi(x)\pi(y) + \pi(y)\pi(x)] + O(\pi^3)\right] \ . \quad (11)$$

The GB fields are contained in the matrix

$$\pi(z) \equiv \lambda^a \pi^a(z) \ . \quad (12)$$

The repeated index in Eq. (12) is implicitly summed over, and $\{\lambda^a\}$ are the generators of SU(N), with $\mathrm{Tr}\lambda^a\lambda^b = \frac{1}{2}\delta^{ab}$. The constant F_0 is determined by properly normalizing the π field [2]. This model Lagrangian contains a nonlocal dynamical quark mass $\Sigma(x-y)$, which is expected to be present in an asymptotically free gauge theory. The dynamical mass is an order parameter for spontaneous chiral-symmetry breaking in gauge theories, and in the model the associated Goldstone bosons appear as fluctuations of this nonlocal order parameter. For more details about the model, the reader is referred to Refs. [1] and [2].

First we will concentrate on coupling photons to quarks with a nonlocal dynamical mass. The action without photons is

$$S = \int d^4x\, \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + \int d^4x\, d^4y\, \bar{\psi}(x)\Sigma(x-y)\psi(y) \ . \quad (13)$$

With the path-ordered exponential prescription (and ignoring the $F_{\mu\nu}F^{\mu\nu}$ term again) the gauged action is

$$S_g = \int d^4x\, \bar{\psi}(x)\gamma^\mu(\partial_\mu - ieQV_\mu)\psi(x) + \int d^4x\, d^4y\, \bar{\psi}(x)\Sigma(x-y) \times \exp\left[-ieQ\int_x^y V_\nu\, dw^\nu\right]\psi(y) \ . \quad (14)$$

We will refer to the local and nonlocal terms as S_{L} and S_{NL} respectively. Since Feynman rules for nonlocal gauge theories are not well known, we will spend some time in deriving the amplitude for a photon to couple to two quarks as shown in Fig. 2.

The local contribution to the amplitude is given by

$$ie\Gamma_L^\mu(y,x,z) = -\left.\frac{\delta^3 S_L}{\delta V_\mu(x)\delta\psi(y)\delta\bar{\psi}(z)}\right|_{V_\mu=0} \ . \quad (15)$$

Fourier transforming with $e^{i(p'z-py-qx)}$ and dropping the usual $(2\pi)^4\delta(p'-p-q)$, we find

$$ie\Gamma_L^\mu(p,q,p+q) = ieQ\gamma_\mu \ . \quad (16)$$

The nonlocal part is harder. First, it will be advantageous to perform a derivative expansion of S_{NL}. Let

$$F(x,y)=\exp(-ieQ\int_x^y V_\nu dw^\nu)\psi(y) ;$$

then by Fourier transforming we have

$$S_{\rm NL}=\int \frac{d^4k\,d^4p}{(2\pi)^8}\bar{\psi}(k)\Sigma(p)F(k-p,p) . \qquad (17)$$

Taylor expanding $\Sigma(p)$ gives

$$S_{\rm NL}=\int \frac{d^4k\,d^4p}{(2\pi)^8}\bar{\psi}(k) \times\left[\sum_{n=0}^{\infty}\frac{1}{n!}\Sigma^{(n)}(0)(p^2)^n\right]F(k-p,p) . \qquad (18)$$

Fourier transforming back to position space gives

$$S_{\rm NL}=\int d^4x\,d^4y\,d^4z\frac{d^4kd^4p}{(2\pi)^8}e^{i[z(k-p)-xk]}\bar{\psi}(x) \times\left[\sum_{n=0}^{\infty}\frac{1}{n!}\Sigma^{(n)}(0)(-\partial_y^2)^n e^{ipy}\right]F(z,y) , \qquad (19)$$

where the derivative acts only inside the large parentheses. Finally, integrating by parts we obtain

$$S_{\rm NL}=\int d^4x\,d^4y\,\delta(x-y)\bar{\psi}(x) \times\left[\sum_{n=0}^{\infty}\frac{1}{n!}\Sigma^{(n)}(0)(-\partial_y^2)^n\right] \times\exp\left[-ieQ\int_x^y V_\nu dw^\nu\right]\psi(y) . \qquad (20)$$

So the nonlocal contribution to the amplitude is

$$\begin{aligned} ie\Gamma^\mu_{\rm NL}(y',x,z)&=-\left.\frac{\delta^3 S_{\rm NL}}{\delta V_\mu(x)\delta\psi(y')\delta\bar{\psi}(z)}\right|_{V_\mu=0}\\ &=ieQ\int d^4y\,\delta(z-y)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\Sigma^{(n)}(0)(-\partial_y^2)^n\right]\int_z^y\delta(x-w)dw^\mu\delta(y'-y)\\ &\equiv ie\sum_{n=0}^{\infty}\Gamma^\mu_{NL,n}(y',x,z) . \qquad (21)\end{aligned}$$

By Fourier transforming with $e^{i(p'z-py'-qx)}$ and performing some tedious differentiation we find

$$ie\Gamma^\mu_{NL,n}(p,q,p+q)=-eQ\frac{1}{n!}\Sigma^{(n)}(0)(2p+q)^\mu f_n , \qquad (22)$$

where

$$\begin{aligned} &f_0=0,\quad f_1=1 ,\\ &f_2=(q+p)^2+p^2 , \qquad (23)\\ &f_3=(q+p)^4+(q+p)^2p^2+p^4 ,\end{aligned}$$

and, in general, for $n>0$,

$$f_n=f_{n-1}(q+p)^2+p^{2(n-1)} . \qquad (24)$$

For this to be useful we must be able to sum the entire series given in Eq. (21). To do this we will make use of some proofs by induction which are relegated to the appendix. In the Appendix it is shown that

$$(q+p)^{2n}-p^{2n}=(q^2+2p\cdot q)f_n . \qquad (25)$$

Inserting this result in Eq. (22) we find

$$\begin{aligned} ie\Gamma^\mu_{NL}(p,q,p+q) &=-eQ(2p+q)^\mu\sum_{n=0}^{\infty}\frac{1}{n!}\Sigma^{(n)}(0)\frac{(q+p)^{2n}-p^{2n}}{2p\cdot q+q^2}\\ &=-eQ\frac{(2p+q)^\mu}{2p\cdot q+q^2}[\Sigma(p+q)-\Sigma(p)] . \qquad (26)\end{aligned}$$

We may check that the full vertex $\Gamma^\mu=\Gamma^\mu_L+\Gamma^\mu_{NL}$ satisfies the Ward-Takahashi (WT) identity

$$-iq_\mu\Gamma^\mu(p,q,p')=S^{-1}(p+q)Q-QS^{-1}(p) . \qquad (27)$$

Using the inverse quark propagator in the NCQM, $S^{-1}(p)=i\not{p}+\Sigma(p)$, both sides reduce to $-iQ\not{q}+Q[\Sigma(p+q)-\Sigma(p)]$.

IV. PHOTONS AND GOLDSTONE BOSONS

Emboldened by this last result, we may consider calculating the amplitudes which involve photons and GB's. The gauge transformation of the GB field is given by

$$\pi^a\lambda^a=\exp(i\alpha Q)\pi'^a\lambda^a\exp(-i\alpha Q) . \qquad (28)$$

Thus, we can easily write down the term in the gauged Lagrangian involving one GB:

$$\mathcal{L}_{g\pi}(x,y)=\frac{-i}{F_0}\bar{\psi}(x)\gamma_5\Sigma(x-y) \times\left[\pi(x)\exp\left[-ieQ\int_x^y V_\nu dw^\nu\right] +\exp\left[-ieQ\int_x^y V_\nu dw^\nu\right]\pi(y)\right]\psi(y) . \qquad (29)$$

Following a very similar analysis as that given above, we find that the vertex which couples one photon, one GB, and two quarks, depicted in Fig. 3, is given by

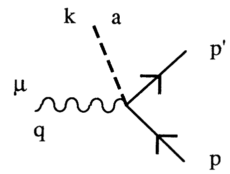

FIG. 3. The photon-GB-quark-quark vertex: $ie\Gamma^{\mu,a}(p,q,k,p+q+k)$. The dashed line represents the GB.

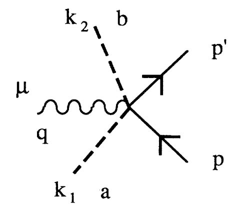

FIG. 4. The two-GB–photon–quark–quark vertex: $ie\Gamma^{\mu,a,b}(p,q,k_1,k_2,p+q+k_1+k_2)$.

$$ie\Gamma^{\mu,a}(p,q,k,p+q+k) = \frac{ie\gamma_5}{F_0}\left[\lambda^a Q\frac{(2p+q)^\mu}{2p\cdot q+q^2}[\Sigma(p+q)-\Sigma(p)] + Q\lambda^a\frac{[2(p+k)+q]^\mu}{2(p+k)\cdot q+q^2} \times[\Sigma(p+k+q)-\Sigma(p+k)]\right]. \tag{30}$$

The equations get cumbersome for two GB's, but introducing the notation

$$E\equiv\exp\left[-ieQ\int_x^y V_\nu dw^\nu\right],\quad E^\dagger\equiv\exp\left[-ieQ\int_y^x V_\nu dw^\nu\right], \tag{31}$$

we find that the term in the gauged Lagrangian with two GB fields is

$$\mathcal{L}_{g\pi^2}(x,y)=\frac{-1}{2F_0^2}\bar\psi(x)\Sigma(x-y)\times[\pi^2(x)E+E\pi^2(y)+\pi(x)E\pi(y)+E\pi(y)E^\dagger\pi(x)E]\psi(y). \tag{32}$$

The vertex with one photon, two GB's, and two quarks, shown in Fig. 4, is

$$\begin{aligned}ie\Gamma^{\mu,a,b}&(p,q,k_1,k_2,p+q+k_1+k_2)\\ =\frac{e}{2F_0^2}\Bigg[&(\lambda^a\lambda^b+\lambda^b\lambda^a)Q\frac{(2p+q)^\mu}{2p\cdot q+q^2}(\Sigma(p+q)-\Sigma(p))\\ &+Q(\lambda^a\lambda^b+\lambda^b\lambda^a)\frac{[2(p+k_1+k_2)+q]^\mu}{2(p+k_1+k_2)\cdot q+q^2}(\Sigma(p+k_1+k_2+q)-\Sigma(p+k_1+k_2))\\ &+(Q\lambda^a\lambda^b+\lambda^a\lambda^bQ+\lambda^bQ\lambda^a-\lambda^aQ\lambda^b)\frac{[2(p+k_1)+q]^\mu}{2(p+k_1)\cdot q+q^2}(\Sigma(p+k_1+q)-\Sigma(p+k_1))\\ &+(Q\lambda^b\lambda^a+\lambda^b\lambda^aQ+\lambda^aQ\lambda^b-\lambda^bQ\lambda^a)\frac{[2(p+k_2)+q]^\mu}{2(p+k_2)\cdot q+q^2}(\Sigma(p+k_2+q)-\Sigma(p+k_2))\Bigg].\end{aligned} \tag{33}$$

V. NON-ABELIAN GAUGE FIELDS

In this section we discuss the inclusion of external non-Abelian vector gauge fields. The considerations of Secs. II–IV still apply, with the only change being the substitution of $P\exp(-igT^a\int_x^y V_\nu^a dw^\nu)$ for $\exp(-ieQ\int_x^y V_\nu dw^\nu)$, where V_ν^a is the gauge field, g is the gauge coupling constant, T^a are the generators of the gauge group, and P denotes path ordering of the exponential, starting with functions of x placed on the left, and working towards functions of y on the right. In Secs. II–IV we only considered interactions that involved one gauge boson, so path ordering the exponential makes no difference in any of the derivations given. However, path ordering does make a difference when two gauge bosons are involved, and we will briefly discuss these differences in this section.

We begin with the two-gauge-boson–two-quark vertex (Fig. 5):

$$
\begin{aligned}
g^2\Gamma^{\mu a,\nu b}(y',x,v,z) &= -\left.\frac{\delta^4 S_{\mathrm{NL}}}{\delta V^a_\mu(x)\,\delta V^b_\nu(v)\,\delta\psi(y')\,\delta\bar{\psi}(z)}\right|_{V^c_\alpha=0} \\
&= g^2\int d^4y\,\delta(z-y)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\Sigma^{(n)}(0)(-\partial_y^2)^n\right] \\
&\qquad \times P\left[\int_z^y T^a\delta(x-w_1)\,dw_1^\mu \int_z^y T^b\delta(v-w_2)\,dw_2^\nu\right]\delta(y'-y) \\
&\equiv g^2\sum_{n=0}^{\infty}\Gamma_n^{\mu a,\nu b}(y',x,v,z)\ .
\end{aligned}
\tag{34}
$$

The path ordering operator in Eq. (34) is taken to order T^a and T^b, according to the relative positions of their associated integration variables (w_1 and w_2) along the path from z to y. The important point is that a derivative with respect to y (the limit of integration) yields the integrand evaluated at y, so the path ordering places the associated generator on the right. Thus, the ordering of T^a and T^b is determined by which integral is differentiated first, and repeated applications of the product rule will generate both orderings.

By Fourier transforming Eq. (34) with $e^{i(p'z-py'-q_1x-q_2v)}$ and performing the derivatives, we find

$$
\begin{aligned}
\Gamma_n^{\mu a,\nu b}(p,q_1,q_2,p+q_1+q_2) = -\frac{1}{n!}\Sigma^{(n)}(0)\{&(T^aT^b+T^bT^a)g^{\mu\nu}d_n(p,q_1,q_2)+T^aT^b(2p+q_2)^\nu[2(p+q_2)+q_1]^\mu j_n(p,q_2,q_1) \\
&+T^bT^a(2p+q_1)^\mu[2(p+q_1)+q_2]^\nu j_n(p,q_1,q_2)\}\ ,
\end{aligned}
\tag{35}
$$

where

$$d_n \equiv d_n(p,q_1,q_2),\quad j_n \equiv j_n(p,q,q_1)\ , \tag{36}$$

and

$$
\begin{aligned}
&d_0=0,\quad d_1=1,\quad d_2=(q_1+q_2+p)^2+p^2,\quad d_3=(q_1+q_2+p)^4+(q_1+q_2+p)^2p^2+p^4\ ,\\
&j_0=0,\quad j_1=0,\quad j_2=1,\quad j_3=(q_1+q_2+p)^2+(q_2+p)^2+p^2\ .
\end{aligned}
\tag{37}
$$

In general we find, for $n>0$,

$$d_n=d_{n-1}(q_1+q_2+p)^2+p^{2(n-1)},\quad j_n=j_{n-1}(q_1+q_2+p)^2+h_{n-1}\ , \tag{38}$$

where

$$h_n=h_{n-1}(q_2+p)^2+p^{2(n-1)}\ , \tag{39}$$

and $h_0=0$, $h_1=1$. Using the results given in the Appendix we find

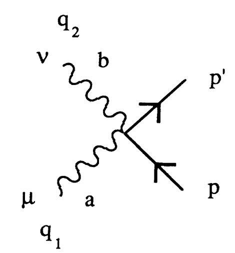

FIG. 5. The vertex coupling two gauge bosons and two quarks: $g^2\Gamma^{\mu a,\nu b}(p,q_1,q_2,p+q_1+q_2)$.

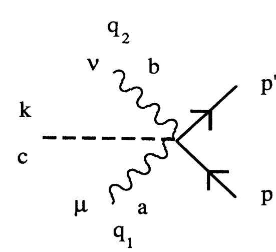

FIG. 6. The vertex coupling two-gauge bosons, two quarks, and one GB: $g^2\Gamma^{\mu a,\nu b,c}(p,q_1,q_2,k,p+k+q_1+q_2)$.

$$
\begin{aligned}
g^2\Gamma^{\mu a,\nu b}&(p,q_1,q_2,p+q_1+q_2)\\
&=-g^2\Bigg[\frac{(T^aT^b+T^bT^a)g^{\mu\nu}}{2p\cdot(q_1+q_2)+(q_1+q_2)^2}[\Sigma(p+q_1+q_2)-\Sigma(p)]\\
&\quad+T^aT^b\frac{(2p+q_2)^\nu[2(p+q_2)+q_1]^\mu}{2(p+q_2)\cdot q_1+q_1^2}\left[\frac{\Sigma(p+q_1+q_2)-\Sigma(p)}{2p\cdot(q_1+q_2)+(q_1+q_2)^2}-\frac{\Sigma(p+q_2)-\Sigma(p)}{2p\cdot q_2+q_2^2}\right]\\
&\quad+T^bT^a\frac{(2p+q_1)^\mu[2(p+q_1)+q_2]^\nu}{2(p+q_1)\cdot q_2+q_2^2}\left[\frac{\Sigma(p+q_1+q_2)-\Sigma(p)}{2p\cdot(q_1+q_2)+(q_1+q_2)^2}-\frac{\Sigma(p+q_1)-\Sigma(p)}{2p\cdot q_1+q_1^2}\right]\Bigg]\;. \qquad (40)
\end{aligned}
$$

As in the case of one gauge boson, the vertices involving two gauge bosons and GB's follow as simple generalizations of the vertex without GB's. The vertex for two gauge bosons, one GB, and two quarks (as shown in Fig. 6) is

$$
\begin{aligned}
g^2\Gamma^{\mu a,\nu b,c}(p,q_1,q_2,k,p+k+q_1+q_2)=-\frac{i\gamma_5 g^2}{F_0}\{&\lambda^c\Gamma^{\mu a,\nu b}(p,q_1,q_2,p+q_1+q_2)\\
&+\Gamma^{\mu a,\nu b}(p+k,q_1,q_2,p+k+q_1+q_2)\lambda^c\}\;. \qquad (41)
\end{aligned}
$$

Introducing the notation

$$
\begin{aligned}
\Gamma^{\mu,\nu}(p,q_1,q_2,p+q_1+q_2)=-\Bigg[&\frac{2g^{\mu\nu}}{2p\cdot(q_1+q_2)+(q_1+q_2)^2}[\Sigma(p+q_1+q_2)-\Sigma(p)]\\
&+\frac{(2p+q_2)^\nu[2(p+q_2)+q_1]^\mu}{2(p+q_2)\cdot q_1+q_1^2}\left[\frac{\Sigma(p+q_1+q_2)-\Sigma(p)}{2p\cdot(q_1+q_2)+(q_1+q_2)^2}-\frac{\Sigma(p+q_2)-\Sigma(p)}{2p\cdot q_2+q_2^2}\right]\\
&+\frac{(2p+q_1)^\mu[2(p+q_1)+q_2]^\nu}{2(p+q_1)\cdot q_2+q_2^2}\left[\frac{\Sigma(p+q_1+q_2)-\Sigma(p)}{2p\cdot(q_1+q_2)+(q_1+q_2)^2}-\frac{\Sigma(p+q_1)-\Sigma(p)}{2p\cdot q_1+q_1^2}\right]\Bigg] \qquad (42)
\end{aligned}
$$

and

$$
p'=p+q_1+q_2+k_1+k_2,\quad p''=p+q_1+q_2\;, \qquad (43)
$$

the vertex for two gauge bosons, two GB's, and two quarks (as shown in Fig. 7) may be written as

$$
\begin{aligned}
\Gamma^{\mu a,\nu b,c,d}(p,q_1,q_2,k_1,k_2,p')=-\frac{g^2}{2F_0^2}[&\lambda^c\lambda^d\Gamma^{\mu a,\nu b}(p,q_1,q_2,p'')+\Gamma^{\mu a,\nu b}(p+k_1+k_2,q_1,q_2,p')\lambda^c\lambda^d\\
&+\lambda^d\Gamma^{\mu a,\nu b}(p+k_1,q_1,q_2,p''+k_1)\lambda^c+\Gamma^{\mu a,\nu b}(p+k_1,q_1,q_2,p''+k_1)\lambda^c\lambda^d\\
&+\lambda^c\lambda^d\Gamma^{\mu a,\nu b}(p+k_1,q_1,q_2,p''+k_1)+\lambda^c\Gamma^{\mu b,\nu a}(p+k_1,q_1,q_2,p''+k_1)\lambda^d\\
&-(T^a\lambda^cT^b\lambda^d+T^b\lambda^cT^a\lambda^d-T^a\lambda^c\lambda^dT^b-T^b\lambda^c\lambda^dT^a+\lambda^cT^a\lambda^dT^b+\lambda^cT^b\lambda^dT^a)\\
&\times\Gamma^{\mu,\nu}(p+k_1,q_1,q_2,p''+k_1)+c\leftrightarrow d\,k_1\leftrightarrow k_2]\;. \qquad (44)
\end{aligned}
$$

Note that in the sixth term, the indices a and b are interchanged. As before, all of these multi-gauge-boson vertices can be checked using the WT identities.

VI. THE EFFECTIVE CHIRAL LAGRANGIAN

We can now calculate the effective action for photons and GB's by integrating out the quarks; i.e.,

$$
\exp(-\Gamma[\pi,V_\mu])\equiv\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left[-\int d^4x\,d^4y\,\mathcal{L}_g(x,y)\right], \qquad (45)
$$

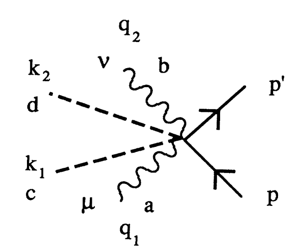

FIG. 7. The vertex coupling two-gauge bosons, two quarks, and two GB's: $g^2\Gamma^{\mu a,\nu b,c,d}(p,q_1,q_2,k_1,k_2,p')$.

where $\mathcal{L}_g$ is the gauged NCQM Lagrangian. The effective action can be written as (dropping a constant term)

$$\Gamma[\pi, V_\mu] = -\mathrm{Tr}\ln\left[\delta(x-y) - \int d^4z\, S(x-z) I(z,y)\right] , \tag{46}$$

where

$$I(z,y) = \frac{-\delta^2 S_{\mathrm{int}}}{\delta\psi(y)\delta\bar{\psi}(z)} , \tag{47}$$

$S(x-z)$ is the quark propagator, and S_{int} is the interaction part of the gauged action. The quark interactions are represented by $I(z,y)$ in Eq. (46), which is just minus the interaction terms in the Lagrangian, with the quarks fields left out. More explicitly,

$$I = I_\pi + \tfrac{1}{2} I_{\pi\pi} + I_V + I_{V\pi} + \tfrac{1}{2} I_{V\pi\pi} + \cdots , \tag{48}$$

where the subscripts indicate the particles involved in the interaction (the factors of $\frac{1}{2}$ indicate that a term in the Lagrangian with two identical fields contributes twice to the corresponding vertex). Expanding out the ln and only keeping terms with one gauge field and two GB fields, we find simply

$$\Gamma[\pi, V_\mu] = \mathrm{Tr}\big(\tfrac{1}{2} S I_{V\pi\pi} + S I_V S I_\pi S I_\pi + \tfrac{1}{2} S I_V S I_{\pi\pi} + S I_{V\pi} S I_\pi \big) . \tag{49}$$

A useful way to extract information from the effective action is to compare its derivative expansion with a chiral Lagrangian. A chiral Lagrangian is obtained by writing down all possible terms for the interactions of GB's among themselves and with external fields that are consistent with the symmetries of the theory. Each term has an arbitrary coefficient which is undetermined by symmetry arguments. Since only symmetry information has been used, the chiral Lagrangian must generate a parametrization of the most general S matrix consistent with the symmetries of the theory (as expressed by the generalized WT identities) [6,7]. The terms in the chiral Lagrangian are arranged in an expansion in powers of derivatives, and usually one keeps only the first few terms in such a derivative (or energy) expansion. Since the gauged NCQM Lagrangian is chirally invariant, the effective action must also enjoy this symmetry. Thus the effective action must be equivalent to (the action associated with) a particular chiral Lagrangian [8], and we may extract the coefficients of this effective chiral Lagrangian for the NCQM by comparing the amplitudes derived from the effective action with those derived from the chiral Lagrangian.

The standard parametrization of the chiral Lagrangian for three flavors of quarks is due to Gasser and Leutwyler [7]. The lowest-order (second-order) term in the energy expansion is [9]

$$\mathcal{L}_2 = \tfrac{1}{4} F_0^2 \mathrm{Tr}(D_\mu U^\dagger D^\mu U) , \tag{50}$$

where

$$U(x) = \exp\left[\frac{-2i\pi(x)}{F_0}\right] , \tag{51}$$

and D_μ is the gauge-covariant derivative:

$$D_\mu U = \partial_\mu U - i(V_\mu^a + A_\mu^a)\lambda^a U + iU\lambda^a(V_\mu^a - A_\mu^a) . \tag{52}$$

Here, V_μ^a and A_μ^a represent external SU(3) vector and axial-vector gauge fields, respectively.

At fourth order in the energy expansion (dropping terms that are irrelevant for our purposes) Gasser and Leutwyler give

$$\mathcal{L}_4 = -iL_9 \mathrm{Tr}(F_{\mu\nu}^R D^\mu U D^\nu U^\dagger + F_{\mu\nu}^L D^\mu U^\dagger D^\nu U) + L_{10}\mathrm{Tr}(U^\dagger F_{\mu\nu}^R U F^{L\mu\nu}) + \cdots , \tag{53}$$

where

$$\begin{aligned} F_{\mu\nu}^R &= \partial_\mu R_\nu - \partial_\nu R_\mu - [R_\mu, R_\nu] , \\ F_{\mu\nu}^L &= \partial_\mu L_\nu - \partial_\nu L_\mu - [L_\mu, L_\nu] , \\ R_\mu &= (V_\mu^a + A_\mu^a)\lambda^a, \quad L_\mu = (V_\mu^a - A_\mu^a)\lambda^a . \end{aligned} \tag{54}$$

It is the coefficients L_9 and L_{10} that we would like to determine from the NCQM. The coefficient L_9 is phenomenologically interesting since it is related to the charge radius of the GB's [10]. Also, L_9 and L_{10} are involved in the determination of the decay $\pi \to \gamma e \nu$ [7].

If we calculate the amplitude which couples two GB's and one photon from the effective action (i.e., calculate $-\delta^3\Gamma/\delta\pi^a\delta\pi^b\delta V^\mu$), we find that it is given by the diagrams in Fig. 8, as we naively expected. It should be noted that, in the calculation of the electromagnetic form factor by Pagels and Stokar [11], these authors only in- the second and third Feynman diagrams shown in Fig. 8. Taking the number of quark "colors" to be N_c, the sum of the diagrams in Fig. 8 gives

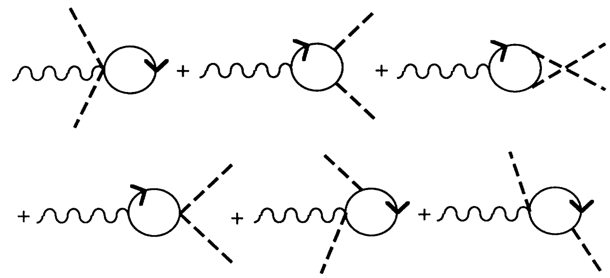

FIG. 8. The two-GB-photon amplitude: $e\Gamma^{\mu,a,b}(p,q,p+q)$.

$$e\Gamma^{\mu,a,b}(p,q,p+q) = \frac{4eN_c}{F_0^2}\mathrm{Tr}Q(\lambda^a\lambda^b - \lambda^b\lambda^a) \times \int \frac{d^4k}{(2\pi)^4} \frac{I^\mu(k,p,q)}{[k^2+\Sigma^2(k)][(k-p)^2+\Sigma^2(k-p)][(k+q)^2+\Sigma^2(k+q)]} , \tag{55}$$

where

$$I^{\mu}(k,p,q)=\frac{(2p+q)^{\mu}}{2k\cdot q+q^{2}}[\Sigma(k+q)-\Sigma(k)]G(k,p,q)+[\Sigma(k)+\Sigma(k-p)][\Sigma(k+q)+\Sigma(k-p)]$$
$$\times\{k^{\mu}[(k+q)\cdot(k-p)+\Sigma(k+q)\Sigma(k-p)]-(k-p)^{\mu}[k\cdot(k+q)+\Sigma(k)\Sigma(k+q)]$$
$$+(k+q)^{\mu}[k\cdot(k-p)+\Sigma(k)\Sigma(k-p)]\}\ , \tag{56}$$

and

$$G(k,p,q)=2\Sigma(k-p)[k^{2}+\Sigma^{2}(k)][(k+q)^{2}+\Sigma^{2}(k+q)]$$
$$-[k\cdot(k-p)+\Sigma(k)\Sigma(k-p)][\Sigma(k)+\Sigma(k-p)][(k+q)^{2}+\Sigma^{2}(k+q)]$$
$$-[(k+q)\cdot(k-p)+\Sigma(k+q)\Sigma(k-p)][\Sigma(k+q)+\Sigma(k-p)][k^{2}+\Sigma^{2}(k)]$$
$$+[\Sigma(k)+\Sigma(k-p)][\Sigma(k+q)+\Sigma(k-p)]$$
$$\times[k\cdot(k-p)\Sigma(k+q)-k\cdot(k+q)\Sigma(k-p)+(k+q)\cdot(k-p)\Sigma(k)+\Sigma(k)\Sigma(k-p)\Sigma(k+q)]\ . \tag{57}$$

Taylor expanding to first order in the external momentum, we find

$$e\Gamma_{1}^{\mu,a,b}(p,q,p+q)=\frac{4eN_{c}}{F_{0}^{2}}\mathrm{Tr}Q[\lambda^{b},\lambda^{a}](2p+q)^{\mu}\times\int\frac{d^{4}k}{(2\pi)^{4}}\frac{4\Sigma^{2}(k)-2k^{2}\Sigma(k)\Sigma'(k)}{[k^{2}+\Sigma^{2}(k)]^{2}}\ . \tag{58}$$

Using Eq. (16) of Ref. [2] for F_0^2 we find

$$e\Gamma_{1}^{\mu,a,b}(p,q,p+q)=2e\ \mathrm{Tr}Q[\lambda^{b},\lambda^{a}](2p+q)^{\mu}\ , \tag{59}$$

which is the same as the gauge vertex derived from the lowest-order chiral Lagrangian [Eq. (50)] (when Wick rotated to Euclidean space). The fact that to this order in the energy expansion the amplitude comes out properly normalized and independent of $\Sigma(p)$ is a consistency check on our gauging procedure.

To go further we must note that in obtaining $\mathcal{L}_4$, Gasser and Leutwyler [7] have used the equations of motion (since the terms in $\mathcal{L}_4$ are only needed at tree order) to eliminate two other possible interaction terms from $\mathcal{L}_4$:

$$\mathcal{L}_{\mathrm{EQM}}=L_{11}\mathrm{Tr}(D^{2}UD^{2}U^{\dagger})+L_{12}\mathrm{Tr}(\chi^{\dagger}D^{2}U+\chi D^{2}U^{\dagger})\ . \tag{60}$$

For our procedure to be consistent we must make the comparison of our effective action to a chiral Lagrangian that contains these terms. Once the coefficients L_{11} and L_{12} are determined, then the equations of motion may be used to rewrite $\mathcal{L}_{\mathrm{EQM}}$ as a linear function of terms already contained in $\mathcal{L}_4$.

By Taylor expanding Eq. (55) to third order in momentum we find

$$e\Gamma_{3}^{\mu,a,b}(p,q,p+q)=\frac{-eN_{c}}{4\pi^{2}F_{0}^{2}}\mathrm{Tr}Q[\lambda^{b},\lambda^{a}]\{p^{\mu}(I_{1}p^{2}+I_{2}p\cdot q+I_{3}q^{2})+q^{\mu}(I_{4}p^{2}+I_{5}p\cdot q+I_{6}q^{2})\}\ , \tag{61}$$

where the coefficients I_1 through I_6 are a series of quite messy integrals involving $\Sigma(p)$. We will not reproduce these integrals here; they are given in Ref. [12]. The fourth-order chiral Lagrangian [Eqs. (53) and (60)] gives

$$e\Gamma_{3}^{\mu,a,b}(p,q,p+q)=\frac{-8e}{F_{0}^{2}}\mathrm{Tr}Q[\lambda^{b},\lambda^{a}]\{p^{\mu}[4L_{11}p^{2}+4L_{11}p\cdot q+(2L_{11}+L_{9})q^{2}]$$
$$+q^{\mu}[2L_{11}p^{2}+(2L_{11}-L_{9})p\cdot q+L_{11}q^{2}]\}\ . \tag{62}$$

Thus we obtain the coefficients L_9 and L_{11}, with several independent checks on their values.

We now turn to calculating L_{10}. The term with fewest fields in the L_{10} term in the fourth-order chiral Lagrangian, Eq. (53), involves one GB, one vector gauge field, and one axial-vector gauge field. We could calculate the coefficient of this term in the effective chiral Lagrangian derived from the underlying NCQM (and thus determine L_{10}) by introducing vector and axial-vector gauge fields into the model. Since the GB fields do not transform linearly under axial transformations, an expansion in powers of GB fields proves to be awkward. An alternative method for calculating L_{10} can be seen by noting that there is a term involving two GB fields, and two vector gauge fields present in the L_{10} term. The coefficient of this term can be calculated using the techniques we have already developed.

Calculating the amplitude which couples two GB's and two vector gauge fields from this effective action (i.e., calculating $-\delta^4\Gamma/\delta\pi^c\delta\pi^d\delta V^a_\mu\delta V^b_\nu$), we find that the amplitude is given by the diagrams in Fig. 9. Our calculations will be simplified a great deal if we consider the limit where the momenta of the GB's vanish. In this limit most of the contributions to the two-GB, two-gauge-boson vertex vanish. For vanishing GB momenta, we Taylor expand the sum of diagrams in Fig. 9 in the gauge-boson momentum q. Keeping only terms of zeroth order in q, we find

$$g^2\Gamma_1^{\mu a,\nu b,c,d}(0,q,-q,0) = \frac{ig^2N_c}{16\pi^2F_0^2}J_1 g^{\mu\nu}\mathrm{Tr}(2\lambda^c\lambda^a\lambda^d\lambda^b+2\lambda^c\lambda^b\lambda^d\lambda^a -\lambda^c\lambda^d\lambda^a\lambda^b-\lambda^c\lambda^a\lambda^b\lambda^d -\lambda^c\lambda^d\lambda^b\lambda^a-\lambda^c\lambda^b\lambda^a\lambda^d)\ , \quad (63)$$

where J_1 represents a messy integral. The second-order chiral Lagrangian [Eq. (50)] gives the amplitude

$$g^2\Gamma_1^{\mu a,\nu b,c,d}(0,q,-q,0) = -2ig^2g^{\mu\nu}\mathrm{Tr}(2\lambda^c\lambda^a\lambda^d\lambda^b+2\lambda^c\lambda^b\lambda^d\lambda^a -\lambda^c\lambda^d\lambda^a\lambda^b-\lambda^c\lambda^a\lambda^b\lambda^d -\lambda^c\lambda^d\lambda^b\lambda^a-\lambda^c\lambda^b\lambda^a\lambda^d)\ . \quad (64)$$

The requirement that

$$F_0^2=-\frac{J_1N_c}{32\pi^2} \quad (65)$$

(which can be verified numerically) is another check on our calculations.

The terms of second order in q give an amplitude of the form

$$g^2\Gamma_2^{\mu a,\nu b,c,d}(0,q,-q,0) = \frac{g^2N_c}{16\pi^2F_0^2}(J_2q^2g^{\mu\nu}+J_3q^\mu q^\nu) \times\mathrm{Tr}(2\lambda^c\lambda^a\lambda^d\lambda^b+2\lambda^c\lambda^b\lambda^d\lambda^a-\lambda^c\lambda^d\lambda^a\lambda^b -\lambda^c\lambda^a\lambda^b\lambda^d-\lambda^c\lambda^d\lambda^b\lambda^a-\lambda^c\lambda^b\lambda^a\lambda^d)\ , \quad (66)$$

where J_2 and J_3 are again some messy integrals which will not be given here (see Ref. [12]). The fourth-order chiral Lagrangian [Eqs. (53) and (60)], taking vanishing GB momentum, gives

$$g^2\Gamma_2^{\mu a,\nu b,c,d}(0,q,-q,0) = \frac{8g^2}{F_0^2}(L_{10}q^2g^{\mu\nu}-(L_{11}+L_{10})q^\mu q^\nu) \times\mathrm{Tr}(2\lambda^c\lambda^a\lambda^d\lambda^b+2\lambda^c\lambda^b\lambda^d\lambda^a-\lambda^c\lambda^d\lambda^a\lambda^b -\lambda^c\lambda^a\lambda^b\lambda^d-\lambda^c\lambda^d\lambda^b\lambda^a-\lambda^c\lambda^b\lambda^a\lambda^d)\ . \quad (67)$$

Thus we have determined L_{10}, and, since we have already determined L_{11}, we again have some checks on our calculations.

Finally, we will present some numerical results. To do this we must specify the functional form of the dynamical mass $\Sigma(p)$. From Politzer's study of the operator-product expansion [13], it is known that in an asymptotically free gauge theory $\Sigma(q)$ should fall off asymptotically as $1/q^2$. As a simple extrapolation of this asymptotic form, which does not behave wildly as q^2 approaches zero, we have used

$$\Sigma(q)=\frac{(A+1)m^3}{m^2+Aq^2}\ . \quad (68)$$

The normalization is chosen such that $\Sigma(q=m)=m$.

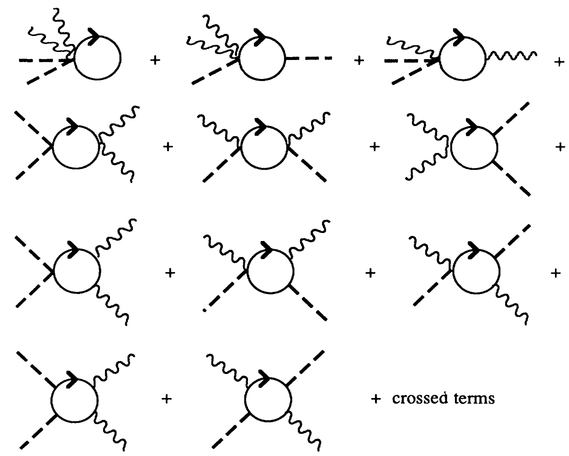

FIG. 9. The amplitude coupling two gauge bosons and two GB: $g^2\Gamma^{\mu a,\nu b,c,d}(p,q_1,q_2,p+q_1+q_2)$. The crossed terms correspond to the 15 other ways to assign particles to the legs.

TABLE I. Coupling constants of the effective chiral Lagrangian for different values of the parameter A.

	$A=1$	$A=\frac{1}{3}$	$A=0$
L_9	5.94×10^{-3}	5.84×10^{-3}	$\frac{N_c}{48\pi^2}=6.33\times10^{-3}$
L_{10}	-6.76×10^{-3}	-4.96×10^{-3}	$\frac{-N_c}{96\pi^2}=-3.17\times10^{-3}$
L_{11}	2.98×10^{-3}	3.03×10^{-3}	$\frac{N_c}{96\pi^2}=3.17\times10^{-3}$

The necessary Taylor expansions were done using MAPLE, and the resulting expressions were integrated numerically using Eq. (68) for the dynamical mass with different values of A. The results (with $N_c=3$) are given in Table I.

We note that the column $A=0$ corresponds to the local limit of the model. This special case is essentially equivalent to the nonlinear σ model, and equivalent results for L_9 and L_{10} have been obtained in this limit by other authors [14]. We also note that the coefficient L_{10} is the most sensitive to how slowly or quickly $\Sigma(p)$ falls off.

To make a meaningful comparison with the experimentally determined values [7] of L_9 and L_{10}, both sets of values should be renormalized at the same scale. The values given in Table I are renormalized at the "matching scale" [15] where the quarks are integrated out (this scale should be taken to be roughly $2m$). Thus the value of m must first be determined, and then L_9 and L_{10} can be renormalized down to the scale used by Gasser and Leutwyler in Ref. [7]. This procedure has been carried out in Ref. [2]. It is found that the results of the NCQM are in surprisingly good agreement with experiment.

VII. CONCLUSIONS

We have described how to gauge nonlocal Lagrangians, and how to derive the Feynman rules. The method has been applied in detail to the NCQM. In particular, this technique has enabled us to extract the coupling constants L_9, L_{10}, and L_{11} of the effective chiral Lagrangian for the model. The value of L_{10} may also be of interest in technicolor models, where the analogue of L_{10} can induce large radiative corrections to the left-right asymmetry [16]. As seen in Table I, L_{10} is smaller for a more slowly falling form of $\Sigma(p)$, and a more slowly falling $\Sigma(p)$ is expected in most "walking" technicolor theories.

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APPENDIX

We present here some proofs by induction which are used above. First we wish to prove that

$$(q+p)^{2n}-p^{2n}=(q^2+2p\cdot q)g_n\ , \tag{A1}$$

where g_n is defined by

$$g_n=(q+p)^{2(n-1)}+p^2 g_{n-1} \tag{A2}$$

and $g_1=1$. Note that $(q+p)^4-p^4=(q^2+2p\cdot q)[(q+p)^2+p^2]$, so that [from Eq. (23)] $g_2=(q+p)^2+p^2=f_2$. Suppose that Eq. (A1) is true for some particular n, and consider

$$\begin{aligned}(q+p)^{2(n+1)}-p^{2(n+1)}&=(q^2+2p\cdot q+p^2)(q+p)^{2n}-p^{2(n+1)}\\&=(q^2+2p\cdot q)(q+p)^{2n}+p^2[(q+p)^{2n}-p^{2n}]\ .\end{aligned} \tag{A3}$$

By assumption we then have

$$(q+p)^{2(n+1)}-p^{2(n+1)}=(q^2+2p\cdot q)[(q+p)^{2n}+p^2 g_n]\ , \tag{A4}$$

and by the definition of g_{n+1} we have

$$(q+p)^{2(n+1)}-p^{2(n+1)}=(q^2+2p\cdot q)g_{n+1}\ , \tag{A5}$$

which completes the proof. We would now like to prove that $f_n=g_n$, for $n>0$, where f_n is defined in Eqs. (23) and (24). Suppose that this is true for a particular n (we have demonstrated this above for $n=1$ and $n=2$); then by the definition of f_{n+1} we have

$$f_{n+1}=f_n(q+p)^2+p^{2n}\ . \tag{A6}$$

Then by assumption we have

$$f_{n+1}=g_n(q^2+2p\cdot q+p^2)+p^{2n}\ . \tag{A7}$$

Use of the preceding lemma yields

$$f_{n+1}=(q+p)^{2n}-p^{2n}+g_n p^2+p^{2n}\ , \tag{A8}$$

which, by the definition of g_{n+1}, gives us

$$f_{n+1}=g_{n+1}\ , \tag{A9}$$

which was to be proved.

We would now like to find analogous results for d_n and j_n, which are defined in Eqs. (36) through (39). From Eqs. (A1) and (A9), we know that

$$d_n=\frac{(q_1+q_2+p)^{2n}-p^{2n}}{2p\cdot(q_1+q_2)+(q_1+q_2)^2} \tag{A10}$$

and

$$j_n=\frac{(q_2+p)^{2n}-p^{2n}}{2p\cdot q_2+q_2^2}\ . \tag{A11}$$

Note that $d_1-h_1=0$, $d_2-h_2=[q_1^2+2(p+q_2)\cdot q_1]j_2$, and $d_3-h_3=[q_1^2+2(p+q_2)\cdot q_1]j_3$. We will again proceed by induction. Suppose that for some particular n, $[q_1^2+2(p+q_2)\cdot q_1]j_{n-1}=d_{n-1}-h_{n-1}$; then, by the definition of j_n [Eq. (38)],

$$\begin{aligned}[q_1^2+2(p+q_2)\cdot q_1]j_n&=[q_1^2+2(p+q_2)\cdot q_1]\\&\quad\times[j_{n-1}(q_1+q_2+p)^2+h_{n-1}]\ .\end{aligned} \tag{A12}$$

By assumption, we then have

$$\begin{aligned}[q_1^2+2(p+q_2)\cdot q_1]j_n &= (d_{n-1}-h_{n-1})(q_1+q_2+p)^2+[q_1^2+2(p+q_2)\cdot q_1]h_{n-1} \\ &= d_{n-1}(q_1+q_2+p)^2-h_{n-1}(q_2+p)^2 \\ &= d_{n-1}(q_1+q_2+p)^2+p^{2(n-1)}-[h_{n-1}(q_2+p)^2+p^{2(n-1)}] \\ &= d_n-h_n\ , \end{aligned} \tag{A13}$$

where we have used the definitions of d_n and h_n [Eqs. (38) and (39)]. Thus,

$$j_n=\frac{d_n-h_n}{2(p+q_2)\cdot q_1+q_1^2}\ . \tag{A14}$$

As shown above for d_n, we also have

$$h_n=\frac{(q_2+p)^{2n}-p^{2n}}{2p\cdot q_2+q_2^2}\ . \tag{A15}$$

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